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A NOTE ON PERINORMAL DOMAINS

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ABSTRACT. Recently, N. Epstein and J. Shapiro introduced and studied the perinormal domains: those domains A whose overrings satisfying going down over A are flat A-modules. We show that every Prüfer v-multiplication domain is perinormal and has no proper lying over overrings. Conversely, we show that a w-treed perinormal domain is a Prüfer v-multiplication domain. We give two pull-back constructions that produce perinormal/non-perinormal domains.

1. Introduction. In their recent paper **[6]**, Epstein and Shapiro introduced and studied *perinormal domains*: those domains A whose local GD-overrings are localizations of A (equivalently, whose GD-overrings are flat A-modules). Here, a GD-*overring* B of A means a ring between A and its quotient field such that going down holds for $A \subseteq B$, that is, the induced spectral map

$$\mathrm{Spec}(B_Q) \longrightarrow \mathrm{Spec}(A_{Q\cap A})$$

is surjective for each $Q \in \mathrm{Spec}(B)$. Due to **[18,** Theorem 2], A is perinormal if and only if whenever B is a local GD-overring of A, we have $B = A_{N\cap A}$, where N is the maximal ideal of B. Perinormality is a local property, cf., **[6,** Theorem 2.3]. Also studied in **[6]** is the subclass of *globally perinormal domains*: those domains A whose GD-overrings are fraction rings of A. Thus, A is globally perinormal if and only if A is perinormal and its flat overrings are fraction rings of A.

The purpose of this note is to extend some of the results in **[6]**. We obtain the following results (all necessary definitions, although standard, are recalled in the next section where used). In Theorem 2.1, we show that an essential domain has no proper LO-overrings (by an LO-*overring* B of a domain A, we mean an overring such that lying

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over holds for $A \subseteq B$, that is, the map

$$\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$$

is surjective). We show that every P-domain **[17]** is perinormal (Theorem 2.2). Consequently, a Prüfer v-multiplication domain (PvMD) is perinormal and has no proper LO-overrings (Corollary 2.5). In particular, we retrieve **[6,** Theorem 3.10], which states that a generalized Krull domain is perinormal (Corollary 2.5). Using Heinzer's example **[12]**, we remark that an essential domain is not necessarily perinormal (Remark 2.3).

A local perinormal domain with linearly ordered spectrum is a valuation domain (Proposition 2.8). We thus extend **[6,** Proposition 3.2], which states essentially that a one-dimensional local perinormal domain is a valuation domain. In particular, a w-treed perinormal domain is a PvMD, and a treed perinormal domain is Prüfer (Corollary 2.9). The pullback construction in **[6,** Theorem 5.2] produces perinormal domains starting from semilocal Krull domains. We extend this construction, relaxing the Krull domain hypothesis to a P-domain and removing the semilocal restriction (Theorem 2.10). We also give a construction producing non-perinormal domains (Theorem 2.11). In **[6,** Theorem 6.4], it was shown that a Krull domain with torsion divisor class group is globally perinormal. We extend this result by showing that a PvMD with torsion class group, e.g., a GCD domain, is globally perinormal (Theorem 2.13). Consequently, an AGCD domain is globally perinormal if and only if it is integrally closed (Corollary 2.14). In Theorem 2.15, we slightly improve Theorem 2.13.

Throughout this paper, all rings are (commutative unitary) integral domains. Any unexplained terminology is standard, as in **[9, 11, 14]**.

2. Results. Let A be a domain. Call a prime ideal P of A a *valued prime* if A_P is a valuation domain. Recall that A is said to be an *essential domain* if $A = \cap_{P \in G} A_P$, where G is a set of valued primes of A.

Theorem 2.1. *If A is an essential domain and B is an* LO*-overring of A, then $A = B$.*

Proof. Since A is essential, $A = \cap_{P \in G} A_P$, where G is the set of valued primes of A. As $A \subseteq B$ satisfies lying over, for each $P \in G$, we can choose $P' \in \operatorname{Spec}(B)$ such that $P' \cap A = P$. Since A_P is a valuation domain, it follows that $A_P = B_{P'}$, cf., [**9,** Theorem 26.1]. Then,

$$A \subseteq B \subseteq \bigcap_{P \in G} B_{P'} = \bigcap_{P \in G} A_P = A.$$

Thus, $B = A$. □

Recall [**17**] that a domain A is called a P-*domain* if A_P is a valuation domain for every prime ideal P which is minimal over an ideal of the form $Aa : b$ with $a, b \in A$. A P-domain is an essential domain, but not conversely, cf., [**17,** Proposition 1.1] and [**12**]. Moreover, a fraction ring of a P-domain is still a P-domain, cf., [**17,** Corollary 1.2].

Now, we state the main result of this paper.

Theorem 2.2. *Every* P-*domain is perinormal.*

Proof. Let A be a P-domain and B a GD-overring of A. Let $Q \in \operatorname{Spec}(B)$ and $P = Q \cap A$. By [**17,** Proposition 1.1] and [**6,** Lemma 2.2], A_P is an essential domain. By going down, B_Q is an LO-overring of A_P. Thus, Theorem 2.1 applies to yield $A_P = B_Q$. Hence, $A \subseteq B$ is flat due to [**18,** Theorem 2]. □

Remark 2.3. An essential domain is not necessarily perinormal. Indeed, the essential domain D constructed in [**12**] has a one-dimensional localization D_P, which is not a valuation domain. It follows that D is not perinormal, cf., [**6,** Proposition 3.2].

Remark 2.4. The underlying idea of the proof of Theorem 2.2 is very simple. Let $\mathcal{D}$ be a class of domains which is closed under localizations at prime ideals. If every $A \in \mathcal{D}$ has no proper LO-overring, then every $A \in \mathcal{D}$ is perinormal. Indeed, suppose that B is a GD-overring of A, $Q \in \operatorname{Spec}(B)$ and $P = Q \cap A$. Then, $A_P \subseteq B_Q$ satisfies lying over; thus, $A_P = B_Q$.

Let A be a domain with the quotient field K. Recall that A is a *Prüfer* v-*multiplication domain* (PvMD) if, for every finitely generated nonzero ideal I, there exists a finitely generated nonzero ideal J such that $(IJ)_v$ is a principal ideal. Here, as usual, for a fractional nonzero ideal H of A, its divisorial closure (v-closure) is the fractional ideal $H_v := (H^{-1})^{-1}$, where H^{-1} is fractional ideal

$$A : H = \{x \in K \mid xH \subseteq A\}.$$

By [**17,** Corollary 1.4 and Example 2.1], a PvMD is a P-domain but not conversely. It is well known that a GCD domain is a PvMD, (see, for instance, [**17,** Proposition 6.1]). From these remarks and Theorem 2.1, we have the following.

Corollary 2.5. *A* PvMD*, e.g., a* GCD *domain, is perinormal and has no proper* LO*-overrings.*

Let A be a domain and $X^1(A)$ the set of height one prime ideals of A. Recall [**9,** section 43] that A is a *generalized Krull domain*, respectively, *Krull domain*, if

$$A = \bigcap_{P \in X^1(A)} A_P,$$

the intersection has finite character, and A_P is a valuation domain, respectively, a discrete valuation domain, for each $P \in X^1(A)$. Clearly, a Krull domain is a generalized Krull domain. Since a generalized Krull domain is a PvMD, cf., [**10,** Theorem 7]), we recover the following.

Corollary 2.6 ([**6,** Theorem 3.10])**.** *A* (*generalized*) *Krull domain is perinormal.*

Remark 2.7. Recall that an FC (*finite conductor*) *domain* is a domain in which every intersection of two principal ideals is finitely generated. By [**16,** Theorem 2], an integrally closed FC domain has no proper LO-overrings. As both normality and FC conditions localize, we derive that an integrally closed FC domain is perinormal, cf., Remark 2.4. However, this is in fact a consequence of Corollary 2.5 since an integrally closed FC domain is a PvMD, cf., [**8,** Corollary 2.5].

By [**6,** Proposition 3.2], the localization of a perinormal domain at a height one prime ideal is a valuation domain. We extend this result. Since perinormality localizes, it suffices to present the local case.

Proposition 2.8. *A local perinormal domain A with linearly ordered spectrum is a valuation domain.*

Proof. By [**13**], there exists a lying over valuation overring V of A. Then, V is a GD-overring of A. Since A is perinormal, it follows that $A = V$ due to the fact the maximal ideal of V lies over the maximal ideal of A. $\square$

Let A be a domain. Recall that a nonzero ideal I is a t-*ideal* if $H_v \subseteq I$ for every nonzero finitely generated subideal H of I. It is well known that every proper t-ideal is contained in a maximal t-ideal, and every maximal t-ideal is prime, see [**11,** Theorem 6.4]. Also, recall that A is called a *treed domain*, respectively, w-*treed domain*, if $\mathrm{Spec}(A_M)$ is linearly ordered for each maximal ideal, respectively, maximal t-ideal, M of A. As the Prüfer domains, respectively PvMDs, have the localizations at their maximal ideals, respectively, maximal t-ideals, valuation domains, see [**17,** Corollary 4.3], the Prüfer domains, respectively, PvMDs, are treed, respectively, w-treed. In addition, the PvMDs are perinormal, cf., Corollary 2.5. We prove the converse.

Corollary 2.9.

(a) *A* w-*treed perinormal domain is a* PvMD*.*

(b) *A treed, e.g., one-dimensional, perinormal domain is a Prüfer domain.*

Proof.

(a) Let A be a w-treed perinormal domain and M a maximal t-ideal of A. Then, $\mathrm{Spec}(A_M)$ is linearly ordered; thus, A_M is a valuation domain, cf., Proposition 2.8. Thus, A is a PvMD, cf., [**17,** Corollary 4.3].

The proof of (b) is similar to (a) and thus will be omitted. $\square$

Our next result extends the pullback construction in [**6,** Theorem 5.2] producing perinormal domains. While our proof uses the same idea, we relax the Krull domain hypothesis to the P-domain and remove the semilocal restriction.

Let B be a P-domain and $M_1, \dots, M_n$ maximal ideals of B such that none of them is a valued prime. Assume further that all fields B/M_i

are isomorphic to the same field K by isomorphisms

$$\sigma_i : B/M_i \longrightarrow K, \quad i = 1, \ldots, n.$$

Set $I = M_1 \cap \cdots \cap M_n$. Let

$$\pi : B \longrightarrow K^n$$

be the composition of the canonical morphism $B \to B/I$, the Chinese remainder theorem morphism

$$B/I \longrightarrow B/M_1 \times \cdots \times B/M_n$$

and

$$(\sigma_1, \ldots, \sigma_n) : B/M_1 \times \cdots \times B/M_n \longrightarrow K^n.$$

Finally, identify K with its diagonal image in K^n.

Theorem 2.10. *In the setup above, the pullback domain $A = \pi^{-1}(K)$ is perinormal.*

Proof. Note that B is perinormal by Theorem 2.2. Clearly, $I = \ker(\pi)$ is a common ideal of A and B. Let (C, N) be a GD-overring of A, and set $P = N \cap A$. Assume that $P \not\supseteq I$. By usual pullback arguments, we have

$$C \supseteq A_P = B_Q,$$

where $Q = PA_P \cap B$. Moreover, since C is a going down extension of the perinormal domain B_Q and $N \cap B_Q = QB_Q$, we obtain

$$A_P = B_Q = C;$$

thus, we are done in this case. Assume now that $P \supseteq I$. By [**7,** Lemma 1.1.6], we may localize A and B in $A - P$, and thus assume that A is local with maximal ideal P. Then, $A \subseteq C$ satisfies lying over since it satisfies going down and $P \subseteq N$. Let G be the set of valued primes of B. For every $H \in G$, select $H' \in \mathrm{Spec}(C)$ such that $H' \cap A = H \cap A$. Since none of $M_1, \ldots, M_n$ is a valued prime, we obtain

$$H \not\supseteq I;$$

thus,

$$B_H = A_{H \cap A} = C_{H'}$$

since B_H is a valuation domain. We have

$$A \subseteq C \subseteq \bigcap_{H \in G} C_{H'} = \bigcap_{H \in G} B_H = B$$

due to the fact that B is an essential domain. Therefore, $A \subseteq C \subseteq B$.

We claim that $A = C$. Indeed, since I is a common ideal of A, B and C, we get

$$A/I = K \subseteq C/I \subseteq B/I = K^n.$$

Since C/I is local and K is the only local ring between K and K^n, we derive that $C/I = K$; thus, $C = A$. $\square$

The following pullback construction provides examples of non-perinormal domains.

Theorem 2.11. *Let B be a domain, M a maximal ideal of B,*

$$\pi : B \longrightarrow B/M$$

the canonical map and K a proper subfield of B/M. Then, the pullback domain $A = \pi^{-1}(K)$ is not perinormal.

Proof. Clearly, M is a maximal ideal of both A and B. We claim that going down holds for $A \subseteq B$. Let $Q \in \operatorname{Spec}(B)$ and $P = Q \cap A$. It suffices to prove that the map

$$\operatorname{Spec}(B_Q) \longrightarrow \operatorname{Spec}(A_P)$$

is surjective. If $Q \neq M$ and $f \in M - Q$, then $A_f = B_f$. Thus, $A_P = (A_f)_{PA_f} = (B_f)_{QB_f} = B_Q$. Assume that $Q = M$, so $P = M$. Let $N \in \operatorname{Spec}(A)$ be a proper subideal of M. As above, we obtain $A_N = B_H$, where $H = NA_N \cap B$. In particular, H lies over N. We show that $H \subseteq M$. If not, select $g \in H - M$ and $h \in B$ such that $\pi(gh) = 1$. Then, $gh \in N - M$, a contradiction. It follows that

$$H \subseteq M;$$

thus, going down holds for $A \subseteq B$. However, $A_M \neq B_M$ since $A_M/MA_M = K$, which is a proper subfield of B/M. Thus, A is not perinormal. $\square$

Let A be a domain. Recall ([**19, 21**]) that A is called an *almost* GCD *domain* or AGCD if, for each $x, y \in A$, there exists an $n \geq 1$ such that $x^n A \cap y^n A$ is a principal ideal.

Recall that the class group of a domain was introduced by Bouvier and Zafrullah [**2, 3**] in order to extend the divisor class group concept from the Krull domains case to arbitrary domains. For simplicity, we choose to recall this definition only in the PvMD case. Let A be a PvMD. The set

$$D(A) = \{H_v \mid H \text{ finitely generated nonzero fractional ideal of } A\}$$

is a group under the operation

$$(H_1, H_2) \longmapsto (H_1 H_2)_v,$$

called the v-multiplication. The *class group* $Cl(A)$ of A is defined as $D(A)$ modulo the subgroup of all principal nonzero fractional ideals. By [**3,** Corollary 1.5], the GCD domains are exactly the PvMDs with zero class group.

The next lemma summarizes some known facts.

Lemma 2.12 ([**1, 21**]).

(a) *Let A be an* AGCD *domain and A' its integral closure. Then, $A \subseteq A'$ is a root extension, that is, every $x \in A'$ has some power in A.*

(b) *The* PvMD*s with torsion class group are exactly the integrally closed* AGCD *domains.*

(c) *Let A be an* AGCD *domain. Then, every flat overring of A is a fraction ring of A.*

Proof. Part (a) is [**21,** Theorem 3.1], part (b) is [**21,** Theorem 3.9] and part (c) is [**1,** Theorem 3.5]. □

According to [**6**], a domain A is called *globally perinormal* if every GD-overring of A is a fraction ring of A. In [**6,** Theorem 6.4], it was shown that a Krull domain with torsion divisor class group is globally perinormal. In Theorem 2.13, we extend this result to PvMDs.

Theorem 2.13. *If A is a* PvMD *with torsion class group, e.g., a* GCD *domain, then A is globally perinormal.*

Proof. By Corollary 2.5, A is perinormal. Combine parts (b) and (c) of Lemma 2.12 to complete the proof. □

Corollary 2.14. *For an* AGCD *domain A, the following assertions are equivalent.*

(a) *A is globally perinormal*;

(b) *A is perinormal*;

(c) *A is integrally closed.*

Proof.

(a) $\Rightarrow$ (b). Clear.

(b) $\Rightarrow$ (c). Let A' be the integral closure of A. By part Lemma 2.12 (a), it follows that $A \subseteq A'$ is a root extension. By [**1,** Theorem 2.1], the natural map

$$\mathrm{Spec}(A') \longrightarrow \mathrm{Spec}(A)$$

is an order isomorphism; hence, $A \subseteq A'$ satisfies going down. Since A is perinormal, it follows that $A \subseteq A'$ is flat; thus, $A = A'$, cf., [**18,** Proposition 2].

(c) $\Rightarrow$ (a). Apply part (b) of Lemma 2.12 and Proposition 2.13. □

In the last result of this paper we slightly improve Theorem 2.13 (note that the Prüfer domain case of Theorem 2.15 is [**9,** Theorem 27.5]). Let A be a domain. An overring B of A is called t-*linked* (over A) [**4**] if, for each finitely generated nonzero ideal I of A such that $I^{-1} = A$, we have $(IB)^{-1} = B$. By [**4,** Proposition 2.2], every flat overring of A is t-linked over A. Hence, a perinormal domain whose t-linked overrings are fraction rings of A is globally perinormal.

Theorem 2.15. *Assume that A is a* PvMD *which satisfies the following condition*: *for each finitely generated nonzero ideal I of A, we have*

$$I^n \subseteq bA \subseteq I_v$$

for some $n \geq 1$ and $b \in A$. Then, A is globally perinormal.

Proof. By [**5,** Theorem 1.3], every t-linked overring of A is a fraction ring of A. As a PvMD is perinormal (due to Corollary 2.5), the paragraph preceding this theorem applies to complete the proof. □

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