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COHOMOLOGY OF FINITE MODULES OVER SHORT GORENSTEIN RINGS

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ABSTRACT. Let R be a Gorenstein local ring with maximal ideal $\mathfrak{m}$ satisfying $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. Set $\mathsf{k} = R/\mathfrak{m}$ and $e = \operatorname{rank}_{\mathsf{k}}(\mathfrak{m}/\mathfrak{m}^2)$. If $e > 2$ and M, N are finitely generated R-modules, we show that the formal power series

$$\sum_{i=0}^{\infty} \operatorname{rank}_{\mathsf{k}} \left(\operatorname{Ext}_R^i(M,N) \otimes_R \mathsf{k}\right) t^i$$

and

$$\sum_{i=0}^{\infty} \operatorname{rank}_{\mathsf{k}} \left(\operatorname{Tor}^R_i(M,N) \otimes_R \mathsf{k}\right) t^i$$

are rational, with denominator $1 - et + t^2$.

Introduction. Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian commutative local ring; $\mathfrak{m}$ denotes the maximal ideal and $\mathsf{k} = R/\mathfrak{m}$. If L is an R-module, we set $\nu(L) = \operatorname{rank}_{\mathsf{k}}(L/\mathfrak{m}L)$. Let M and N be finite (meaning finitely generated) R-modules.

We consider the formal power series

$$\mathrm{E}_R^{M,N}(t) = \sum_{i=0}^{\infty} \nu\left(\operatorname{Ext}_R^i(M,N)\right) t^i$$

and

$$\mathrm{T}^R_{M,N}(t) = \sum_{i=0}^{\infty} \nu\left(\operatorname{Tor}^R_i(M,N)\right) t^i.$$

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Note that $\mathrm{E}_R^{M,\mathsf{k}}(t) = \mathrm{T}^R_{M,\mathsf{k}}(t) = \mathrm{T}^R_{\mathsf{k},M}(t)$; this series is usually called the *Poincaré series* of M, denoted $\mathrm{P}^R_M(t)$. The series $\mathrm{E}_R^{\mathsf{k},N}(t)$ is called the *Bass series* of N.

Although rings with transcendental Poincaré series exist, significant classes of rings R are known to satisfy the property that the Poincaré series of all finite R-modules are rational, sharing a common denominator; see, for example, **[9]** for a recent development. If this property holds, then the Bass series of all finite R-modules are also rational, sharing a common denominator, see [**10,** Lemma 1.2].

Less is known about the series $\mathrm{E}_R^{M,N}(t)$ and $\mathrm{T}^R_{M,N}(t)$ for arbitrary M, N. If $\mathfrak{m}^2 = 0$, then it is an easy exercise to show that $(1-et)\cdot \mathrm{E}_R^{M,N}(t) \in \mathbb{Z}[t]$, where $e = \nu(\mathfrak{m})$. When R is a complete intersection of codimension c, Avramov and Buchweitz [**1,** Proposition 1.3] showed that $(1-t^2)^c \cdot \mathrm{E}_R^{M,N}(t) \in \mathbb{Z}[t]$ for all finite M, N.

We consider R to be Gorenstein, with $\mathfrak{m}^3 = 0$. In this case, Sjödin **[11]** showed that the Poincaré series of all finite R-modules are rational, sharing a common denominator. We prove that Sjödin's result can be extended as follows:

Theorem. Let $(R,\mathfrak{m},\mathsf{k})$ be a local Gorenstein ring with $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$, and set $e = \nu(\mathfrak{m})$. If $e > 2$ and M, N are finite R-modules, then

$$(1-et+t^2)\cdot \mathrm{E}_R^{M,N}(t) \in \mathbb{Z}[t]$$

and

$$(1-et+t^2)\cdot \mathrm{T}^R_{M,N}(t) \in \mathbb{Z}[t].$$

When $l(M\otimes_R N) < \infty$, with $l(-)$ denoting length, modified versions of the series $\mathrm{E}_R^{M,N}(t)$ and $\mathrm{T}^R_{M,N}(t)$ may be defined as follows:

$$\mathcal{E}_R^{M,N}(t) = \sum_{i=0}^{\infty} l\left(\operatorname{Ext}^i_R(M,N)\right) t^i$$

and

$$\mathcal{T}^R_{M,N}(t) = \sum_{i=0}^{\infty} l\left(\operatorname{Tor}_i^R(M,N)\right) t^i.$$

Under the assumptions of the Theorem, our proof reveals that $\mathfrak{m}\operatorname{Ext}_R^i(M,N) = 0$ and $\mathfrak{m}\operatorname{Tor}_i^R(M,N) = 0$ for $i \gg 0$; hence, we also have, cf., Corollary 3.2:

$$(1 - et + t^2) \cdot \mathcal{E}_R^{M,N}(t) \in \mathbb{Z}[t]$$

and

$$(1 - et + t^2) \cdot \mathcal{T}_{M,N}^R(t) \in \mathbb{Z}[t].$$

When R is a complete intersection, rationality of $\mathcal{E}_R^{M,N}(t)$ and $\mathcal{T}_{M,N}^R(t)$ is known, due to Gulliksen [**4**]. On the other hand, Roos [**8**] gave an example of a (non-Gorenstein) ring R with $\mathfrak{m}^3 = 0$ and a module M such that $\mathcal{E}_R^{M,M}(t)$ is rational, while $\mathcal{T}_{M,M}^R(t)$ is transcendental. We refer to [**8**] for the connections of such results with homology of free loop spaces and cyclic homology.

The rings considered in this paper, i.e., Gorenstein rings with radical cube zero, are homomorphic images of a hypersurface, via a Golod homomorphism (see [**2,** 1.4]). As indicated by Roos, it is reasonable to expect that the series $\mathcal{E}_R^{M,N}(t)$ and $\mathcal{T}_{M,N}^R(t)$ are rational for all M, N with $l(M \otimes_R N) < \infty$ whenever R is a homomorphic image of a complete intersection via a Golod homomorphism. Along the same lines, we may also expect that the series $\mathrm{E}_R^{M,N}(t)$ and $\mathrm{T}_{M,N}^R(t)$ are rational for such R, and any finite R-modules M, N.

An important aspect of our arguments is the use of the notion of the Koszul module. The structure of Koszul modules in the case of Gorenstein rings R with $\mathfrak{m}^3 = 0$ is well understood, and is used heavily in the proofs. The main ingredient in the proof consists of showing that, under the hypotheses of the Theorem, the homomorphism

$$\operatorname{Tor}_i^R(\mathfrak{m}M, N) \longrightarrow \operatorname{Tor}_i^R(M,N)$$

induced by the inclusion $\mathfrak{m}M \hookrightarrow M$ is zero for $i \gg 0$ whenever the module M is Koszul. This is the statement of Proposition 2.6, proven in Section 2. The proof of the main theorem is given in Section 3.

1. Preliminaries. In this section we introduce notation and discuss some background. We introduce the notion of the Koszul module, and we give equivalent characterizations in the case of interest. Lemmas 1.1 and 1.4 will become instrumental in Section 2 in setting up an induction argument towards the proof of the main result, while Lemma 1.2 provides one of the key ideas in constructing the proof.

Throughout, $(R, \mathfrak{m}, \mathsf{k})$ denotes a local commutative ring with maximal ideal $\mathfrak{m}$ and $\mathsf{k} = R/\mathfrak{m}$, and M, N are finite R-modules. We set

$$\overline{M} = M/\mathfrak{m}M \quad \text{and} \quad \nu(M) = \operatorname{rank}_{\mathsf{k}}(\overline{M}).$$

Lemma 1.1. *Assume* k *is algebraically closed and* $\nu(\mathfrak{m}) \geq 2$. *Let* M *be a finite* R*-module with* $\mathfrak{m}^2 M = 0$ *and such that* $\nu(M) \geq \nu(\mathfrak{m}M)$. *There exists then* $x \in M \smallsetminus \mathfrak{m}M$ *such that* $\operatorname{ann}(x) \neq \mathfrak{m}^2$.

Proof. Assume that $\operatorname{ann}(x) = \mathfrak{m}^2$ for every $x \in M \smallsetminus \mathfrak{m}M$. If $a \in R$, we denote by $\overline{a}$ its image in $\mathsf{k} = R/\mathfrak{m}$. Set $\nu(\mathfrak{m}M) = n$. Since $\mathfrak{m}^2 M = 0$, note that $\mathfrak{m}M$ has a vector space structure over k and $\operatorname{rank}_{\mathsf{k}}(\mathfrak{m}M) = n$. The structure is given by $\overline{a}x = ax$ for $x \in \mathfrak{m}M$ and $a \in R$.

By hypothesis, we have $\nu(M) \geq n$. Let $x_1, \dots, x_n$ be part of a minimal generating set of M.

Claim. If $\alpha \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$, then $\alpha x_1, \dots, \alpha x_n$ is a basis of $\mathfrak{m}M$ over k.

In order to prove this claim, assume that

$$\overline{b_1}(\alpha x_1) + \cdots + \overline{b_n}(\alpha x_n) = b_1(\alpha x_1) + \cdots + b_n(\alpha x_n) = 0$$

for some $b_i \in R$. Set $x = b_1 x_1 + \cdots + b_n x_n$. We thus have $\alpha x = 0$; hence, $\alpha \in \operatorname{ann}(x)$. If $x \notin \mathfrak{m}M$, then $\operatorname{ann}(x) = \mathfrak{m}^2$ by assumption, and thus, $\alpha \in \mathfrak{m}^2$, a contradiction. Consequently, $x \in \mathfrak{m}M$, and hence $b_i \in \mathfrak{m}$ and thus $\overline{b_i} = \overline{0}$ for all i. This shows that $\alpha x_1, \dots, \alpha x_n$ is linearly independent over k. Since $\operatorname{rank}_{\mathsf{k}}(\mathfrak{m}M) = n$, this set is a basis of $\mathfrak{m}M$, and the claim is proved.

Assume now that α, β is part of a minimal set of generators of $\mathfrak{m}$. By the above, the sets $\alpha x_1, \dots, \alpha x_n$ and $\beta x_1, \dots, \beta x_n$ are both bases of $\mathfrak{m}M$ over k. We then have relations

$$\beta x_j = \sum_{i=1}^{n} p_{ij} \alpha x_i \quad \text{for all } j \text{ with } 1 \leq j \leq n, \tag{1.1}$$

where $p_{ij} \in R$, and the change of basis matrix $P = (\overline{p_{ij}})$ is invertible. Recall that k is algebraically closed, and let $\lambda \in \mathsf{k}$ be an eigenvalue of P. Since P is invertible, we have $\lambda \neq 0$. Next, choose $\gamma \in R$ so that $\overline{\gamma} = -\lambda^{-1}$. Since $\lambda = -(\overline{\gamma})^{-1}$ is an eigenvalue, we have $\det(P +$

$(\overline{\gamma})^{-1}I) = \overline{0}$, where I is the $n \times n$ identity matrix, and it follows that $\det(I + \overline{\gamma}P) = \overline{0}$, and hence, the matrix equation

$$(I + \overline{\gamma}P)\boldsymbol{b} = \overline{0}$$

has a nontrivial solution $\boldsymbol{b} \in \mathsf{k}^n$, where $\boldsymbol{b} = (\overline{b_1}, \ldots, \overline{b_n})^T$ with $b_i \in R$. With this choice of γ and b_i, we thus have

$$b_i + \gamma \sum_{j=1}^{n} p_{ij} b_j \in \mathfrak{m} \quad \text{for all } i \text{ with } 1 \leq i \leq n. \tag{1.2}$$

Equations (1.2) and (1.1) yield:

$$\begin{aligned}
&(\alpha + \beta\gamma)(b_1 x_1 + \cdots + b_n x_n) \\
&\quad = \sum_{i=1}^{n} b_i(\alpha x_i) + \gamma \sum_{j=1}^{n} b_j(\beta x_j) \\
&\quad = \sum_{i=1}^{n} (b_i + \gamma \sum_{j=1}^{n} p_{ij} b_j)(\alpha x_i) \in \mathfrak{m}^2 M \\
&\quad = 0.
\end{aligned}$$

Set $x = b_1 x_1 + \cdots + b_n x_n$, and note that $x \notin \mathfrak{m}M$, since the vector $\boldsymbol{b} \in \mathsf{k}^n$ is nontrivial, and thus, $b_i \notin \mathfrak{m}$ for at least some i. We thus have $\alpha + \beta\gamma \in \operatorname{ann}(x)$ and, since $\operatorname{ann}(x) = \mathfrak{m}^2$, it follows that $\alpha + \beta\gamma \in \mathfrak{m}^2$. This is a contradiction, since α, β is part of a minimal set of generators for $\mathfrak{m}$. □

Let $\varphi\colon M \to N$ be a homomorphism. We denote by $\overline{\varphi}$ the induced map $\overline{\varphi}\colon \overline{M} \to \overline{N}$. If A is a finite R-module, then, for each i, we let

$$\begin{aligned}
\operatorname{Tor}_i^R(\varphi, A)&\colon \operatorname{Tor}_i^R(M, A) \longrightarrow \operatorname{Tor}_i^R(N, A) \\
\operatorname{Ext}_R^i(\varphi, A)&\colon \operatorname{Ext}_R^i(N, A) \longrightarrow \operatorname{Ext}_R^i(M, A)
\end{aligned}$$

denote the induced maps. We also let

$$\iota_M \colon \mathfrak{m}M \longrightarrow M \quad \text{and} \quad \pi_M \colon M \longrightarrow \overline{M}$$

denote the inclusion, respectively, the canonical projection.

For each i, we set $\beta_i^R(M) = \operatorname{rank}_{\mathsf{k}} \operatorname{Tor}_i^R(M, \mathsf{k})$; this number is the ith *Betti number* of M over R.

The main ingredient in proving rationality of the series defined in the introduction consists of showing that the maps $\operatorname{Tor}_i^R(\iota_M, N)$ become zero for large values of i, under certain assumptions on the ring and on the modules. The next lemma is a first step in this direction, and it will be further extended in Section 2.

Lemma 1.2. *Let M and N be finite R-modules with $\nu(M) = 1$. Assume that there exists an integer $i \geq 0$ such that $\operatorname{Tor}_i^R(\iota_M, \mathsf{k}) = 0$ and $\beta_i^R(M) > \beta_i^R(N)$.*

(1) *If $\varphi \in \operatorname{Hom}_R(M, N)$, then $\varphi(M) \subseteq \mathfrak{m}N$.*
(2) *If $\mathfrak{m}^2 N = 0$, then $\operatorname{Hom}_R(\iota_M, N) = 0$.*

Proof. Assume $\varphi(M) \not\subseteq \mathfrak{m}N$. Since M is cyclic, the induced map $\overline{\varphi}\colon \overline{M} \to \overline{N}$ is injective. Since it is a homomorphism of vector spaces, it has a splitting; hence, the induced maps $\operatorname{Tor}_i^R(\overline{\varphi}, \mathsf{k})\colon \operatorname{Tor}_i^R(\overline{M}, \mathsf{k}) \to \operatorname{Tor}_i^R(\overline{N}, \mathsf{k})$ are injective for $i \geq 0$. The short exact sequence

$$0 \longrightarrow \mathfrak{m}M \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0$$

induces the top exact row in the commutative diagram below:
(1.3)

$$\begin{tikzcd}
\operatorname{Tor}_i^R(\mathfrak{m}M, \mathsf{k}) \arrow[r, "\operatorname{Tor}_i^R(\iota_M, \mathsf{k})"] & \operatorname{Tor}_i^R(M, \mathsf{k}) \arrow[r, "\operatorname{Tor}_i^R(\pi_M, \mathsf{k})"] \arrow[d, "\operatorname{Tor}_i^R(\varphi, \mathsf{k})"'] & \operatorname{Tor}_i^R(\overline{M}, \mathsf{k}) \arrow[d, "\operatorname{Tor}_i^R(\overline{\varphi}, \mathsf{k})"] \\
& \operatorname{Tor}_i^R(N, \mathsf{k}) \arrow[r, "\operatorname{Tor}_i^R(\pi_N, \mathsf{k})"] & \operatorname{Tor}_i^R(\overline{N}, \mathsf{k})
\end{tikzcd}$$

If $\operatorname{Tor}_i^R(\iota_M, \mathsf{k}) = 0$, then $\operatorname{Tor}_i^R(\pi_M, \mathsf{k})$ is injective. The commutative square then gives that $\operatorname{Tor}_i^R(\varphi, \mathsf{k})$ is injective. We conclude that $\beta_i^R(M) \leq \beta_i^R(N)$, a contradiction.

Thus, we have $\varphi(M) \subseteq \mathfrak{m}N$; hence, (1) is established. In order to prove (2), note that the image of φ under the map $\operatorname{Hom}_R(\iota_M, N)$ is the composition $\varphi\iota_M\colon \mathfrak{m}M \to N$. We have:

$$\varphi\iota_M(\mathfrak{m}M) \subseteq \varphi(\mathfrak{m}M) \subseteq \mathfrak{m}\varphi(M) \subseteq \mathfrak{m}^2 N.$$

When $\mathfrak{m}^2 N = 0$, we conclude that $\varphi\iota_M = 0$, and thus, $\operatorname{Hom}_R(\iota_M, N) = 0$. $\square$

1.1. Hilbert and Poincaré series. The *Hilbert series* of M (over R) is the formal power series

$$H_M(t) = \sum_{i=0}^{\infty} \operatorname{rank}_{\mathsf{k}} \left(\frac{\mathfrak{m}^i M}{\mathfrak{m}^{i+1} M} \right) t^i \,.$$

The *Poincaré series* of M is the formal power series

$$\mathrm{P}_M^R(t) = \sum_{i=0}^{\infty} \beta_i^R(M) t^i.$$

The next remark clarifies the attention we will give in Section 2 to the vanishing of the maps $\operatorname{Tor}_i^R(\iota_M, N)$; such vanishing allows for computations of the series of interest.

Remark 1.3. Assume that $\mathfrak{m}^2 M = 0$. The short exact sequence

$$0 \longrightarrow \mathfrak{m}M \xrightarrow{\iota_M} M \xrightarrow{\pi_M} \overline{M} \longrightarrow 0$$

induces for each $i > 0$ the following exact sequence:

$$\begin{aligned} 0 \longrightarrow L_i \longrightarrow \operatorname{Tor}_i^R(M, N) &\xrightarrow{\operatorname{Tor}_i^R(\pi_M, N)} \operatorname{Tor}_i^R(\overline{M}, N) \\ &\xrightarrow{\Delta_i} \operatorname{Tor}_{i-1}^R(\mathfrak{m}M, N) \longrightarrow L_{i-1} \longrightarrow 0, \end{aligned}$$

where L_i is the image of the map $\operatorname{Tor}_i^R(\iota_M, N)$. A length count gives:

$$l(\operatorname{Tor}_i^R(M,N)) = l(\operatorname{Tor}_i^R(\overline{M},N)) - l(\operatorname{Tor}_{i-1}^R(\mathfrak{m}M,N)) + l(L_i) + l(L_{i-1}).$$

Since both $\overline{M}$ and $\mathfrak{m}M$ are k-vector spaces, we have

$$\begin{aligned} l(\operatorname{Tor}_i^R(\overline{M},N)) &= \operatorname{rank}_{\mathsf{k}}(\operatorname{Tor}_i^R(\mathsf{k}^{\nu(M)},N)) = \nu(M)\beta_i^R(N); \\ l(\operatorname{Tor}_{i-1}^R(\mathfrak{m}M,N)) &= \operatorname{rank}_{\mathsf{k}}(\operatorname{Tor}_{i-1}^R(\mathsf{k}^{\nu(\mathfrak{m}M)},N)) = \nu(\mathfrak{m}M)\beta_{i-1}^R(N). \end{aligned}$$

Thus, we have

$$l(\operatorname{Tor}_i^R(M,N)) \geq \nu(M)\beta_i^R(N) - \nu(\mathfrak{m}M)\beta_{i-1}^R(N). \tag{1.4}$$

Equality holds in (1.4) if and only if $L_i = 0 = L_{i-1}$, and hence, if and only if $\operatorname{Tor}_i^R(\iota_M, N) = 0 = \operatorname{Tor}_{i-1}^R(\iota_M, N)$. In particular, from here we obtain that the following two statements are equivalent when $\mathfrak{m}^2 M = 0$:

(1) $\operatorname{Tor}_i^R(\iota_M, N) = 0$ for all $i \geq 0$;

(2) $\sum_{i=0}^{\infty} l(\operatorname{Tor}_i^R(M,N))t^i = H_M(-t)\,\mathrm{P}_N^R(t)$.

Also, note that $L_i = 0$ implies that $\operatorname{Tor}_i^R(M,N)$ is isomorphic to a submodule of $\operatorname{Tor}_i^R(\overline{M},N)$, and hence, $\mathfrak{m}\operatorname{Tor}_i^R(M,N) = 0$. In particular, condition (1) also implies:

(3) $\sum_{i=0}^{\infty} \nu(\operatorname{Tor}_i^R(M,N))t^i = H_M(-t)\,\mathrm{P}_N^R(t)$.

1.2. Koszul rings and modules. As defined in [**5**], an R-module M is said to be *Koszul* if its linearity defect is 0; we refer to loc. cit. for the definition of linearity defect, and we note that M is Koszul if and only if the associated graded module $\operatorname{gr}_{\mathfrak{m}}(M)$ has a linear resolution over $\operatorname{gr}_{\mathfrak{m}}(R)$. As noted in [**5,** 1.8], if M is Koszul, then

$$\mathrm{P}_M^R(t) = \frac{H_M(-t)}{H_R(-t)}. \tag{1.5}$$

The ring R is said to be *Koszul* if k is a Koszul module.

If R is Koszul and $\mathfrak{m}^2M = 0$, then the following are equivalent:

(1) M is Koszul;
(2) $\operatorname{Tor}_i^R(\iota_M, \mathsf{k}) = 0$ for all $i \geq 0$;
(3) formula (1.5) holds.

See [**2,** 3.1] for the equivalence (1) $\Leftrightarrow$ (2) and Remark 1.3 for (2) $\Leftrightarrow$ (3).

Lemma 1.4. *Assume that there exists a short exact sequence*

$$0 \longrightarrow A \xrightarrow{\varphi} M \xrightarrow{\psi} B \longrightarrow 0$$

of finite R-modules such that $\overline{\varphi}\colon \overline{A} \to \overline{M}$ is injective. Let N be a finite R-module.

(1) *If $\operatorname{Tor}_i^R(\iota_A, N) = 0$ for some i, then $\operatorname{Tor}_i^R(\varphi, N)$ is injective and $\operatorname{Tor}_{i+1}^R(\psi, N)$ is surjective.*
(2) *If $\operatorname{Tor}_i^R(\iota_B, N) = \operatorname{Tor}_{i-1}^R(\iota_A, N) = \operatorname{Tor}_i^R(\iota_A, N) = 0$ for some i, then we also have $\operatorname{Tor}_i^R(\iota_M, N) = 0$.*
(3) *If R is a Koszul ring, $\mathfrak{m}^2M = 0$ and M is Koszul, then B is Koszul.*

Proof. The hypothesis that $\overline{\varphi}$ is injective yields a commutative diagram with exact rows and columns:

(1.6)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathfrak{m}A \arrow[r] \arrow[d, "\iota_A"] & \mathfrak{m}M \arrow[r, "\psi'"] \arrow[d, "\iota_M"] & \mathfrak{m}B \arrow[r] \arrow[d, "\iota_B"] & 0 \\
0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\pi_A"] & M \arrow[r, "\psi"] \arrow[d, "\pi_M"] & B \arrow[r] \arrow[d, "\pi_B"] & 0 \\
0 \arrow[r] & \overline{A} \arrow[r, "\overline{\varphi}"] \arrow[d] & \overline{M} \arrow[r, "\overline{\psi}"] \arrow[d] & \overline{B} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Note that the bottom row in this diagram is an exact sequence of vector spaces; hence, it is split, and it remains exact when applying $\operatorname{Tor}_i^R(-,N)$. In particular, $\operatorname{Tor}_i^R(\overline{\varphi},N)$ is injective and $\operatorname{Tor}_i^R(\overline{\psi},N)$ is surjective. Diagram (1.6) then induces the following commutative diagram with exact rows and columns:

(1.7)
$$\begin{tikzcd}
 & \operatorname{Tor}_i^R(\mathfrak{m}A,N) \arrow[r] \arrow[d, "\operatorname{Tor}_i^R(\iota_A,N)"] & \operatorname{Tor}_i^R(\mathfrak{m}M,N) \arrow[r, "\operatorname{Tor}_i^R(\psi',N)"] \arrow[d, "\operatorname{Tor}_i^R(\iota_M,N)"] & \operatorname{Tor}_i^R(\mathfrak{m}B,N) \arrow[d, "\operatorname{Tor}_i^R(\iota_B,N)"] & \\
 & \operatorname{Tor}_i^R(A,N) \arrow[r, "\operatorname{Tor}_i^R(\varphi,N)"] \arrow[d, "\operatorname{Tor}_i^R(\pi_A,N)"] & \operatorname{Tor}_i^R(M,N) \arrow[r, "\operatorname{Tor}_i^R(\psi,N)"] \arrow[d, "\operatorname{Tor}_i^R(\pi_M,N)"] & \operatorname{Tor}_i^R(B,N) \arrow[d, "\operatorname{Tor}_i^R(\pi_B,N)"] & \\
0 \arrow[r] & \operatorname{Tor}_i^R(\overline{A},N) \arrow[r, "\operatorname{Tor}_i^R(\overline{\varphi},N)"] & \operatorname{Tor}_i^R(\overline{M},N) \arrow[r, "\operatorname{Tor}_i^R(\overline{\psi},N)"] & \operatorname{Tor}_i^R(\overline{B},N) \arrow[r] & 0.
\end{tikzcd}$$

We then have:

(1) If $\operatorname{Tor}_i^R(\iota_A,N)=0$, it follows that $\operatorname{Tor}_i^R(\pi_A,N)$ is injective. Since $\operatorname{Tor}_i^R(\overline{\varphi},N)$ is injective, the bottom left commutative square yields that $\operatorname{Tor}_i^R(\varphi,N)$ is injective as well. The fact that $\operatorname{Tor}_{i+1}^R(\psi,N)$ is surjective follows from the long exact sequence associated in homology with the exact sequence from the statement.

(2) In view of part (1), the hypothesis that $\operatorname{Tor}_{i-1}^R(\iota_A,N) = \operatorname{Tor}_i^R(\iota_A,N)=0$ shows that $\operatorname{Tor}_i^R(\varphi,N)$ is injective and $\operatorname{Tor}_i^R(\psi,N)$

is surjective. The hypothesis also implies that $\operatorname{Tor}_i^R(\pi_A, N)$ and $\operatorname{Tor}_i^R(\pi_B, N)$ are injective. A snake lemma argument using the bottom two rows of (1.7) gives that $\operatorname{Tor}_i^R(\pi_M, N)$ is injective as well, and thus, $\operatorname{Tor}_i^R(\iota_M, N) = 0$.

(3) The additional hypothesis that $\mathfrak{m}^2 M = 0$ gives that the top row in (1.6) is an exact sequence of vector spaces. Consequently, it is split, and in particular, $\operatorname{Tor}_i^R(\psi', \mathsf{k})$ is surjective for all $i \geq 0$. Since M is Koszul, we have $\operatorname{Tor}_i^R(\iota_M, \mathsf{k}) = 0$ for all $i \geq 0$. The upper right commutative square in (1.7) with $N = \mathsf{k}$ then yields $\operatorname{Tor}_i^R(\iota_B, \mathsf{k}) = 0$ for all $i \geq 0$, and hence, B is Koszul, in view of 1.2. $\square$

2. Koszul modules over short Gorenstein rings. In this section, we focus our attention on Gorenstein local rings with $\mathfrak{m}^3 = 0$. We first present some necessary background material. The bulk of the section concentrates upon the proof of Proposition 2.6 and its supporting lemmas.

Throughout this section, $(R, \mathfrak{m}, \mathsf{k})$ denotes a Gorenstein local ring with $\mathfrak{m}^3 = 0$ and $\mathfrak{m}^2 \neq 0$. We set $e = \nu(\mathfrak{m})$, and we assume that $e \geq 2$.

Let M and N be finite R-modules. For any N, set $N^* = \operatorname{Hom}_R(N, R)$. If $\varphi\colon M \to N$ is a homomorphism, then $\varphi^*\colon N^* \to M^*$ denotes the induced map.

2.1. Syzygies. Let M be a finitely generated R-module, and let

$$(2.1) \qquad \cdots \longrightarrow F_i \xrightarrow{\partial_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0$$

be a minimal free resolution of M over R. Note that the Betti numbers of M can be read off this resolution, namely, $\beta_i^R(M) = \operatorname{rank}_R(F_i)$ for all $i \geq 0$. We set $M_0 = M$ and, for each $i > 0$, we set

$$M_i = \operatorname{Im}(\partial_i).$$

The module M_i is called the ith *syzygy* of M. Since $\mathfrak{m}^3 = 0$, the minimality of the resolution shows that $\mathfrak{m}^2 M_i = 0$ for all $i > 0$. Now, let

$$(2.2) \qquad \cdots \longrightarrow G_i \xrightarrow{d_i} G_{i-1} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{d_1} G_0 \longrightarrow 0$$

be a minimal free resolution of M^*. Since R is Gorenstein and Artinian, the dual of this resolution is also exact and gives a minimal injective resolution of M^{**}:

$$0 \longrightarrow F_{-1} \xrightarrow{\partial_{-1}} F_{-2} \longrightarrow \cdots \longrightarrow F_{-i} \xrightarrow{\partial_{-i}} F_{-i-1} \longrightarrow \cdots \tag{2.3}$$

with $F_{-i} = G^*_{i-1}$ and $\partial_{-i} = d^*_i$.

Since $M \cong M^{**}$, note that the resolutions in (2.1) and (2.3) can be "glued" together through a map ∂_0, yielding a *complete resolution* of M:

$$\begin{aligned}\cdots \longrightarrow F_i \xrightarrow{\partial_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} F_{-1} \xrightarrow{\partial_{-1}} F_{-2} \longrightarrow \cdots \\ \longrightarrow F_{-i} \xrightarrow{\partial_{-i}} F_{-i-1} \longrightarrow \cdots .\end{aligned}$$

This complex is acyclic, that is, its homology is zero in each degree. If $i > 0$, we set

$$M_{-i} = \mathrm{Im}(\partial_{-i}).$$

If $\mathfrak{m}^2 M = 0$, then $\partial_0(F_0) \subseteq \mathfrak{m}F_{-1}$, and the complete resolution is minimal. Consequently, if M and N are two R-modules with $\mathfrak{m}^2 M = 0 = \mathfrak{m}^2 N$, the minimal complete resolution shows that

$$M_{-i} \cong N \iff M \cong N_i \quad \text{for all } i.$$

In a similar manner, we define negative Betti numbers, when $\mathfrak{m}^2 M = 0$, by setting $\beta^R_{-i}(M) = \mathrm{rank}_R(F_{-i})$ for all $i > 0$. In particular, we have:

$$\begin{gathered}\beta^R_{-i}(M) = \mathrm{rank}_R(F_{-i}) = \mathrm{rank}_R(G_{i-1}) = \beta^R_{i-1}(M^*) \\ M_{-i} = \mathrm{Im}(\partial_{-i}) = \mathrm{Im}(d^*_i) \cong (\mathrm{Im}\, d_i)^* = (M^*)_i\end{gathered}$$

Furthermore, since $\mathsf{k}^* \cong \mathsf{k}$, we have $\beta^R_{-i}(\mathsf{k}) = \beta^R_{i-1}(\mathsf{k})$ and $\mathsf{k}_{-i} \cong \mathsf{k}_i$.

2.2. Koszul modules over short Gorenstein rings. With R as above, the following statements are equivalent (see [**2**, 4.6]):

(1) M is Koszul;
(2) the syzygy M_i does not split off a copy of k for any $i > 0$ (equivalently, M is *exceptional*, using the terminology of Lescot [**6**]);
(3) M has no direct summand isomorphic to k_{-i} for all $i > 0$.

In particular, it follows, as noted in [**2**, 4.6], that an indecomposable module M over the short Gorenstein ring R is Koszul if and only if M

is not isomorphic to k_{-i} for any $i > 0$. Also, note that, if M is Koszul, then M_i is Koszul for all $i > 0$.

Löfwall [**7**] showed that a Gorenstein ring with $\mathfrak{m}^3 = 0$ and $e \geq 2$ satisfies

$$\mathrm{P}^R_{\mathsf{k}}(t) = \frac{1}{1 - et + t^2}. \tag{2.4}$$

This result was recovered and used by Sjödin [**11**] to show that, for every finitely generated R-module, one has:

$$(1 - et + t^2) \cdot \mathrm{P}^R_M(t) \in \mathbb{Z}[t]. \tag{2.5}$$

The results of [**11**] were recovered in [**2**], where it is also noted that any such R is Koszul.

Note that formula (2.4) shows that the Betti numbers $b_i = \beta^R_i(\mathsf{k})$ satisfy the relations $b_0 = 1$, $b_1 = e$ and $b_{i+1} = eb_i - b_{i-1}$ for all $i \geq 1$. Since we have assumed that $e \geq 2$, it inductively follows that the sequence $\{\beta^R_i(\mathsf{k})\}_{i\geq 0}$ is strictly increasing.

Remark 2.1. It is known that $\mathfrak{m}^2 M = 0$ when M is indecomposable and not free; see, for instance, the proof of [**2,** 4.6].

Also, if $\mathfrak{m}^2 M = 0$ and M_1 does not split off a copy of k, then the following formulas hold, cf., [**6,** 3.3]:

$$\nu(M_1) = \nu(M)e - \nu(\mathfrak{m}M); \tag{2.6}$$

$$\nu(\mathfrak{m}M_1) = \nu(M). \tag{2.7}$$

Lemma 2.2. *Let I be an ideal of R. Then, R/I is not Koszul if and only if $I = \mathfrak{m}^2$.*

Proof. Since R is Gorenstein with socle $\mathfrak{m}^2$, note that $\mathfrak{m}^2 \cong \mathsf{k}$. If $I = \mathfrak{m}^2$, it follows that $\mathsf{k} \cong (R/I)_1$; hence, R/I is not Koszul.

Now, assume that R/I is not Koszul. Then, $R/I \cong \mathsf{k}_{-i}$ for some $i > 0$; hence,

$$\beta^R_{i-1}(\mathsf{k}) = \beta^R_{-i}(\mathsf{k}) = \beta^R_0(R/I) = 1.$$

Since the Betti numbers of k are strictly increasing, the equality $\beta^R_{i-1}(\mathsf{k}) = 1$ implies $i = 1$. We thus have $R/I \cong \mathsf{k}_{-1}$. Since $\mathsf{k}_{-1} \cong R/\mathfrak{m}^2$, we conclude $I = \mathfrak{m}^2$. $\square$

Lemma 2.3. *If I is a proper ideal of R and $e > 2$, then the sequence $\{\beta_i^R(R/I)\}_{i\geq 1}$ is strictly increasing and $\beta_i^R(R/I) \geq i$ for all $i \geq 0$.*

Proof. If $I = \mathfrak{m}^2$, then $\beta_i^R(R/\mathfrak{m}^2) = \beta_{i-1}^R(\mathfrak{m}^2) = \beta_{i-1}^R(\mathsf{k})$ for all $i \geq 1$, and the conclusion follows from the fact that the sequence $\{\beta_i^R(\mathsf{k})\}_{i\geq 0}$ is strictly increasing.

Assume now that $I \neq \mathfrak{m}^2$. Since $I \subseteq \mathfrak{m}$, we have $\mathfrak{m}I \subseteq \mathfrak{m}^2$. Since $I \neq 0$, and R is Gorenstein with socle $\mathfrak{m}^2$, it follows that $\mathfrak{m}I = \mathfrak{m}^2$. Hence, $\mathfrak{m}^2(R/I) = 0$. Set $a = \operatorname{rank}_{\mathsf{k}}(\mathfrak{m}/I)$. The assumption that $I \neq \mathfrak{m}^2$ gives $a < e$. The Hilbert series of R/I is $H_{R/I}(t) = 1 + at$. Since R/I is Koszul by Lemma 2.2, we have:

$$\mathrm{P}^R_{R/I}(t) = \frac{1-at}{1-et+t^2}. \tag{2.8}$$

Set $b_i = \beta_i^R(R/I)$ for $i \geq 0$. We then have:

$$1 - at = \left(b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \dots\right)\left(1 - et + t^2\right). \tag{2.9}$$

From this equation, we derive the following: $b_0 = 1$, $b_1 = e - a$ and $b_{i+2} = eb_{i+1} - b_i$ for $i \geq 0$. Note that $b_1 - b_0 \geq 0$ since $a < e$. Let $n \geq 1$, and assume that $b_n - b_{n-1} \geq n - 1$. Since $e > 2$, we have

$$b_{n+1} - b_n = (eb_n - b_{n-1}) - b_n = b_n(e-1) - b_{n-1} > b_n - b_{n-1} \geq n-1;$$

hence, $b_{n+1} - b_n \geq n$. This inductive argument yields that $b_{i+1} - b_i \geq i$ for all $i \geq 0$. In particular, $b_i \geq i$ for all $i \geq 0$, and the sequence $\{b_i\}_{i\geq 1}$ is strictly increasing. □

Lemma 2.4. *Assume that $e > 2$. If M is Koszul with $\nu(M) = 1$, then* $\operatorname{Ext}^i_R(\iota_M, N) = 0$ *for all i with $i > \nu(N)$. Equivalently,* $\operatorname{Tor}_i^R(\iota_M, N) = 0$ *for all i with $i > \nu(N^*)$.*

Proof. We may assume that M and N are indecomposable and not free. In particular, it follows that $\mathfrak{m}^2 M = 0 = \mathfrak{m}^2 N$. Let i be such that $i > \nu(N)$, and set $L = N_{-i}$. Note that $\mathfrak{m}^2 L = 0$ and

$$\nu(N) = \beta_0(N) = \beta_i(N_{-i}) = \beta_i(L).$$

Since M is cyclic, we have $M \cong R/I$ for a proper ideal I. Lemma 2.3 gives that $\beta_i^R(M) \geq i$, and hence, $\beta_i^R(M) > \beta_i^R(L)$ since $i > \nu(N)$.

Since M is Koszul, we have $\operatorname{Tor}_i^R(\iota_M, \mathsf{k}) = 0$ for all i, and Lemma 1.2 gives that $\operatorname{Hom}_R(\iota_M, L) = 0$.

For each $n \geq 0$, extract from a minimal complete resolution of N the short exact sequence

$$0 \longrightarrow N_{-i+n+1} \longrightarrow R^c \longrightarrow N_{-i+n} \longrightarrow 0 \tag{2.10}$$

with $c = \beta^R_{-i+n}(N)$, and consider the induced commutative diagram with exact rows:
(2.11)

$$\begin{array}{ccccc}
\operatorname{Ext}^n_R(M,N_{-i+n}) & \xrightarrow{\Delta_{n+1}} & \operatorname{Ext}^{n+1}_R(M,N_{-i+n+1}) & \longrightarrow & \operatorname{Ext}^{n+1}_R(M,R^c) \\
\big\downarrow{\scriptstyle \operatorname{Ext}^n_R(\iota_M,N_{-i+n})} & & \big\downarrow{\scriptstyle \operatorname{Ext}^{n+1}_R(\iota_M,N_{-i+n+1})} & & \big\downarrow \\
\operatorname{Ext}^n_R(\mathfrak{m}M,N_{-i+n}) & \longrightarrow & \operatorname{Ext}^{n+1}_R(\mathfrak{m}M,N_{-i+n+1}) & \longrightarrow & \operatorname{Ext}^{n+1}_R(\mathfrak{m}M,R^c).
\end{array}$$

We prove by induction on n that $\operatorname{Ext}^n_R(\iota_M, N_{-i+n}) = 0$ for all $n \geq 0$. This holds for $n = 0$ since we know that $\operatorname{Hom}_R(\iota_M, L) = 0$.

Assume now that $n \geq 0$ and $\operatorname{Ext}^n_R(\iota_M, N_{-i+n}) = 0$. The connecting homomorphism Δ_{n+1} in (2.11) is surjective due to the fact that we have $\operatorname{Ext}^{n+1}_R(M, R^c) = 0$, since R is Gorenstein Artinian. (It is an isomorphism when $n \geq 1$.) The commutative square on the left yields that $\operatorname{Ext}^{n+1}_R(\iota_M, N_{-i+n+1}) = 0$.

We thus have $\operatorname{Ext}^n_R(\iota_M, N_{-i+n}) = 0$ for all $n \geq 0$. Taking $n = i$ and noting that $N_0 = N$, we obtain the desired conclusion that $\operatorname{Ext}^i_R(\iota_M, N) = 0$ for all $i > \nu(N)$. In particular, we have $\operatorname{Ext}^i_R(\iota_M, N^*) = 0$ for all $i > \nu(N^*)$. Finally, note that $\operatorname{Ext}^i_R(\iota_M, N^*) = 0$ if and only if $\operatorname{Tor}_i^R(\iota_M, N) = 0$, in view of the canonical isomorphisms given by duality. □

Lemma 2.5. *Assume that $\mathfrak{m}^2 M = 0$. If M_1 does not split off a copy of k, then $\operatorname{Tor}_i^R(\iota_M, N) = 0$ for $i \gg 0$ if and only if $\operatorname{Tor}_i^R(\iota_{M_1}, N) = 0$ for $i \gg 0$.*

Proof. By (1.4), we have inequalities

$$l(\operatorname{Tor}^R_{i+1}(M, N)) \geq \nu(M)\beta_{i+1}(N) - \nu(\mathfrak{m}M)\beta_i(N); \tag{2.12}$$

$$l(\operatorname{Tor}^R_i(M_1, N)) \geq \nu(M_1)\beta_i(N) - \nu(\mathfrak{m}M_1)\beta_{i-1}(N). \tag{2.13}$$

We have $\operatorname{Tor}_i^R(\iota_M, N) = 0$ for $i \gg 0$ if and only if (2.12) is an equality for $i \gg 0$, and $\operatorname{Tor}_i^R(\iota_{M_1}, N) = 0$ for all $i \gg 0$ if and only if (2.13) is an equality for $i \gg 0$. Since $\operatorname{Tor}_i^R(M_1, N) \cong \operatorname{Tor}_{i+1}^R(M, N)$, it suffices to show that

$$\nu(M)\beta_{i+1}(N) - \nu(\mathfrak{m}M)\beta_i(N) = \nu(M_1)\beta_i(N) - \nu(\mathfrak{m}M_1)\beta_{i-1}(N) \tag{2.14}$$

for $i \gg 0$. Since the Poincaré series of N is rational with denominator $1 - et + t^2$, we have

$$\beta_{i+1}(N) = e\beta_i(N) - \beta_{i-1}(N) \quad \text{for } i \gg 0. \tag{2.15}$$

Let i be large enough so that (2.15) holds. Using first (2.15) and then (2.6) and (2.7), we establish (2.14) as follows:

$$\begin{aligned}
&\nu(M)\beta_{i+1}(N) - \nu(\mathfrak{m}M)\beta_i(N) \\
&\quad = \nu(M)(e\beta_i(N) - \beta_{i-1}(N)) - \nu(\mathfrak{m}M)\beta_i(N) \\
&\quad = (\nu(M)e - \nu(\mathfrak{m}M))\beta_i(N) - \nu(M)\beta_{i-1}(N) \\
&\quad = \nu(M_1)\beta_i(N) - \nu(\mathfrak{m}M_1)\beta_{i-1}(N).
\end{aligned}$$

As noted above, this finishes the proof. □

We are now ready to eliminate the assumption that $\nu(M) = 1$ in Lemma 2.4.

Proposition 2.6. *If $e > 2$ and M is Koszul, then $\operatorname{Tor}_i^R(\iota_M, N) = 0$ for $i \gg 0$.*

Remark 2.7. If $e = 2$, then the conclusion of the proposition may not hold. Indeed, if $R = \mathsf{k}[x, y]/(x^2, y^2)$ and $N = R/(x)$, then a minimal free resolution of N over R is

$$\cdots \longrightarrow R \xrightarrow{x} R \longrightarrow \cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R;$$

hence, $\operatorname{Tor}_i^R(M, N) \cong M$ for any M with $xM = 0$. When $M = R/(x)$ as well, the map $\operatorname{Tor}_i^R(\iota_M, N)$ can thus be identified with the inclusion $\mathfrak{m}M \hookrightarrow M$.

Proof of Proposition 2.6. Let $(R', \mathfrak{m}', \mathsf{k}')$ be a local ring with k' algebraically closed, where $R \to R'$ is an inflation in the sense of [**3,** Appendix, Thm. 1, Corollaire], that is, R' is flat over R and $\mathfrak{m}' = R'\mathfrak{m}$. For each finite R-module we set $M' = M \otimes_R R'$. As noted in [**2,** 1.8],

M is Koszul if and only if M' is Koszul over R'. Also, note that we can make the identifications $(\overline{M})' = M'/\mathfrak{m}'M'$ and $(\mathfrak{m}M)' = \mathfrak{m}'M'$. The maps $\operatorname{Tor}_i^R(\pi_M, N)$ and $\operatorname{Tor}_i^{R'}(\pi_{M'}, N')$ are simultaneously injective, since $R \to R'$ is faithfully flat; hence, $\operatorname{Tor}_i^R(\iota_M, N) = 0$ if and only if $\operatorname{Tor}_i^{R'}(\iota_{M'}, N') = 0$. We may thus assume that k is algebraically closed.

We may also assume that M is indecomposable and non-free, and this implies $\mathfrak{m}^2 M = 0$ as in Remark 2.1.

We prove by induction on n the following statement:

> If M is a Koszul R-module such that $\mathfrak{m}^2 M = 0$ and $\nu(\mathfrak{m}M) = n$, then $\operatorname{Tor}_i^R(\iota_M, N) = 0$ for $i \gg 0$.

The statement is trivially true when $n = 0$, since $\mathfrak{m}M = 0$ in this case. Let $n \geq 1$, and assume that $\operatorname{Tor}_i^R(\iota_M, N) = 0$ for $i \gg 0$ for all Koszul modules M with $\mathfrak{m}^2 M = 0$ and $\nu(\mathfrak{m}M) \leq n - 1$.

Let M be a Koszul R-module with $\mathfrak{m}^2 M = 0$ and $\nu(\mathfrak{m}M) = n$. We will show that $\operatorname{Tor}_i^R(\iota_M, N) = 0$ for $i \gg 0$. It suffices to establish the conclusion when M is indecomposable; thus, we will assume this.

Case 2.8. Assume that $\nu(M) \leq n - 1$. In this case, we have $\nu(\mathfrak{m}M_1) = \nu(M) \leq n - 1$ by Remark 2.1. Since M is Koszul, note that M_1 is Koszul and M_1 does not split off a copy of k. The induction hypothesis, applied to M_1, shows that $\operatorname{Tor}_i^R(\iota_{M_1}, N) = 0$ for $i \gg 0$, and then, Lemma 2.5 gives $\operatorname{Tor}_i^R(\iota_M, N) = 0$ for $i \gg 0$.

Case 2.9. Assume that $\nu(M) \geq n$. By Lemma 1.1, there exists an $x \in M \smallsetminus \mathfrak{m}M$ such that $\operatorname{ann}(x) \neq \mathfrak{m}^2$. Set $A = Rx$ and $B = M/A$. Note that the map $\overline{A} \to \overline{M}$ induced by the inclusion $A \hookrightarrow M$ is injective since $x \notin \mathfrak{m}M$. If $\operatorname{ann}(x) = \mathfrak{m}$, then $A \cong \mathsf{k}$, and this implies that M splits off a copy of k; hence, $M \cong \mathsf{k}$ since M is assumed indecomposable. In this case, the statement trivially holds since $\mathfrak{m}M = 0$. We may thus assume that $\operatorname{ann}(x) \neq \mathfrak{m}$ as well.

Since $\operatorname{ann}(x) \neq \mathfrak{m}^2$, Lemma 2.2 shows that A is Koszul. It follows that $\operatorname{Tor}_i^R(\iota_A, N) = 0$ for $i \gg 0$ by Lemma 2.4. Since $\mathfrak{m}^2 M = 0$, we also have $\mathfrak{m}^2 A = 0 = \mathfrak{m}^2 B$, and the top exact row in the commutative diagram (1.6) is an exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{m}A \longrightarrow \mathfrak{m}M \longrightarrow \mathfrak{m}B \longrightarrow 0,$$

which gives

$$n = \nu(\mathfrak{m}M) = \nu(\mathfrak{m}A) + \nu(\mathfrak{m}B).$$

Since $\operatorname{ann}(x) \neq \mathfrak{m}$, we have $\nu(\mathfrak{m}A) \neq 0$, and hence, $\nu(\mathfrak{m}B) \leq n-1$. Note that B is Koszul by Lemma 1.4 (c), and the induction hypothesis gives $\operatorname{Tor}_i^R(\iota_B, N) = 0$ for $i \gg 0$. Lemma 1.4 (b) then yields $\operatorname{Tor}_i^R(\iota_M, N) = 0$ for $i \gg 0$.

The induction argument is finished, thus establishing the conclusion. $\square$

3. Proof of the main theorem. In this section, we prove the main theorem stated in the introduction.

Theorem 3.1. *Let $(R, \mathfrak{m}, \mathsf{k})$ be a local Gorenstein ring with $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$, and set $e = \nu(\mathfrak{m})$. If $e > 2$ and M, N are finitely generated R-modules, then the following hold*:

(1) $\mathfrak{m}\operatorname{Tor}_i^R(M, N) = 0$ *for* $i \gg 0$;
(2) $\mathfrak{m}\operatorname{Ext}_R^i(M, N) = 0$ *for* $i \gg 0$;
(3) $(1 - et + t^2) \cdot \mathrm{T}_{M,N}^R(t) \in \mathbb{Z}[t]$;
(4) $(1 - et + t^2) \cdot \mathrm{E}_R^{M,N}(t) \in \mathbb{Z}[t]$.

Proof. Statements (2) and (4) follow from statements (1), respectively (3), by duality. Below, we prove (1) and (3).

We may assume that both M and N are indecomposable and not free. In particular, $\mathfrak{m}^2 M = 0 = \mathfrak{m}^2 N$. Let $j \geq 0$. Since $\operatorname{Tor}_{i+j}^R(M, N) \cong \operatorname{Tor}_i^R(M_j, N)$ for all $i \geq 1$, statement (1) holds if and only if $\mathfrak{m}\operatorname{Tor}_i^R(M_j, N) = 0$ for $i \gg 0$, and statement (3) holds if and only if $(1 - et + t^2) \cdot \mathrm{T}_{M_j,N}^R(t) \in \mathbb{Z}[t]$.

Assume first that M is not Koszul; hence, $\mathsf{k} \cong M_j$ for some $j \geq 1$ (see subsection 2.2). In view of the above observation, it suffices to prove the statement for $M = \mathsf{k}$ and, in this case, (1) is clear, and (3) follows from the fact that $\mathrm{T}_{\mathsf{k},N}^R(t) = \mathrm{P}_N^R(t)$ is rational with denominator $1 - et + t^2$, as proven by Sjödin [**11**].

Now assume that M is a Koszul module. Proposition 2.6 gives that there exists an integer s such that $\operatorname{Tor}_i^R(\iota_M, N) = 0$ for $i \geq s$. By Remark 1.3, we have that $\mathfrak{m}\operatorname{Tor}_i^R(M, N) = 0$ for all $i \geq s$, proving (1), and

$$\nu(\operatorname{Tor}_i^R(M,N)) = l(\operatorname{Tor}_i^R(M,N)) = \nu(M)\beta_i^R(N) - \nu(\mathfrak{m}M)\beta_{i-1}^R(N)$$

for all $i > s$. We thus have:

$$\begin{aligned}\mathrm{T}_{M,N}^R(t) = \sum_{i=0}^{s} \nu(\operatorname{Tor}_i^R(M,N))t^i + \nu(M) \sum_{i\geq s+1} \beta_i^R(N)t^i \\ - \nu(\mathfrak{m}M)t \sum_{i\geq s+1} \beta_{i-1}^R(N)t^{i-1}.\end{aligned}$$

It follows from here that $\mathrm{T}_{M,N}^R(t) - H_M(-t)\,\mathrm{P}_N^R(t) \in \mathbb{Z}[t]$. The conclusion of (3) follows, again using the fact that $\mathrm{P}_N^R(t)$ is rational with denominator $1 - et + t^2$. □

When $l(M \otimes_R N) < \infty$, we define a modified version of the series $\mathrm{E}_R^{M,N}(t)$ and $\mathrm{T}_{M,N}^R(t)$ as follows:

$$\mathcal{E}_R^{M,N}(t) = \sum_{i=0}^{\infty} l\left(\operatorname{Ext}_R^i(M,N)\right) t^i \in \mathbb{Z}[[t]],$$

$$\mathcal{T}_{M,N}^R(t) = \sum_{i=0}^{\infty} l\left(\operatorname{Tor}_i^R(M,N)\right) t^i \in \mathbb{Z}[[t]].$$

Under the assumptions of Theorem 3.1, parts (1) and (2) of its statement give that

$$\nu(\operatorname{Ext}_R^i(M,N)) = l(\operatorname{Ext}_R^i(M,N))$$

and

$$\nu(\operatorname{Tor}_i^R(M,N)) = l(\operatorname{Tor}_i^R(M,N))$$

for $i \gg 0$; hence, we have the following corollary.

Corollary 3.2. *Under the hypotheses of Theorem* 3.1*, the following hold*:

(1) $(1 - et + t^2) \cdot \mathcal{E}_R^{M,N}(t) \in \mathbb{Z}[t]$;
(2) $(1 - et + t^2) \cdot \mathcal{T}_{M,N}^R(t) \in \mathbb{Z}[t]$.

Remark 3.3. Several classes of local rings, including that discussed in this paper, are known to satisfy the property that the Poincaré series of all finite modules are rational, sharing a common denominator; see **[9]** for a large class of Gorenstein Artinian rings. In all known cases, such rings are homomorphic images of a complete intersection via a Golod homomorphism. As also mentioned in **[8]**, it seems reasonable to expect that similar rationality results for the series $\mathcal{T}^R_{M,N}(t)$, $\mathcal{E}_R^{M,N}(t)$, $\mathrm{T}^R_{M,N}(t)$ and $\mathrm{E}_R^{M,N}(t)$ hold for other such classes.

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