JOURNAL OF COMMUTATIVE ALGEBRA
Volume 9, Number 4, Winter 2017

GRADED VERSION OF LOCAL COHOMOLOGY WITH RESPECT TO A PAIR OF IDEALS

P.H. LIMA AND V.H. JORGE PÉREZ

ABSTRACT. In this paper, we prove some well-known results on local cohomology with respect to a pair of ideals in graded version, such as the Independence theorem, Lichtenbaum-Harshorne vanishing theorem, Basic finiteness and vanishing theorem, among others. In addition, we present a generalized version of the Melkersson theorem regarding the Artinianness of modules, and a result concerning Artinianness of local cohomology modules.

1. Introduction. Local cohomology with respect to a pair of ideals was first defined in **[11]**, where the authors generalized the usual notion of local cohomology module and studied its various properties such as the relation between the usual local cohomology module $H_I^i(M)$ and that defined to a pair of ideals, $H_{I,J}^i(M)$, vanishing and nonvanishing theorems, the generalized version of the Lichtenbaum-Hartshorne theorem, among others.

Using the above results as motivation, the aim of this paper is to present in graded version some basic theorems on cohomology with respect to a pair of ideals, such as the Independence theorem, Lichtenbaum-Harshorne vanishing theorem, basic finiteness and vanishing theorems, and assertions concerning Artinianness and depth with respect to a pair of ideals.

We shall let R be a graded ring, $I \subseteq R$ a graded ideal, J an arbitrary ideal and M a graded R-module. We will also let R_+ denote the irrelevant ideal of R, that is, the ideal generalized by elements of positive degree.

2010 AMS *Mathematics subject classification.* Primary 13A02, 13D45, 13E10.
Keywords and phrases. Local cohomology, asymptotical stability, Artinianness.
The second author was partially supported by CNPq-Brazil, grant No. 245872/2012-4 and FAPESP, grant No. 2012/20304-1. This work was partially supported by CAPES-Brazil 10056/12-2.
Received by the editors on March 17, 2015, and in revised form on September 30, 2015.
DOI:10.1216/JCA-2017-9-4-545

The organization of this paper is as follows. In Section 2, we view the local cohomology ${}^*H^i_{I,J}(M)$ as a graded module and express it in terms of usual local cohomology modules. In order to be more precise, we denote by ${}^*\widetilde{W}(I,J)$ the set of homogeneous ideals $\mathfrak{c}$ of R such that $I^n \subseteq \mathfrak{c} + J$ for some integer n, and then, show that

$$ {}^*H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} {}^*H^i_{\mathfrak{c}}(M). $$

We also present the graded version of the Independence and Lichtenbaum-Harshorne vanishing theorems for a pair of ideals.

In Section 3, we suppose that R is a positively graded Noetherian ring which is standard, that is, $R = R_0[R_1]$, where R_0 is local ring. In [**5**], the number

$$ \operatorname{cd}(I,J,M) := \sup\{i \mid H^i_{I,J}(M) \neq 0\} $$

is defined. We prove that

$$ \dim M/(\mathfrak{m}_0 R + J)M = \sup\{i \mid H^i_{R_+,\mathfrak{m}_0 R + J}(M) \neq 0\}, $$

and thus, it is obtained that $\operatorname{cd}(R_+,\mathfrak{m}_0 R, M) = \operatorname{cd}(R_+,M)$. Also, in [**10,** Theorem 2.1], the authors show that, if $n = \sup\{i \mid H^i_{R_+}(M) \neq 0\}$, then the R-module $H^n_{R_+}(M)/\mathfrak{m}_0 H^n_{R_+}(M)$ is Artinian. We prove a similar result for the case of local cohomology with respect to a pair of homogeneous ideals. If $c := \operatorname{cd}(R_+,\mathfrak{m}_0 R, M)$, then the R-module $H^c_{R_+,\mathfrak{m}_0 R}(M)/\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0 R}(M)$ is Artinian.

Furthermore, a new version, with respect to a pair of ideals, is also presented for Melkersson's theorem regarding Artinianness.

In Section 4, the module M and the ring R are assumed to be a Cohen-Macaulay, and then, an expression to the number

$$ \inf\{i \in \mathbb{N}_0 \mid H^i_{R_+,J}(M) \neq 0\} $$

is obtained.

In Section 5, it is proved that $H^i_{R_+,J}(M)_n$ is a finite R_0-module for all $n \in \mathbb{Z}$ and all $i \geq 0$ if and only if $H^i_{R_+,J}(M)_n = 0$ for $n \gg 0$ and all $i \geq 0$. Finally, we prove a result regarding asymptotical stability (Theorem 5.6).

2. Graded versions for a pair of ideals. In this section, we introduce a grading to $H^i_{I,J}(M)$, making this a graded module. The results [**11,** Theorem 3.2], the Independence theorem for a pair of ideals and the generalized version of the Lichtenbaum-Hartshorne theorem are again presented from the point of view of graded modules.

(A) Let R be a graded ring, let $I \subseteq R$ be a graded ideal and let J be an arbitrary ideal. If M is a graded R-module, then $\Gamma_{I,J}(M)$ is a graded submodule of M. In fact, set $m = (m_k) \in \Gamma_{I,J}(M)$, such that $I^n \subseteq (0 : m) + J$. It is easy to see that $I^n \subseteq (0 : m_k) + J$, for each k. Then define $\Gamma_{I,J}(M)_i = \{m \in M_i : mI^n \subseteq mJ$ for some positive integer $n \geq 1\}$.

(B) For a homomorphism $f : M \to N$, we have $f(\Gamma_{I,J}(M)) \subseteq \Gamma_{I,J}(N)$ such that there is a mapping $\Gamma_{I,J}(f) : \Gamma_{I,J}(M) \to \Gamma_{I,J}(N)$, which is the restriction of f to $\Gamma_{I,J}(M)$. Thus, $\Gamma_{I,J}$ is an additive functor on the category of all graded R-modules.

(C) Since the category of the graded modules has enough injectives, we can form the ith right derived functor of $\Gamma_{I,J}$ (on the category of the graded modules), which will be denoted by ${}^*H^i_{I,J}$, $i \geq 0$. For a graded R-module M, we shall refer to ${}^*H^i_{I,J}(M)$ as the ith graded local cohomology module of M with respect to the pair of ideals (I, J).

(D) Through the use of functor properties, given an exact sequence $0 \to M \to N \to P \to 0$ of graded R-modules, a long exact sequence can be derived

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}^*H^0_{I,J}(M) & \longrightarrow & {}^*H^0_{I,J}(N) & \longrightarrow & {}^*H^0_{I,J}(P) & \longrightarrow & \\ & \longrightarrow & {}^*H^1_{I,J}(M) & \longrightarrow & {}^*H^1_{I,J}(N) & \longrightarrow & {}^*H^1_{I,J}(P) & \longrightarrow & \cdots, \end{array}$$

of graded modules with respect to a pair of ideals.

Definition 2.1. We denote by ${}^*\widetilde{W}(I, J)$ the set of homogeneous ideals $\mathfrak{c}$ of R such that $I^n \subseteq \mathfrak{c} + J$ for some integer n. We also define a partial order for this set:

$$\mathfrak{c} \leq \mathfrak{d} \text{ if } \mathfrak{c} \supseteq \mathfrak{d}, \quad \text{for } \mathfrak{c}, \mathfrak{d} \in {}^*\widetilde{W}(I, J).$$

If $\mathfrak{a} \leq \mathfrak{b}$, then we obtain the inclusion map $\Gamma_{\mathfrak{a}}(M) \hookrightarrow \Gamma_{\mathfrak{b}}(M)$. The order relation on ${}^*\widetilde{W}(I, J)$ and the inclusion maps turn $\{\Gamma_{\mathfrak{a}}(M)\}_{\mathfrak{a} \in {}^*\widetilde{W}(I,J)}$ into a direct system of graded R-modules.

Proposition 2.2. *Let R be a graded ring, I a graded ideal, J an arbitrary ideal of R and M a graded R-module. Then, there is a natural graded isomorphism*

$$^*H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{c}\in {}^*\widetilde{W}(I,J)} {}^*H^i_{\mathfrak{c}}(M).$$

Proof. Firstly, observe that $\Gamma_{I,J}(M) = \bigcup_{\mathfrak{c}\in {}^*\tilde{W}(I,J)} \Gamma_{\mathfrak{c}}(M)$. In fact, if $x \in \bigcup_{\mathfrak{c}\in {}^*\tilde{W}(I,J)} \Gamma_{\mathfrak{c}}(M)$, we have $x\mathfrak{c}^n = 0$ and $I^m \subseteq \mathfrak{c} + J$ for positive integers n, m and some $\mathfrak{c} \in^* \widetilde{W}(I,J)$. Since $I^{mn} \subseteq (\mathfrak{c}+J)^n \subseteq \mathfrak{c}^n + J$, we have $I^{mn}x \subseteq Jx$, that is, $x \in \Gamma_{I,J}(M)$.

Now, let $x \in \Gamma_{I,J}(M)$. Write $x = x_1 + \cdots + x_r$, where x_i is homogeneous. Then, for each i, we have that there exists an n_i such that $I^{n_i} \subseteq \operatorname{ann}(x_i) + J$. Set $\mathfrak{a}_i = \operatorname{ann}(x_i)$. This yields $x\mathfrak{a}_1 \cdots \mathfrak{a}_r = 0$. In addition, note that

$$I^{n_1+\cdots+n_r} = I^{n_1}\cdots I^{n_r} \subseteq (\mathfrak{a}_1 + J)\cdots(\mathfrak{a}_r + J) \subseteq \mathfrak{a}_1\cdots\mathfrak{a}_r + J.$$

This means that $\mathfrak{a}_1 \cdots \mathfrak{a}_r \in \widetilde{W}(I,J)$. Therefore, $x \in \Gamma_{\mathfrak{c}}(M)$, where $\mathfrak{c} = \mathfrak{a}_1 \cdots \mathfrak{a}_r$.

The rest of the proof follows by using [**2,** Theorem 12.3.1]. $\square$

Remark 2.3. Consider the above setup. Since

$$\Gamma_{I,J}(M) = \bigcup_{\mathfrak{c}\in {}^*\tilde{W}(I,J)} \Gamma_{\mathfrak{c}}(M),$$

it may be concluded similarly to the above proof that

$$H^i_{I,J}(M) \cong \varinjlim_{\mathfrak{c}\in {}^*\tilde{W}(I,J)} H^i_{\mathfrak{c}}(M).$$

The next result is obtained from the previous proposition since ${}^*H^i_{\mathfrak{c}}(E) = 0$ for any graded ideal $\mathfrak{c}$ and an *injective graded R-module E and $i > 0$.

Proposition 2.4. *Let R be a graded ring, I a graded ideal and J an arbitrary ideal. Let E be an *injective graded R-module. Then, $H^i_{I,J}(E) = 0$.*

By Proposition 2.4, we have the following

Proposition 2.5. *Let R be a graded ring, I a graded ideal and J an arbitrary ideal. Let M be an R-graded module. There is an isomorphism*

$$ {}^*H^i_{I,J}(M) \cong H^i_{I,J}(M), $$

for all i as underlying R-modules.

The next result is a graded version for the independence theorem with respect to a pair of ideals, see [**11,** Theorem 2.7].

Theorem 2.6. *Assume that $R = \bigoplus_{n\in\mathbb{Z}} R_n$ is a graded ring, $I \subset R$ a graded ideal and $J \subset R$ an arbitrary ideal. Let $R' = \bigoplus_{n\in\mathbb{Z}} R'_n$ be another Noetherian graded ring and $f : R \to R'$ a graded homomorphism of rings such that $f(J) = JR'$ and M' an R'-module.*

(i) *For each $i \in \mathbb{N}_0$, both the cohomology modules $H^i_{IR',JR'}(M')$ and $H^i_{I,J}(M')$ are graded R-modules*;

(ii) *For each $i \in \mathbb{N}_0$, there exists a graded isomorphism*

$$ H^i_{IR',JR'}(M') \cong H^i_{I,J}(M') $$

of graded R-modules.

Proof. The first item follows since f is homogeneous.

The assumption $f(J) = JR'$ gives $\Gamma_{IR',JR'}(M') = \Gamma_{I,J}(M')$. By [**11,** Theorem 2.7], for each graded R-module M', an isomorphism

$$ H^i_{IR',JR'}(M') \cong H^i_{I,J}(M') $$

is obtained. By using the grading on $H^i_{I,J}(M')$ obtained from the first item, we can turn this isomorphism into a graded isomorphism.

On the other hand, given an R-graded homomorphism $f : M' \to N'$, we have a natural commutative diagram

$$ \begin{array}{ccc} H^i_{I,J}(M') & \longrightarrow & H^i_{I,J}(N') \\ \downarrow & & \downarrow \\ H^i_{IR',JR'}(M') & \longrightarrow & H^i_{IR',JR'}(N'). \end{array} $$

Hence, as $H^i_{IR',JR'}(E)$ equals zero (by Proposition 2.4), these (new) gradings coincide with those from item (i). Thus, item (ii) holds. □

Next, we provide a type of generalization for the graded version of the Lichtenbaum-Hartshorne theorem, see [**2,** 13.1.16].

Theorem 2.7. *Let $R = \oplus_{n\geq 0} R_n$ be a positively graded ring of dimension d which is an integral domain. Assume that R_0 is a complete local ring. Let I be a graded ideals of R such that* $\dim R/I > 0$ *and J an arbitrary ideal of R. Then, $H^d_{I,J}(R)/JH^d_{I,J}(R) = 0$.*

Proof. By using the exact sequence

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

and the Grothendieck vanishing theorem, we obtain a surjective map

$$H^d_I(R) \longrightarrow H^d_I(R/J).$$

From the ordinary graded Lichtenbaum-Hartshorne vanishing theorem, we have $H^d_I(R) = 0$, thus $H^d_I(R/J) = 0$.

Now we use [**11,** Corollary 2.5 and Lemma 4.8] to obtain

$$0 = H^d_I(R/J) \cong H^d_{I,J}(R/J) \cong H^d_{I,J}(R) \otimes_R R/J \cong H^d_{I,J}(R)/JH^d_{I,J}(R).$$

□

3. Top local cohomology. In this section, we give a version for Melkersson's theorem concerning Artinianness of a module with respect to a pair of ideals (Proposition 3.1). Also, a result is obtained regarding Artinianness of local cohomology with respect to a pair of ideals (Theorem 3.8). To conclude the section, we find the top of the local cohomology with respect to a pair of ideals (Theorem 3.5).

Proposition 3.1 ([**9,** Theorem 1.3])**.** *Let M be an (I, J)-torsion R-module for which $(JM :_M I)$ is an Artinian module. Then M is an Artinian module.*

Proof. If M is an (I, J)-torsion R-module, then M/JM is an I-torsion module by [**11,** Corollary 1.9]. Since by assumption $(JM :_M I) = (0 :_{M/JM} I)$ is an Artinian module we can use a result due to Melkersson (see [**9,** Theorem 1.3]) to obtain that M/JM is an Artinian module. Moreover, $JM \subseteq (JM :_M I)$ is also an Artinian module. By an exact sequence, we can conclude that M is an Artinian module. □

We now enunciate two results from **[11]** in the graded case. The proofs are essentially the same.

Lemma 3.2 ([**11,** Corollary 4.2])**.** *Let M be a finite module over a graded local ring R. Let I and J be graded ideals of R. Then the following are equivalent*:

(i) *M is (I,J)-torsion R-module.*
(ii) *$H^i_{I,J}(M) = 0$ for all integers $i > 0$.*

Theorem 3.3 ([**11,** Theorem 4.3])**.** *Let M be a finite module over a graded local ring R. Let I and J be graded ideals of R. Then $H^i_{I,J}(M) = 0$ for any $i > \dim M/JM$.*

Throughout the remainder of this section let

$$R = \oplus_{d\geq 0} R_d$$

denote a positively graded commutative Noetherian ring, which is standard, that is, $R = R_0[R_1]$. Assume that R_0 is a local ring of maximal ideal $\mathfrak{m}_0$. Set $R_+ = \oplus_{i>0} R_i$, the irrelevant ideal of R. Let $M = \oplus_{d\in\mathbb{Z}} M_d$ be a finitely graded R-module. Let $M[a]$ denote the graded module a-shift of M, defined by $M[a]_i = M_{i+a}$.

Remark 3.4. From Proposition 2.2, $H^i_{I,J}(M[a]) \cong H^i_{I,J}(M)[a]$ can be derived as homogeneous modules.

Let I and J be ideals of R. The number

$$\operatorname{cd}(I,J,M) := \sup\{i \mid H^i_{I,J}(M) \neq 0\}$$

is defined in [**5**]. If $J = 0$, we just denote it by $\operatorname{cd}(I,M)$.

Theorem 3.5. *Let J be a graded ideal of R. Then,*

$$\dim M/(\mathfrak{m}_0 R + J)M = \sup\{i \mid H^i_{R_+,\mathfrak{m}_0 R+J}(M) \neq 0\}.$$

Proof. Set $d := \dim M/(\mathfrak{m}_0 R + J)M$. Consider the exact sequence

$$0 \longrightarrow (\mathfrak{m}_0 R + J)M \longrightarrow M \longrightarrow M/(\mathfrak{m}_0 R + J)M \longrightarrow 0,$$

which yields the exact sequence

$$H^i_{R_+,\mathfrak{m}_0R+J}(M) \longrightarrow H^i_{R_+,\mathfrak{m}_0R+J}\left(\frac{M}{(\mathfrak{m}_0R+J)M}\right) \longrightarrow H^{i+1}_{R_+,\mathfrak{m}_0R+J}((\mathfrak{m}_0R+J)M).$$

Now note that

$$\dim \frac{(\mathfrak{m}_0R+J)M}{(\mathfrak{m}_0R+J)(\mathfrak{m}_0R+J)M} \leq \dim \frac{M}{(\mathfrak{m}_0R+J)^2M} = \dim \frac{M}{(\mathfrak{m}_0R+J)M} = d;$$

thus, due to Theorem 3.3, we obtain $H^i_{R_+,\mathfrak{m}_0R+J}((\mathfrak{m}_0R+J)M) = 0$ for $i > d$.

On the other hand, by using [**6,** Corollary 35.20] and [**11,** Corollary 2.5], it is easily seen that

$$H^d_{R_+,\mathfrak{m}_0R+J}(M/(\mathfrak{m}_0R+J)M) \cong H^d_{R_+}(M/(\mathfrak{m}_0R+J)M) \cong H^d_{(R/\mathfrak{m}_0R)_+}(M/(\mathfrak{m}_0R+J)M).$$

This last cohomology module is nonzero [**6,** Corollary 36.19]. By observing the above long exact sequence we obtain the desired result. □

Corollary 3.6. $\operatorname{cd}(R_+,\mathfrak{m}_0R,M) = \operatorname{cd}(R_+,M)$.

Proof. The proof follows from [**3,** Lemma 3.4]. □

Remark 3.7. Observe that, if $\operatorname{cd}(R_+,\mathfrak{m}_0R,M) > 0$, then, by Theorem 3.5, we can choose a homogeneous element $x \in R_+$ avoiding all the minimal primes of $(\mathfrak{m}_0M :_R M)$, so that

$$\operatorname{cd}(R_+,\mathfrak{m}_0R,M/xM) = \operatorname{cd}(R_+,\mathfrak{m}_0R,M) - 1.$$

In addition, if $\operatorname{grade}(R_+,M) > 0$, the element x may be chosen to be M-regular.

The next theorem is similar to [**10,** Theorem 2.1].

Theorem 3.8. *Set* $c := \operatorname{cd}(R_+, \mathfrak{m}_0 R, M)$*. The* R*-module*

$$H^c_{R_+,\mathfrak{m}_0 R}(M)/\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0 R}(M)$$

is Artinian.

Proof. We proceed by induction on $c = \operatorname{cd}(R_+, \mathfrak{m}_0 R, M)$. If $c = 0$, then $M = \Gamma_{R_+,\mathfrak{m}_0 R}(M)$. As M is a finite R-module, we have

$$R^n_+ \Gamma_{R_+,\mathfrak{m}_0 R}(M) \subseteq \mathfrak{m}_0 \Gamma_{R_+,\mathfrak{m}_0 R}(M)$$

for some integer n. Hence, $\Gamma_{R_+,\mathfrak{m}_0 R}(M)/\mathfrak{m}_0 \Gamma_{R_+,\mathfrak{m}_0 R}(M)$ is annihilated by a power of $\mathfrak{m}_0 + R_+$; thus, it is Artinian.

Now assume by induction that $c > 0$. We have shown the result for any finite graded R-module N such that $\operatorname{cd}(R_+, \mathfrak{m}_0 R, N) < c$.

Since $c > 0$, due to [**11,** Corollary 1.13], we have $\operatorname{cd}(R_+, \mathfrak{m}_0 R, M) = \operatorname{cd}(R_+, \mathfrak{m}_0 R, M/\Gamma_{R_+,\mathfrak{m}_0}(M))$. Thus, we can assume that M is an (R_+, J)-torsion free R-module. Therefore, M is also an R_+-torsion free R-module, as $\Gamma_{R_+}(M) \subseteq \Gamma_{R_+,J}(M)$. By using Remark 3.7, there exists a homogeneous element $x \in R_+$ which is M-regular and such that $\operatorname{cd}(R_+, \mathfrak{m}_0 R, M/xM) = c - 1$, say, $\deg(x) = a$. The exact sequence

$$0 \longrightarrow M \xrightarrow{x} M[-a] \longrightarrow (M/xM)[-a] \longrightarrow 0$$

induces an exact sequence

$$\cdots \to H^{c-1}_{R_+,\mathfrak{m}_0 R}(M/xM)$$
$$\to H^c_{R_+,\mathfrak{m}_0 R}(M) \xrightarrow{x} H^c_{R_+,\mathfrak{m}_0 R}(M)[-a] \to H^c_{R_+,\mathfrak{m}_0 R}(M/xM)[-a] \to \cdots .$$

Observe that $H^c_{R_+,\mathfrak{m}_0 R}(M/xM) = 0$. Let L denote the kernel of multiplication by x on $H^c_{R_+,\mathfrak{m}_0 R}(M)$. By inductive hypothesis, we obtain that the R-module $H^{c-1}_{R_+,\mathfrak{m}_0 R}(M/xM)/\mathfrak{m}_0 H^{c-1}_{R_+,\mathfrak{m}_0 R}(M/xM)$ is Artinian and so is $L/\mathfrak{m}_0 L$, as a homomorphic image of this module.

In addition, we have the exact sequence

$$L/\mathfrak{m}_0 L \longrightarrow \frac{H^c_{R_+,\mathfrak{m}_0 R}(M)}{\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0 R}(M)} \xrightarrow{x} \frac{H^c_{R_+,\mathfrak{m}_0 R}(M)}{\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0 R}(M)}[-a] \longrightarrow 0.$$

Note then that the kernel of the multiplication by x on $H^c_{R_+,\mathfrak{m}_0 R}(M)/[\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0 R}(M)]$ is an Artinian R-module. Since $H^c_{R_+,\mathfrak{m}_0 R}(M)/[\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0 R}(M)]$ is an $((x), \mathfrak{m}_0 R)$-torsion R-module, Proposition 3.1

can be used in order to obtain $H^c_{R_+,\mathfrak{m}_0R}(M)/[\mathfrak{m}_0 H^c_{R_+,\mathfrak{m}_0R}(M)]$ is Artinian. $\square$

Theorem 3.9. *Let J be a graded ideal of R. Set $d := \dim M/(\mathfrak{m}_0R + J)M$. Suppose that $H^i_{R_+,J}(M)$ is a finite R-module for all $i > d$. Then,*

$$H^i_{R_+,J}(M) = 0$$

for all $i > d$.

Proof. We argue by induction. Let $n = \dim_R M$. Suppose that $n = 0$; thus, $H^i_{R_+,\mathfrak{m}_0R}(M) = 0$ for $i > 0$ by Theorem 3.3, for example.

Assume then that $n > 0$ and that the result is established for R-modules of dimension smaller than n. By [**11,** Corollary 2.5], $H^i_{R_+,J}(\Gamma_J(M)) \cong H^i_{R_+}(\Gamma_J(M))$ for all $i \geq 0$. By using the fact that

$$\sqrt{\operatorname{ann}\left(\frac{\Gamma_J(M)}{\mathfrak{m}_0\Gamma_J(M)}\right)} = \sqrt{\mathfrak{m}_0R + \operatorname{ann}(\Gamma_J(M))}$$

and

$$\sqrt{\operatorname{ann}\left(\frac{M/JM}{\mathfrak{m}_0(M/JM)}\right)} = \sqrt{\mathfrak{m}_0R + J + \operatorname{ann}(M)},$$

it can be concluded that

$$\dim \frac{\Gamma_J(M)}{\mathfrak{m}_0\Gamma_J(M)} \leq \dim \frac{M/JM}{\mathfrak{m}_0(M/JM)} = d.$$

Next, we use [**3,** Lemma 3.4] to obtain $H^i_{R_+,J}(\Gamma_J(M)) = 0$ for all $i > d$. Moreover, the exact sequence

$$0 \longrightarrow \Gamma_J(M) \longrightarrow M \longrightarrow M/\Gamma_J(M) \longrightarrow 0$$

yields the long exact sequence

$$\cdots \longrightarrow H^i_{R_+,J}(\Gamma_J(M)) \longrightarrow H^i_{R_+,J}(M) \longrightarrow H^i_{R_+,J}(M/\Gamma_J(M)) \longrightarrow \cdots .$$

We then derive

$$H^i_{R_+,J}(M) \cong H^i_{R_+,J}(M/\Gamma_J(M)),$$

for all $i > d$. This yields

$$\dim \frac{M/\Gamma_J(M)}{(\mathfrak{m}_0 R + J)M/\Gamma_J(M)} = \dim \frac{M}{(\mathfrak{m}_0 R + J)M + \Gamma_J(M)} \leq d.$$

In this way, we can assume that M is a J torsion-free module. Thus, there exists a homogeneous element $a \in J$ which is a nonzero divisor on M. The exact sequence

$$0 \longrightarrow M \xrightarrow{a} M \longrightarrow M/aM \longrightarrow 0$$

yields the long exact sequence

$$\cdots \longrightarrow H^i_{R_+,J}(M) \xrightarrow{a} H^i_{R_+,J}(M) \longrightarrow H^i_{R_+,J}(M/aM) \longrightarrow \cdots .$$

Since

$$\dim_R M/aM = n-1 \quad \text{and} \quad \dim_R \frac{M/aM}{(\mathfrak{m}_0 R + J)M/aM} = d,$$

we can use the inductive hypothesis to conclude that $H^i_{R_+,J}(M/aM) = 0$ for all $i > d$. Then, the above long exact sequence yields $aH^i_{R_+,J}(M) = H^i_{R_+,J}(M)$. By Nakayama's lemma, we obtain the desired result. □

4. Depth (I, J) on graded module. In this section, we work with the concept of the depth of a pair homogeneous ideals of R and obtain an expression for it as M and R are both Cohen-Macaulay.

For the typical case, [**2,** Theorem 6.2.7] stated

$$\operatorname{depth}_I(M) = \inf\{i \in \mathbb{N}_0 \mid H^i_I(M) \neq 0\},$$

for an R-ideal I and a finite R-module M such that $IM \neq M$. The case of a pair of (I, J) ideals was defined in [**1,** Definition 3.1]. Now we recall this definition, as follows.

Throughout this section, let $R = \oplus_{d\geq 0} R_d$ denote a positively standard graded Noetherian ring such that R_0 is a local ring of maximal ideal $\mathfrak{m}_0$. Let $M = \oplus_{d\in\mathbb{Z}} M_d$ be a finitely graded R-module.

Definition 4.1. Let I and J be two homogeneous ideals of the graded ring R and M a graded R-module. We define depth of (I, J) on M by

$$\operatorname{depth}(I,J,M)=\inf\{\operatorname{depth}(\mathfrak{a},M)\mid \mathfrak{a}\in \widetilde{W}(I,J)\}$$

if this infimum exists, and ∞ otherwise.

Remark 4.2. In the general case, by [**11,** Theorem 3.2, Theorem 4.1] or [**1,** Proposition 3.3],

$$\operatorname{depth}(I,J,M)=\inf\{i\in\mathbb{N}_0\mid H^i_{I,J}(M)\neq 0\}.$$

Let I,J,K be three homogeneous ideals of R. We introduce

$$\operatorname{ht}(I,J,K):=\inf\{\operatorname{ht}(\mathfrak{p})\mid \mathfrak{p}\in W(I,J)\cap V(K)\}.$$

Note that, when $J=(0)$,

$$\operatorname{ht}(I,J,K)=\operatorname{ht}(I+K).$$

Proposition 4.3. *Assume that M is a Cohen-Macaulay R-module and R is a Cohen-Macaulay ring. Set $I=\sqrt{\operatorname{ann}_R M}$. Then,*

$$\operatorname{depth}(R_+,J,M)=\operatorname{ht}(R_+,J,I)-\operatorname{ht}(I).$$

In particular, $\operatorname{ht}(R_+,J,I)-\operatorname{ht}(I)=\inf\{i\in\mathbb{N}_0\mid H^i_{R_+,J}(M)\neq 0\}$.

Proof. By [**11,** Theorem 4.1] (or [**1,** Proposition 3.3]),

$$\operatorname{depth}(R_+,J,M)=\inf\{\operatorname{depth} M_{\mathfrak{p}}\mid \mathfrak{p}\in W(R_+,J)\}.$$

Note that

$$\operatorname{depth}(R_+,J,M)=\inf\{\operatorname{depth} M_{\mathfrak{p}}\mid \mathfrak{p}\in W(R_+,J)\cap V(I)\},$$

since $M_{\mathfrak{p}}=0$ as $\mathfrak{p}\notin V(I)$, that is, $\operatorname{depth} M_{\mathfrak{p}}=\infty$ and since M is Cohen-Macaulay, in particular,

$$\operatorname{depth} M_{\mathfrak{p}}=\dim M_{\mathfrak{p}}$$

for each $\mathfrak{p}\in W(R_+,J)$. However, R is also Cohen-Macaulay, by hypothesis, so

$$\dim M_{\mathfrak{p}}=\dim R_{\mathfrak{p}}/I_{\mathfrak{p}}=\dim R_{\mathfrak{p}}-\operatorname{ht} I_{\mathfrak{p}}.$$

By [**8,** Lemma 1.2.2], all minimal primes of I have the same height, so that $\operatorname{ht} I_{\mathfrak{p}}=\operatorname{ht} I$ for each $\mathfrak{p}$. Combining the above arguments, we obtain

$$\operatorname{depth}(R_+, J, M) = \inf\{\dim R_{\mathfrak{p}} - \operatorname{ht} I \mid \mathfrak{p} \in W(R_+, J) \cap V(I)\}.$$

This equals $\operatorname{ht}(R_+, J, I) - \operatorname{ht} I$, by definition. The proof is complete. □

Proposition 4.4. *Let J be a graded ideal of R. Then, any integer i for which $H^i_{R_+,\mathfrak{m}_0 R+J}(M) \neq 0$ satisfies*

$$\operatorname{depth}(R_+, \mathfrak{m}_0 R + J, M) \leq i \leq \dim M/(\mathfrak{m}_0 R + J)M.$$

Proof. The proof follows by Remark 4.2 and Theorem 3.5. □

Corollary 4.5. *There is exactly one integer i for which $H^i_{R_+,\mathfrak{m}_0 R+J}(M) \neq 0$ if and only if*

$$\operatorname{depth}(R_+, \mathfrak{m}_0 R + J, M) = \dim M/(\mathfrak{m}_0 R + J)M.$$

5. Basic finiteness, vanishing theorem and asymptotical stability. In this section, we generate two classical results regarding the graded components of local cohomology for the case of local cohomology with respect to a pair of ideals (Proposition 5.2 and Theorem 5.4). This leads to a result on asymptotical stability of the sequence $\{\operatorname{Ass}_{R_0}(H^i_{R_+,J}(M))_n\}_{n\in\mathbb{Z}}$. Throughout this section, we assume R and M are as in Section 4.

It is well known that $H^i_{R_+}(M)_n$ is a finitely generated R_0-module for all $n \in \mathbb{Z}$ and $H^i_{R_+}(M)_n = 0$, for n sufficiently large. The next assertion gives a positive answer for the case of cohomology modules with respect to a pair of ideals.

Remark 5.1. Let I, K, J be arbitrary ideals of R, and let M be a K-torsion R-module. It is easy to verify that $H^i_{I+K,J}(M) \cong H^i_{I,J}(M)$ for all $i \in \mathbb{N}_0$.

Proposition 5.2. *Let J be an ideal generated by elements of zero degree. Set $\mathfrak{a} := \mathfrak{a}_0 + R_+$, where $\mathfrak{a}_0$ denotes an ideal of R_0. Suppose that $H^i_{\mathfrak{a},J}(M)_n$ is a finite R_0-module for all integers n and i. Then, $H^i_{\mathfrak{a},J}(M)_n = 0$ for all $i \geq 0$ and n sufficiently large.*

Proof. We proceed by induction on $\dim M$. If $\dim M = 0$, then by [**11,** Theorem 4.7(1)]. The result is clearly true for all $i > 0$. In addition, $\Gamma_{\mathfrak{a},J}(M)_n = M_n = 0$ for $n \gg 0$.

Suppose now that $\dim M > 0$ and the result is established for finitely generated modules of dimension less than $\dim M$. By [**11,** Corollary 2.5], $H^i_{\mathfrak{a},J}(\Gamma_J(M)) \cong H^i_{\mathfrak{a}}(\Gamma_J(M))$ for all $i \geq 0$. The exact sequence

$$0 \longrightarrow \Gamma_J(M) \longrightarrow M \longrightarrow M/\Gamma_J(M) \longrightarrow 0$$

then yields the long exact sequence

$$H^i_{\mathfrak{a}}(\Gamma_J(M))_n \longrightarrow H^i_{\mathfrak{a},J}(M)_n \longrightarrow H^i_{\mathfrak{a},J}(M/\Gamma_J(M))_n \longrightarrow H^{i+1}_{\mathfrak{a}}(\Gamma_J(M))_n.$$

Using [**7,** Proposition 1.1], the above sequence yields that $H^i_{\mathfrak{a},J}(M)_n$ is isomorphic to $H^i_{\mathfrak{a},J}(M/\Gamma_J(M))_n$ for all n sufficiently large. Hence, we can assume that M is a J torsion-free R-module. As J is generated by homogeneous elements of zero degree, by the prime avoidance lemma, there exists an element x in J of zero degree which is a non zero divisor on M.

The exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

induces an exact sequence

$$\longrightarrow H^i_{\mathfrak{a},J}(M)_n \xrightarrow{x} H^i_{\mathfrak{a},J}(M)_n \longrightarrow H^i_{\mathfrak{a},J}(M/xM)_n \longrightarrow .$$

By the above sequence, we have $H^i_{\mathfrak{a},J}(M/xM)_n$ is finitely generated. Therefore, by the induction hypothesis, for every $i \geq 0$, $H^i_{\mathfrak{a},J}(M/xM)_n = 0$ for all n sufficiently large. The above exact sequence then yields an epimorphism

$$H^i_{\mathfrak{a},J}(M)_n \xrightarrow{x} H^i_{\mathfrak{a},J}(M)_n,$$

for all $n \gg 0$ so that

$$H^i_{\mathfrak{a},J}(M)_n = xH^i_{\mathfrak{a},J}(M)_n.$$

Nakayama's lemma completes the proof. □

Lemma 5.3. *If J is generated by elements of degree* 0 *and $H^i_{R_+,J}(M)_n = 0$ for $n \gg 0$ and all $i \geq 0$, then $H^i_{R_+,J}(M)_n$ is a finitely generated R_0-module for all n integers and for all $i \geq 0$.*

Proof. We proceed by induction on i. Since M is a finite R-module, $H^0_{R_+,J}(M)_n$ is a finite R_0-module for all n.

Suppose now that $i > 0$. The result is established for smaller values of i. Due to the graded isomorphism $H^i_{R_+,J}(M) \cong H^i_{R_+,J}(M/\Gamma_{R_+,J}(M))$ for $i > 0$, we can assume that M is an (R_+, J)-torsion-free R-module (and thus an R_+ torsion-free module). Then there exists an element $x \in R_+$, which is a non-zero divisor on M, say $\deg(x) = a$. The exact sequence

$$0 \longrightarrow M \xrightarrow{x} M[a] \longrightarrow (M/xM)[a] \longrightarrow 0$$

induces an exact sequence

$$H^{i-1}_{R_+,J}(M)_{n+a} \to H^{i-1}_{R_+,J}(M/xM)_{n+a} \to H^i_{R_+,J}(M)_n \xrightarrow{x} H^i_{R_+,J}(M)_{n+a}.$$

From the above exact sequence it can be deduced that $H^i_{R_+,J}(M/xM)_n = 0$ for n sufficiently large. By the inductive hypothesis, it then follows that $H^{i-1}_{R_+,J}(M/xM)_q$ is finitely generated for all $q \in \mathbb{Z}$. There exists $s \in \mathbb{Z}$ such that $H^{i-1}_{R_+,J}(M/xM)_n = 0$ by hypothesis for all $n \geq s$ and $H^i_{R_+,J}(M)_n = 0$ for all $n \geq s - a$.

Fix $n \in \mathbb{Z}$, and let $k \geq 0$ be an integer such that $n + ka \geq s - a$. Thus, $H^i_{R_+,J}(M)_{n+ka} = 0$. For each $j = 0, \ldots, k-1$, we have the exact sequence

$$H^{i-1}_{R_+,J}(M/xM)_{n+(j+1)a} \longrightarrow H^i_{R_+,J}(M)_{n+ja} \xrightarrow{x} H^i_{R_+,J}(M)_{n+(j+1)a}.$$

In conclusion, we obtain that $H^i_{R_+,J}(M)_{n+ja}$ is finitely generated for $j = k-1, k-2, \ldots, 1, 0$ such that $H^i_{R_+,J}(M)_n$ is finitely generated for all $n \in \mathbb{Z}$. □

Theorem 5.4. *If J is generated by elements of zero degree, then the following are equivalent.*

(i) $H^i_{R_+,J}(M)_n$ *is a finite R_0-module for all $n \in \mathbb{Z}$ and all $i \geq 0$.*
(ii) $H^i_{R_+,J}(M)_n = 0$ *for $n \gg 0$ and all $i \geq 0$.*

Proof. The proof follows from Lemma 5.3 and Proposition 5.2. □

Consider the following definition. We say $\mathrm{Ass}_{R_0}(H^i_{R_+,J}(M)_n)$ is *asymptotically increasing* for $n \to -\infty$ if there exists an $n_0 \in \mathbb{Z}$ such that

$$\operatorname{Ass}_{R_0}(H^i_{R_+,J}(M)_n) \subseteq \operatorname{Ass}_{R_0}(H^i_{R_+,J}(M)_{n+1})$$

for all $n \leq n_0$.

Lemma 5.5. *Let M be a finite graded R-module and J an arbitrary ideal. Let $i \in \mathbb{N}_0$ be such that $H^j_{R_+,J}(M)_n$ is a finite R_0-module for all $j < i$ and $n \ll 0$. Then $\operatorname{Ass}_{R_0}(H^i_{R_+,J}(M)_n)$ is asymptotically increasing for $n \to -\infty$.*

Proof. The proof essentially follows as in [**7,** Theorem 3.4] by using the natural graded isomorphism $H^k_{R_+,J}(M) \cong H^k_{R_+,J}(M/\Gamma_{R_+,J}(M))$ for all $k \geq 1$, and observing that, if M is an (R_+, J)-torsion-free R-module, it is also an R_+-torsion-free since $\Gamma_{R_+}(M) \subseteq \Gamma_{R_+,J}(M)$. □

Theorem 5.6. *Let M be a finitely graded R-module, J an ideal of R generated by elements of zero degree, and let $i \in \mathbb{N}$ be such that $H^j_{R_+,J}(M)_n$ is a finite R_0-module for all $j < i$ and $n \ll 0$. If one of the conditions in Theorem 5.4 occurs, then there exists a finite subset X of $\operatorname{Spec}(R_0)$ such that $\operatorname{Ass}_{R_0}(H^i_{R_+,J}(M)_n) = X$ for $n \ll 0$.*

Proof. This is a consequence of Lemma 5.5 and Theorem 5.4, since $\operatorname{Ass}_{R_0}(H^i_{R_+,J}(M)_n)$ is finite for all n integers. □

If the sequence $\{\operatorname{Ass}_{R_0}(H^i_{R_+,J}(M))_n\}_{n\in\mathbb{Z}}$ satisfies the assertion in Theorem 5.6, we say it is *asymptotically stable* for $n \to -\infty$.

Acknowledgments. The first author thanks Sathya Sai Baba for guidance.

REFERENCES

1. M. Aghapournahr, Kh. Ahmadi-Amolie and M.Y. Sadeghi, *The concept of (I, J)-Cohen-Macaulay modules*, J. Alg. Syst., accepted.

2. M. Brodmann and R.Y. Sharp, *Local cohomology*: *An algebraic introduction with geometric applications*, Cambridge University Press, Cambridge, UK, 1998.

3. M.P. Brodmann and M. Hellus, *Cohomological patterns of coherent sheaves over projective schemes*, J. Pure Appl. Alg. **172** (2002), 165–182.

4. Lizhong Chu, *Top local cohomology modules with respect to a pair of ideals*, Proc. Amer. Math. Soc. **139** (2011), 777–782.

5. Lizhong Chu and Wing Wang, *Some results on local cohomology modules defined by a pair of ideals*, J. Math. Kyoto Univ. **49** (2009), 193–200.

6. H. Herrmann, S. Ikeda and U. Orbanz, *Equimultiplicity and blowing up*, *An algebraic study*, Springer-Verlag, Berlin, 1988.

7. M. Jahangiri and H. Zakeri, *Local cohomology modules with respect to an ideal containing the irrelevant ideal*, J. Pure Appl. Alg. **213** (2009), 573–581.

8. C. Lim, *Graded local cohomology and its associated primes*, Ph.D. dissertation, Michigan State University, Ann Arbor, 2002.

9. L. Melkersson, *On asymptotic stability for sets of prime ideals connected with the powers of an ideal*, Math. Proc. Cambr. Philos. Soc. **107** (1990), 267–271.

10. Christel Rotthaus and Liana M. Sega, *Some properties of graded local cohomology modules*, J. Algebra **283** (2005), 232–247.

11. Ryo Takahashi, Yuji Yoshino and Takeshi Yoshizawa, *Local cohomology based on a nonclosed support defined by a pair of ideals*, J. Pure Appl. Alg. **213** (2009), 582–600.

Universidade Federal de Maranhão-UFM, Bacharelado em Ciência e Tecnologia, São Luis, Brazil
Email address: **apoliano27@gmail.com**

University of São Paulo, Department of Mathematics, Institute of Mathematics and Computer Science, ICMC, São Paulo, Brazil
Email address: **vhjperez@icmc.usp.br**