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# BOIJ-SÖDERBERG AND VERONESE DECOMPOSITIONS

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ABSTRACT. Boij-Söderberg theory has had a dramatic impact on commutative algebra. We determine explicit Boij-Söderberg coefficients for ideals with linear resolutions and illustrate how these arise from the usual Eliahou-Kervaire computations for Borel ideals. In addition, we explore a new numerical decomposition for resolutions based on a row-by-row approach; here, the coefficients of the Betti diagrams are not necessarily positive. Finally, we demonstrate how the Boij-Söderberg decomposition of an arbitrary homogeneous ideal with a pure resolution changes when multiplying the ideal by a homogeneous polynomial.

**1. Introduction.** Boij-Söderberg theory deals with the question: Given an ideal $I$, how can one write the Betti table of $I$ as a positive componentwise linear combination of pure Betti tables of Cohen-Macaulay modules? See the survey paper of Fløystad [**8**] for an introduction to the seminal work in this area of Boij and Söderberg [**2, 3**], Eisenbud and Schreyer [**6**], and Eisenbud, Fløystad and Weyman [**5**]. The purpose of this paper is to study the question from a more combinatorial perspective, that is, given an ideal, when can we easily (i.e., without recourse to a recursive algorithm) write down a Boij-Söderberg decomposition of its Betti table?

In the present paper, we consider the following question. If two modules have essentially identical Betti tables, how are their Boij-Söderberg decompositions related? For example, let $I$ be a homogeneous ideal. Then, $I$ is the first syzygy module of $S/I$; thus, their Betti tables differ only by the introduction of an extra row on the top and a column

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on the left, with a one in the upper left corner and zeroes elsewhere. Similarly, let $\mu$ be a homogeneous $k$-form. Then, Betti tables of $S/I$ and $S/\mu I$ differ only by the introduction of $k$ rows of zeroes between the generator and the other Betti numbers. In both cases, since the content of the Betti tables is essentially unchanged, we expect that the Boij-Söderberg decomposition should evolve in a controlled manner. As far as we can tell, the following questions have not been systematically addressed in the literature:

**Question 1.1.** *Let $I$ be a homogeneous ideal. What is the relationship between the Boij-Söderberg coefficients of $I$ and $S/I$?*

**Question 1.2.** *Let $I$ be a homogeneous ideal and $\mu$ a $k$-form. How do the Boij-Söderberg coefficients of $S/\mu I$ evolve as $k$ changes?*

We answer both questions for ideals with pure resolutions. In principle, these answers may be extended to arbitrary ideals by summing over the pure resolutions in a Boij-Söderberg decomposition. However, these questions necessarily involve non-Cohen-Macaulay modules, for which Boij-Söderberg decompositions are not unique, and the decomposition found in this manner is different from the canonical one involving an increasing chain of degree-sequences [**3**]. Consequently, we do not generalize our results from the pure case.

Sections 1 and 2 are devoted to introductory notions and examples of Boij-Söderberg theory. In Section 3, we determine the Boij-Söderberg coefficients for modules $S/I$ with monomial ideals $I$ having linear resolutions. Specifically, we focus on the information given in the minimal generating set of a Borel ideal, which is used to compute the Eliahou-Kervaire resolution of a Borel ideal. In recent independent work, Cook [**4**] also computes these Boij-Söderberg coefficients although his framework is very different from ours. While much of our work in this section is actually a subcase of work in later sections, we have chosen to separate it from the more general work (which is necessarily heavier on notation), in the interest of greater readability. Indeed, the constructions are best understood with basic examples, which we give in the next section.

In Section 4, we generalize our earlier techniques, describing a numerical decomposition of the Betti diagram of $S/I$, where $I$ is

an arbitrary homogeneous ideal. This method develops a row-by-row decomposition of the Betti diagram, using the Betti diagrams of Veronese ideals. While the coefficients in this decomposition are rarely all nonnegative (while Boij-Söderberg coefficients, by definition, are nonnegative), we can use this approach to prove the surprising result that ideals that do not have a linear resolution must contain at least one nonlinear Betti diagram in their Boij-Söderberg decomposition (Corollary 4.5).

Finally, in Sections 5 and 6, we give our most general results. We determine the Boij-Söderberg decomposition of $S/\mu M$, where $M$ is a Veronese ideal. We then generalize this result to modules $S/I$, where $I$ is a Cohen-Macaulay ideal with pure resolution. Finally, we characterize the Boij-Söderberg coefficients in the case where $I$ has a pure resolution but is not Cohen-Macaulay.

**2. Motivating examples.** This section provides two examples that motivate the results presented in the rest of the paper and that demonstrate the main ideas. In our examples, we often substitute the variables $\{a, b, c, \ldots\}$ for $\{x_1, x_2, x_3, \ldots\}$ in the interest of readability.

**Example 2.1.** For an introduction of the material in Sections 3 and 4, we begin by investigating an ideal that can be resolved fairly easily. Let $K$ be a field, and let $I \subseteq K[a, b, c, d]$ be the smallest Borel ideal containing the monomial $ac^2d$. Thus, the unique minimal monomial generating set of $I$ is $\{a^4, a^3b, a^3c, a^3d, a^2b^2, a^2bc, a^2bd, a^2c^2, a^2cd, ab^3, ab^2c, ab^2d, abc^2, abcd, ac^3, ac^2d\}$.

Other than a '1' in the upper-left corner, the Betti table of $S/I$ consists of a single nonzero row: 16 33 24 6. Our first goal is to write the Betti table of $S/I$ as a convex combination of the Betti tables of $(a)^4, (a, b)^4, (a, b, c)^4$, and $(a, b, c, d)^4$, that is, we want nonnegative scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that

$$\begin{aligned}\langle 16, 33, 24, 6\rangle = \lambda_1\langle 1, 0, 0, 0\rangle + \lambda_2\langle 5, 4, 0, 0\rangle \\ + \lambda_3\langle 15, 24, 10, 0\rangle + \lambda_4\langle 35, 84, 70, 20\rangle,\end{aligned}$$

as vectors in $\mathbb{R}^4$; note that the above vectors are the nonzero rows of the Betti tables for the associated ideals. Furthermore, as each of the Betti tables has a '1' in the upper left, we must further require that $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$.

Our first observation is that it is easier to work not with the Betti numbers of the ideal, but rather a different invariant. For a Borel ideal $J$, recall that $w_i(J)$ counts the number of monomials in the minimal monomial generating set of $J$ whose largest variable has index $i$. For $I$ as above, we have $w_1(I) = 1$, $w_2(I) = 3$, $w_3(I) = 6$ and $w_4(I) = 6$. (Here, for example, $w_3(I)$ is the number of minimal monomial generators of $I$ whose largest variable is $c$.) We place these invariants together in a vector: $\langle 1,\ 3,\ 6,\ 6\rangle$.

With Borel ideals, the invariants $w_i$ give rise to the Betti numbers via an invertible linear transformation (Proposition 3.6). Also, the $w_i$ for ideals of the form $J = (x_1, x_2, \ldots, x_j)^d$ can be easily calculated: $w_i(J)$ is simply the number of degree-$(d-1)$ monomials in the variables $x_1, x_2, \ldots x_i$, which is $\binom{d+i-2}{i-1}$. This simplifies the search for the scalars $\lambda_i$: replacing the Betti numbers with the $w_i$, we now wish to solve the following with nonnegative $\lambda_i$s:

$$\begin{aligned}\langle 1,3,6,6\rangle = \lambda_1\langle 1,0,0,0\rangle + \lambda_2\langle 1,4,0,0\rangle \\ + \lambda_3\langle 1,4,10,0\rangle + \lambda_4\langle 1,4,10,20\rangle.\end{aligned}$$

Because all first components are '1,' any solution to the above will satisfy $\lambda_1+\lambda_2+\lambda_3+\lambda_4 = 1$. Moreover, since the nonzero entries of the above vectors are identical, we have an easy solution: we are forced to set $\lambda_4 = 6/20$, and thus, $\lambda_3$ must equal $6/10 - 6/20$, and so on. Thus, our unique convex combination is obtained as:

$$\lambda_1 = \frac{1}{4}, \qquad \lambda_2 = \frac{3}{20}, \qquad \lambda_3 = \frac{3}{10}, \qquad \lambda_4 = \frac{3}{10}.$$

**Example 2.2.** The material of Section 5 is motivated through consideration of $I = (a,b,c,d)^2$. The Betti table of $S/I$ is

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| total: | 1 | 10 | 20 | 15 | 4 |
| 0: | 1 | . | . | . | . |
| 1: | . | 10 | 20 | 15 | 4 |

Since this is a pure resolution, and $S/I$ is Cohen-Macaulay, its Boij-Söderberg decomposition is trivial: that is, one times itself.

Continuing with this example, let $\mu$ be a linear form, and set $J = \mu I$. Then, the resolution of $S/J$ is obtained from the resolution of $S/I$ by

incrementing the degree of every syzygy, so the Betti table of $S/J$ is:

$$\begin{array}{rccccc} & 0 & 1 & 2 & 3 & 4 \\ \text{total:} & 1 & 10 & 20 & 15 & 4 \\ 0: & 1 & . & . & . & . \\ 1: & . & . & . & . & . \\ 2: & . & 10 & 20 & 15 & 4 \end{array}$$

This is still a pure diagram, but $S/J$ is not Cohen-Macaulay; thus, its Boij-Söderberg decomposition is considerably less obvious. In general, non-Cohen-Macaulay modules have nonunique Boij-Söderberg decompositions, but in this case $S/J$ has pure resolution; therefore, the decomposition remains unique:

$$\frac{2}{5}\begin{pmatrix} 1 & . & . & . & . \\ . & . & . & . & . \\ . & 20 & 45 & 36 & 10 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} 1 & . & . & . \\ . & . & . & . \\ . & 10 & 15 & 6 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 & . & . \\ . & . & . \\ . & 4 & 3 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & . \\ . & . \\ . & 1 \end{pmatrix}.$$

Since the relationship between the Betti diagrams of $S/I$ and $S/J$ is so simple, we expect that these coefficients should have some meaning. In order to find it, we let $\mu$ be a $k$-form, compute the Boij-Söderberg decomposition of $S/\mu I$ and clear denominators to obtain relationships of the form:

$$\begin{aligned} \lambda_* B(S/\mu I) = {} & \lambda_4 B(S/(a,b,c,d)^{k+2}) + \lambda_3 B(S/(a,b,c)^{k+2}) \\ & + \lambda_2 B(S/(a,b)^{k+2}) + \lambda_1 B(S/(a)^{k+2}), \end{aligned}$$

where $B(S/J)$ is the Betti table of $S/J$, and the coefficients are as follows:

| $k$ | $\lambda_*$ | $\lambda_4$ | $\lambda_3$ | $\lambda_2$ | $\lambda_1$ |
|---|---|---|---|---|---|
| 0 | 24 | 24 | 0 | 0 | 0 |
| 1 | 60 | 24 | 6 | 10 | 20 |
| 2 | 120 | 24 | 12 | 24 | 60 |
| 3 | 210 | 24 | 18 | 42 | 126 |
| $k$ | $(k+2)(k+3)(k+4)$ | $(2)(3)(4)$ | $(2)(3)(k)$ | $(2)(k)(k+4)$ | $(k)(k+3)(k+4)$ |

In Section 5, we prove that what we see in Example 2.2 is a true pattern. Let $I$ be a Veronese ideal of codimension $c$ and $\mu$ a $k$-form. Then, after clearing denominators, for all $j \leq c$, the coefficient $\lambda_j$ on the codimension $j$ term in the Boij-Söderberg decomposition of $S/\mu I$

is a degree $(c-j)$ polynomial in $k$, with simple integer factorization as in the table above. We also show how these coefficients arise from standard combinatorial invariants of $S$ and $I$.

Additionally, we prove that, if $I$ is an arbitrary Cohen-Macaulay ideal with pure resolution and $\mu$ is a $k$-form, then the Boij-Söderberg decomposition of $S/\mu I$ obeys a similar pattern, namely, if $I$ has codimension $c$, then the coefficient on the codimension $j$ term is a polynomial of degree $c-j$, with integer factorization involving the degrees of the syzygies as in Example 2.2. Unlike in the Veronese case, our proof of this more general theorem is purely computational. We hope that these coefficients will turn out to measure some important invariant of the ideal.

**3. Boij-Söderberg decompositions of equigenerated Borel ideals.** For the duration of this section, let $I \subseteq K[x_1, x_2, \ldots, x_n]$ be an ideal with linear resolution, generated in degree $d$. Let $\mathfrak{m} = (x_1, \ldots, x_n)$. The results in Sections 5 and 6 subsume the material in this section. However, this section is included to demonstrate the motivation for Sections 5 and 6, which would not be clear without this treatment. In recent independent work, Cook [**4**] obtains results very similar to those in this section but with a much different approach and perspective, and Nagel and Sturgeon prove related theorems in [**10**].

Let $B_I(t)$ be the generating function of the Betti numbers of $I$, that is,

$$B_I(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \cdots + \beta_{n-1} t^{n-1}.$$

Initially, we restrict to Borel ideals. If $m$ is a monomial, write $\max(m)$ for the largest index of a variable dividing $m$. For each $i$, let $w_i(I)$ be the number of minimal generators $m$ of a Borel ideal $I$ with $\max(m) = i$. When $I$ is clear from the context, we will sometimes write $w_i(I)$ as $w_i$. Let $W_I(t)$ denote the generating function for the $w_i(I)$:

$$W_I(t) = w_1 + w_2 t + w_3 t^2 + \cdots + w_n t^{n-1}.$$

Borel ideals are one of the few classes of ideals for which an explicit minimal free resolution is known, and the $w_i$ determine the Betti numbers.

**Theorem 3.1. [7, 12]** *Let $I$ be a Borel ideal. Then*:

(i) *The minimal resolution of $I$ (viewed as a module) has basis consisting of Eliahou-Kervaire symbols, which are pairs $(m, \alpha)$ such that $m$ is a minimal monomial generator of $I$ and $\alpha$ is a squarefree monomial with $\max(\alpha) < \max(m)$. The symbol $(m, \alpha)$ has homological degree $\deg(\alpha)$ and internal degree $\deg(m\alpha)$.*

(ii) *The minimal resolution of the quotient $S/I$ has basis consisting of 1 and the Eliahou-Kervaire symbols. The numeral 1 has both homological and internal degree 0, while the Eliahou-Kervaire symbol $(m, \alpha)$ has homological degree $1 + \deg(\alpha)$ and internal degree $\deg(m\alpha)$.*

If we view the Borel ideal $I$ as a module, Theorem 3.1 (i) is the key to its Boij-Söderberg decomposition. If we sort the basis elements $(m, \alpha)$ by the generating monomial $m$, we see that, if $\max(m) = i$, $m$ appears in exactly $\binom{i}{j}$ Eliahou-Kervaire symbols of homological degree $j$. Now, recall that the Koszul complex resolving the (Cohen-Macaulay) module $S/(x_1, \ldots, x_i)$ also has Betti number $\binom{i}{j}$ in homological degree $j$. Thus, each generator contributes a copy of the appropriate Koszul complex to the Betti table of $I$, and the $i$th Koszul complex appears $w_i$ times. This proves the following.

**Theorem 3.2.** *Let $I$ be a Borel ideal. Then the Boij-Söderberg decomposition of $I$ (viewed as a module) is given by $W_I$.*

*More precisely, for each $i$ and $d$, let $K_i(-d)$ be the Betti table of the $i$th Koszul complex, generated in degree $d$. Then*:

(i) *If $I$ is equigenerated in degree $d$, the Boij-Söderberg decomposition of $I$ is given by the $w_i(I)$*:

$$B_I = \sum_i w_i(I) K_i(-d).$$

(ii) *In general, let $w_{i,d}(I)$ count the number of minimal generators $m$ of $I$ with $\max(m) = i$ and $\deg(m) = d$. Then a Boij-Söderberg decomposition of $I$ is given by the $w_{i,d}(I)$*:

$$B_I = \sum_{i,d} w_{i,d}(I) K_i(-d).$$

**Remark 3.3.** We note that the usual proof of the Eliahou-Kervaire resolution, see, for example, [**11,** Chapter 28], involves a mapping cone argument that yields precisely the Koszul complexes discussed above. Thus, in this case, the Boij-Söderberg coefficients actually count something intrinsic to the resolution.

**Remark 3.4.** The Eliahou-Kervaire resolution and the statistics $w_i$ apply to a somewhat larger class of ideals called *stable* ideals, which are defined by a slightly weaker combinatorial condition. Thus, all of our results involving Borel ideals, and their proofs, hold verbatim for stable ideals as well.

With the situation for Borel-fixed ideals (viewed as modules) well understood, we move on to study the quotient $S/I$ for Borel ideals $I$. Here, the situation is considerably more murky, but we will show that the $w_i(I)$ continue to play a key role. The Koszul complexes, however, are replaced by Veronese ideals.

**Notation 3.5.** For $i \leq n$, let $V_{i,d} = (x_1, x_2, \ldots, x_i)^d$. In this section, $d$ is fixed and thus redundant in the notation, but later we will allow $d$ to vary.

First, let $I$ be a Borel ideal with all generators in degree $d$. Since $S/V_{i,d}$ is Cohen-Macaulay and $S/I$ has a pure $d$-linear resolution, the Boij-Söderberg decomposition of $S/I$ is the linear combination

$$B_{S/I}(t) = \lambda_1 B_{S/V_{1,d}}(t) + \lambda_2 B_{S/V_{2,d}}(t) + \cdots + \lambda_n B_{S/V_{n,d}}(t), \tag{3.1}$$

where each $\lambda_i \geq 0$.

The Eliahou-Kervaire resolution of Borel ideals immediately implies the following.

**Proposition 3.6.** *If $I$ is a Borel ideal generated in degree $d$, we have $W_I(t+1) = B_I(t)$.*

Proposition 3.6 is key to the calculation of the Boij-Söderberg decomposition of a Borel ideal: because each $V_{i,d}$ is a Borel ideal

generated in a single degree, equation (3.1) is equivalent to

$$W_I(t) = \lambda_1 W_{V_{1,d}}(t) + \lambda_2 W_{V_{2,d}}(t) + \cdots + \lambda_n W_{V_{n,d}}(t). \tag{3.2}$$

In order to calculate the polynomials $W_{V_{i,d}}$, observe that the number of monomials in $V_{i,d}$ whose largest variable is $x_j$ is simply the number of $(d-1)$-degree monomials in $\{x_1, x_2, \ldots, x_j\}$, which is $\binom{\binom{j}{d-1}}{} = \binom{(d-1)+j-1}{d-1} = \binom{d+j-2}{j-1}$. Writing $w_j(V_{i,d})$ for this count, we have

$$W_{V_{i,d}}(t) = \sum_{j=1}^{i} w_j(V_{i,d}) t^{j-1} = \sum_{j=1}^{i} \binom{d+j-2}{j-1} t^{j-1}.$$

If $i < j$, we say $w_j(V_{i,d}) = 0$.

**Proposition 3.7.** *Let $I$ be a Borel ideal generated in degree $d$, and define $w_i$ as above. Then, the Boij-Söderberg decomposition of $S/I$ has coefficients $\{\lambda_i\}$ given by*

$$\lambda_n = \frac{w_n(I)}{w_n(\mathfrak{m}^d)} \quad \text{and} \quad \lambda_i = \frac{w_i(I)}{w_i(\mathfrak{m}^d)} - \frac{w_{i+1}(I)}{w_{i+1}(\mathfrak{m}^d)} \quad \text{for } i < n.$$

*Equivalently,*

$$\lambda_n = \frac{w_n(I)}{\binom{d+n-2}{n-1}} \quad \text{and} \quad \lambda_i = \frac{w_i(I)}{\binom{d+i-2}{i-1}} - \frac{w_{i+1}}{\binom{d+i-1}{i}} \quad \text{for } i < n.$$

*Proof.* Fix $j-1$. By definition, $t^{j-1}$ only appears in $W_{V_{i,d}}(t)$ for $i \geq j$, since the coefficient of $t^{j-1}$ counts the number of monomials in $V_{i,d}$ whose largest variable is $x_j$. Moreover, in each such $W_{V_{i,d}}$, the coefficient of $t^{j-1}$ is $\binom{d+j-2}{j-1} = w_j(V_{i,d}) = w_j(\mathfrak{m}^d)$. Substituting our expressions for each $\lambda_i$ into equation (3.2) shows that the coefficient of $t^{j-1}$ on the right-hand side is:

$$\begin{aligned} &w_j(\mathfrak{m}^d)\left[\left(\tfrac{w_j(I)}{w_j(\mathfrak{m}^d)} - \tfrac{w_{j+1}(I)}{w_{j+1}(\mathfrak{m}^d)}\right) + \left(\tfrac{w_{j+1}(I)}{w_{j+1}(\mathfrak{m}^d)} - \tfrac{w_{j+2}(I)}{w_{j+2}(\mathfrak{m}^d)}\right) + \cdots + \tfrac{w_n(I)}{w_n(\mathfrak{m}^d)}\right] \\ &\quad = w_j(\mathfrak{m}^d)\left[\tfrac{w_j(I)}{w_j(\mathfrak{m}^d)}\right] = w_j(I), \end{aligned}$$

which is the coefficient of $t^{j-1}$ on the left-hand side. □

**Example 3.8.** Let $I$ be the smallest Borel ideal containing the monomial $x_1x_2\cdots x_n$. Then, each $w_i(I)$ is given as the $i$th entry of the $n$th row of *Catalan's triangle* (where we begin counting rows and entries at 1 rather than 0):

$$w_i(I) = \frac{(n+i-2)!(n-i+1)}{(i-1)!n!}.$$

Then, $d = n$, and a straightforward calculation shows that $\lambda_i = 1/n$ for all $i$.

**Remark 3.9.** Example 3.8 shows that there are ideals at the orthocenters of at least some of the Boij-Söderberg surfaces. We speculate that the ideals of Example 3.8 may be central to the theory in other ways as well. However, these ideals have received little attention until very recently. In [**9**], we proved that their Betti numbers have a combinatorial interpretation, counting objects in rigid geometry called *pointed pseudotriangulations*.

We can also adapt Proposition 3.7 to give a formula for the coefficients $\{\lambda_i\}$ in the case when a Borel ideal is multiplied by a $k$-form $\mu$.

**Proposition 3.10.** *Let $I$ be as in Proposition* 3.7*, and let $\mu$ be a $k$-form. Then, the Boij-Söderberg coefficients of $S/\mu I$ are*:

$$\lambda_n = \frac{w_n(I)}{\binom{d+n+k-2}{n+k-1}} \quad \textit{and} \quad \lambda_i = \frac{w_i(I)}{\binom{d+i+k-2}{i+k-1}} - \frac{w_{i+1}(I)}{\binom{d+i+k-1}{i+k}} \textit{ for } i < n.$$

*Proof.* The only difference between this proposition and Proposition 3.7 is that the constituent ideals are $V_{i,d+k} = (x_1, x_2, \ldots, x_i)^{d+k}$. The rest of the proof is analogous to that of Proposition 3.7. □

The following observation plays an important role in the rest of the paper.

**Remark 3.11.** The computation of the coefficients $\lambda_i$ still works for any ideal $I$ with a linear resolution, where we define the constants $w_i$ by the relation $W_I(t) = B_I(t-1)$. Note that there exists a Borel ideal $B$ with the same graded Betti numbers as $I$. For example, take $B$ to

be the reverse-lex gin of $I$. Because $B$ and $I$ have the same Hilbert function, the lowest degree of a generator is the same for each, and, by [**1,** Theorem 2.4], the regularity of $B$ and $I$ are the same.

**4. Veronese decompositions.** We generalize our results for ideals with a linear resolution to any homogeneous ideal, decomposing the Betti diagram using Betti diagrams of powers of monomial prime ideals. We no longer expect the $\lambda_i$ to be positive and yield a Boij-Söderberg decomposition. However, we can give a new decomposition that has some interesting properties, and we derive a corollary about Boij-Söderberg decompositions of ideals that do not have a linear resolution.

Let $I$ be a homogeneous ideal. We decompose the Betti diagram of $S/I$ row-by-row as a $\mathbb{Q}$-linear combination of the Betti diagrams of ideals of the form $V_{i,j+1} = (x_1, \ldots, x_i)^{j+1}$, allowing negative coefficients. For each row $j$ of the Betti diagram of $S/I$, define constants $w_{i,j}(I)$ by the formula $W_I(t) = B_{S/I}(t-1)$, as in Remark 3.11, using the Betti numbers in row $j$ to form the polynomial $B_{S/I}(t)$ for row $j$. As in Proposition 3.7, for each row $j$ of the Betti diagram of $S/I$ (in which any minimal generators are of degree $j+1$), we define

$$\lambda_{i,j} = \frac{w_{i,j}(I)}{\binom{j+1+i-2}{i-1}} - \frac{w_{i+1,j}(I)}{\binom{j+1+i-1}{i}} = \frac{w_{i,j}(I)}{\binom{j+i-1}{i-1}} - \frac{w_{i+1,j}(I)}{\binom{j+i}{i}}$$

for $1 \le i \le n$ (noting that $w_{n+1,j}(I)$ is always zero).

**Example 4.1.** Let $I \subset S$ be the monomial ideal $(a^2, b^2, abc)$. The Betti diagram of $S/I$ is:

| | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| total: | 1 | 3 | 3 | 1 |
| 0: | 1 | . | . | . |
| 1: | . | 2 | . | . |
| 2: | . | 1 | 3 | 1 |

For row 1, we have $w_{1,1} = 2 = \lambda_{1,1}$. The $w$-values in row 2 are obtained from the polynomial $1+3(t-1)+(t-1)^2 = -1+t+t^2$, giving $w_{1,2} = -1$, $w_{2,2} = 1$ and $w_{3,2} = 1$. Consequently, $\lambda_{1,2} = -4/3$, $\lambda_{2,2} = 1/6$ and $\lambda_{3,2} = 1/6$.

Note that the Betti diagram of $S/I$ is

$$B(S/I) = \underbrace{2 \cdot B(S/(x_1)^2)}_{\text{row 1}} \underbrace{-\frac{4}{3} \cdot B(S/(x_1)^3) + \frac{1}{6} \cdot B(S/(x_1,x_2)^3) + \frac{1}{6} \cdot B(S/(x_1,x_2,x_3)^3)}_{\text{row 2}}.$$

**Theorem 4.2.** *The process described above gives a $\mathbb{Q}$-linear decomposition of the Betti diagram of $S/I$ in terms of the Betti diagrams of the Veronese quotients $S/V_{i,s}$ (possibly with negative coefficients),*

$$B(S/I) = \sum_{i,s} \lambda_{i,s} B(S/V_{i,s}).$$

*Proof.* It is immediate that each row $j > 0$ of the Betti diagram of $S/I$ can be constructed in this way. Note that, for each $i$, the projective dimension of $S/(x_1,\dots,x_i)^{j+1}$ is $i$, meaning that finding the $\lambda_{i,j}$ amounts to solving a triangular system of linear equations in which the coefficient matrix has all nonzero entries on the diagonal. The formulae for the $\lambda_{i,j}$ are as in the Borel case, noted in Remark 3.11.

The only question is whether these values of $\lambda_{i,j}$ yield a 1 in the upper left corner of the Betti diagram, that is, do the $\lambda_{i,j}$ sum to 1? For each row $j$, the sum $\lambda_{1,j} + \cdots + \lambda_{n,j}$ telescopes to $w_{1,j}$. Each $w_{1,j}$ is the alternating sum of the Betti numbers in row $j$ of the Betti diagram of $S/I$. Thus, if $\beta_i$ is the $i$th total Betti number of $S/I$, then

$$\sum_{i,j\geq 1} \lambda_{i,j} = \sum_{i\geq 1} (-1)^{i+1} \beta_i = 1$$

because the alternating sum of all total Betti numbers of $S/I$ is 1. □

**Definition 4.3.** We call the decomposition of Theorem 4.2 the *Veronese decomposition* of the Betti diagram of $S/I$ since each of the ideals $V_{i,s} = (x_1,\dots,x_i)^s$ are Veronese ideals.

The following is a consequence of the proof of Theorem 4.2, following from the telescoping sum $\lambda_{1,j} + \cdots + \lambda_{n,j}$.

**Corollary 4.4.** *For each row $j$, the sum of the $\lambda_{i,j}$ is an integer.*

We conclude by using Veronese decompositions to prove a result about Boij-Söderberg decompositions.

**Corollary 4.5.** *If $I$ does not have a linear resolution, in particular, if $I$ is not equigenerated, then any Boij-Söderberg decomposition of the Betti diagram of $S/I$ contains at least one nonlinear diagram.*

*Proof.* By Corollary 4.4, for a fixed row $j$, the sum of the $\lambda_{i,j}$ is an integer, and the sum of all $\lambda$ is one. Thus, if $S/I$ does not have a linear resolution, there must be at least one negative $\lambda_{i,j}$ because $S/I$ has multiple nonzero rows (in addition to row 0) in its Betti diagram. □

**5. Boij-Söderberg decompositions and multiplication by homogeneous forms.** This section addresses the question, "what happens to the Boij-Söderberg decomposition of $S/I$ when $I$ is multiplied by a homogeneous form?" In particular, we prove the pattern observed in Example 2.2. Our primary tools are the Veronese decomposition and the $W$-polynomial of $I$. In order to allow readers to skip earlier sections, we provide proofs of some results from Section 3 in the new notation.

We begin by focusing on one of the few classes of modules whose Boij-Söderberg decompositions can be described explicitly, namely the Borel ideals.

**Notation 5.1.** From this point forward, we treat Betti tables as bigraded Poincaré series; for a module $M$, the series is

$$P_M(t,u) = \sum_{i,j} B_{i,j}(M) t^i u^j.$$

In particular, the variable $t$ counts homological degree, and the variable $u$ counts internal degree. For example, the quotient $S/(a^2, b^2, ab)$, which has Betti table

| | 0 | 1 | 2 |
|---|---|---|---|
| total: | 1 | 3 | 2 |
| 0: | 1 | . | . |
| 1: | . | 3 | 2 |

has Poincaré series $1+3tu^2+2t^2u^3$. In particular, moving one row down in a Betti diagram corresponds to multiplication by $u$, while moving one column to the right corresponds to multiplication not by $t$ but by $tu$.

**Notation 5.2.** Fix an increasing degree sequence $\mathbf{d} = (d_0, d_1, \ldots, d_n)$. Then the (normalized) *pure Betti diagram* associated to $\mathbf{d}$ is the diagram arising from the Herzog-Kühl relations,

$$P_{\mathbf{d}}(t,u) = \sum_{i=0}^{d} \frac{\prod_{\ell\neq 0} |d_\ell - d_0|}{\prod_{\ell\neq i} |d_\ell - d_i|} t^i u^{d_i}.$$

We set $V_{r,d}$ equal to the Betti diagram of the Veronese quotient $S/(x_1,\ldots,x_r)^d$, namely, $V_{r,d} = P_{(0,d,d+1,\ldots,d+r-1)}$.

Initially, we restrict our attention to the resolution of the ideal rather than the quotient. Let $I$ be a Borel ideal. The basis Eliahou-Kervaire symbols may be sorted by their monomial generators; we see that a generator $m$ with degree $d$ and $\max(m) = r$ contributes $\binom{r-1}{i}$ basis elements of homological degree $i$ and internal degree $d+i$ for every $i$, and no basis elements of any other degree. Thus, the Poincaré series of $I$ is

$$\begin{aligned}
P_I(t,u) &= \sum_m \left( \sum_{i=0}^{\max(m)-1} \binom{\max(m)-1}{i} t^i u^{i+\deg(m)} \right) \\
&= \sum_m u^{\deg(m)} (1+tu)^{\max(m)-1} \\
&= \sum_{r=1}^{\infty} \sum_{\max(m)=r} u^{\deg(m)} (1+tu)^{r-1} \\
&= \sum_{r=1}^{\infty} \sum_{d=0}^{\infty} \sum_{\substack{\max(m)=r \\ \deg(m)=d}} u^d (1+tu)^{r-1} \\
&= \sum_{r=1}^{\infty} \sum_{d=0}^{\infty} |\{m : \max(m)=r, \deg(m)=d\}| u^d (1+tu)^{r-1} \\
&= \sum_{r=1}^{\infty} \sum_{d=0}^{\infty} w_{r,d}(I) u^d (1+tu)^{r-1}.
\end{aligned}$$

The Boij-Söderberg decomposition of a Borel ideal is now almost immediate.

**Theorem 5.3.** *Let $I$ be a Borel ideal. Then the Boij-Söderberg decomposition of $I$ is given by its $W$-polynomial, that is,*

$$P_I(t,u) = \sum_{r,d} w_{r,d}(I) P_{(d,d+1,\dots,d+r-1)}(t,u).$$

*Proof.* We need only observe that $P_{(d,d+1,\dots,d+r-1)}(t,u) = u^d(1+tu)^{r-1}$. □

The Boij-Söderberg decomposition of the quotient $S/I$ is more interesting, but since the Betti numbers are closely related, we expect that this should also have something to do with the $W$-polynomial of $I$. Indeed, we have the following generalization of Proposition 3.7.

**Theorem 5.4.** *Let $I$ be a Borel ideal, and let*

$$P_{S/I} = \sum_{r,d} \nu_{r,d} V_{r,d}$$

*be the Veronese decomposition of its quotient. Then we have*

$$\nu_{r,d} = \frac{w_{r,d}(I)}{w_{r,d}(\mathfrak{m}^d)} - \frac{w_{r+1,d}(I)}{w_{r+1,d}(\mathfrak{m}^d)}.$$

(*Note that* $w_{r,d}(\mathfrak{m}^d) = \binom{r+d-2}{d-1}$.)

*If $I$ is equigenerated, this is also its Boij-Söderberg decomposition.*

Expanding to the case of arbitrary homogeneous ideals, we note that the relationship between the Poincaré series and the $W$-polynomial is invertible. We take this as the definition.

**Definition 5.5.** Let $I$ be a homogeneous ideal, with Betti table $P_I(t,u)$. Then the *$W$-polynomial of $I$* is

$$W_I(t,u) = \sum w_{r,d}(I) t^r u^d = P_I\left(\frac{t-1}{u}, u\right).$$

Now, let $M = M_{r,d} = (x_1, \ldots, x_r)^d$ be a Veronese ideal and $\mu$ a homogeneous $k$-form. We compute the Boij-Söderberg decomposition of $S/\mu M$.

Observe that $P_{\mu M} = u^k P_M$; thus, $W_{r,d}(\mu M) = u^k W_{r,d}(M)$. Since we know the $W$-polynomial of $\mu M$, we can compute its Veronese decomposition as above. We obtain the following:

**Theorem 5.6.** *Let $M = M_{n,d}$ be a Veronese ideal and $\mu$ a $k$-form. Then the Boij-Söderberg decomposition of $S/\mu M$ is*

$$P_{S/\mu M} = \sum_{i=1}^{n} \nu_{i,d+k} V_{i,d+k},$$

*where*

$$\nu_{n,d+k} = \frac{(d)(d+1)\cdots(d+n-2)}{(k+d)(k+d+1)\cdots(k+d+n-2)}$$
$$\nu_{i,d+k} = \frac{(d)(d+1)\cdots(d+i-2)(k)}{(k+d)(k+d+1)\cdots(k+d+i-1)} \quad \text{for } i \neq n.$$

Our proof of Theorem 5.6 is an unenlightening computation, so we delay for some remarks and speculation.

**Remark 5.7.** If we clear the denominators, we obtain

$$(k+d)(k+d+1)\cdots(k+d+n-2)P_{S/\mu M} = \sum_{i=1}^{n} \widetilde{\nu}_{i,d+k} V_{i,d+k},$$

where

$$\widetilde{\nu}_{n,d+k} = (d)(d+1)\cdots(d+n-2)$$
$$\widetilde{\nu}_{i,d+k} = (d)(d+1)\cdots(d+i-2)(k)(k+d+i)(k+d+i+1)\cdots (k+d+n-2) \quad \text{for } i \neq n.$$

Here the coefficient on $P_{S/\mu M}$ is a degree $n-1$ polynomial in $k$, and the coefficient $\widetilde{\nu}_{i,d+k}$ on the codimension-$i$ diagram is a degree $(n-i)-1$ polynomial in $k$. (In particular, the coefficient on the top-dimension diagram $V_{n,k+d}$ does not depend upon $k$.)

**Remark 5.8.** The factors in the various coefficients are so simple, and so strongly reminiscent of the factors in the Herzog-Kühl relationships, that we struggle to believe this is a mere numerical miracle. We speculate that there is some hidden structure underlying everything.

**Remark 5.9.** As $k$ tends to infinity, the coefficient $\nu_{1,k+d}$ approaches 1, and all other coefficients approach 0. This appears to correspond to the fact that, as $k$ grows, $S/\mu M$ looks more and more like the quotient by an (unmixed, height one) $k$-form, and less and less like the (unmixed, height $n$) Veronese $S/M$. We propose that the vector $(\nu_{1,k+d}, \ldots, \nu_{n,k+d})$ might serve to simultaneously refine the ideas of height and projective dimension, and we wonder if there is some related algebraic structure whose components are in the same proportion as the entries of this vector.

*Proof of Theorem* 5.6. We observe that $w_{i,k+d}(\mu M) = w_{i,d}(M) = \binom{i+d-2}{d-1}$. This allows computation of the Veronese decomposition of $S/\mu M$ using Theorem 5.4. We obtain:

$$\begin{aligned}\nu_{n,d} &= \frac{w_{n,d}(M)}{\binom{k+n+d-2}{k+d-1}} = \frac{\binom{n+d-2}{d-1}}{\binom{k+n+d-2}{k+d-1}} \\ &= \frac{(d)(d+1)\cdots(d+n-2)}{(k+d)(k+d+1)\cdots(k+d+n-2)},\end{aligned}$$

and, for $i \neq n$,

$$\begin{aligned}\nu_{i,d} &= \frac{w_{i,d}(M)}{\binom{k+i+d-2}{k+d-1}} - \frac{w_{i+1,d}(M)}{\binom{k+i+d-1}{k+d}} \\ &= \frac{\binom{i+d-2}{d-1}}{\binom{k+i+d-2}{k+d-1}} - \frac{\binom{i+d-1}{d}}{\binom{k+i+d-1}{k+d}} \\ &= \frac{(d)(d+1)(d+2)\cdots(d+i-2)k}{(k+d)(k+d+1)\cdots(k+d+i-1)}.\end{aligned}$$

Since these coefficients are all nonnegative, the Veronese decomposition is the Boij-Söderberg decomposition. □

**6. Pure resolutions.** In this section, we extend Theorem 5.6 to arbitrary Cohen-Macaulay ideals with pure resolution. Theorem 6.1 below generalizes Theorem 5.6, although its proof is less transparent.

Suppose, throughout the section, that $I$ is a Cohen-Macaulay ideal with pure resolution in degrees $(d_0 = 0, d_1, \ldots, d_n)$. Then, $P_{S/I} = P_{(0,d_1,\ldots,d_n)}$; we describe the Boij-Söderberg decomposition of $S/\mu I$.

**Theorem 6.1.** *Let $\mu$ be a $k$-form and $I$ a Cohen-Macaulay ideal with pure resolution supported in degrees $(0, d_1, \ldots, d_n)$. Then, the Boij-Söderberg decomposition of $S/\mu I$ is*

$$P_{S/\mu I} = \sum_{i=1}^{n} \lambda_i P_{(0,d_1+k,\ldots,d_i+k)},$$

*where the coefficients $\lambda_i$ are given by*

$$\lambda_n = \frac{(d_1)\cdots(d_{n-1})}{(d_1+k)\cdots(d_{n-1}+k)},$$
$$\lambda_i = \frac{(d_1)\cdots(d_{i-1})\quad(k)}{(d_1+k)\cdots(d_i+k)} \qquad \text{for } i \neq n.$$

**Remark 6.2.** All of the same remarks as in the Veronese case apply. In particular, the coefficient $\lambda_i$ is a degree $(1-i)$ rational function in $k$, and $\lambda_1$ approaches 1 as $k$ grows large.

**Corollary 6.3.** *Let $J$ be an arbitrary ideal with pure resolution supported in degrees $(0, d_1, \ldots, d_n)$, and let $\mu$ be a $k$-form. Then, after clearing denominators, the Boij-Söderberg decomposition of $S/\mu J$ satisfies*

$$\widetilde{\lambda_*} P_{S/\mu J} = \sum_{i=1}^{n} \widetilde{\lambda_i} P_{(0,d_1+k,\ldots,d_i+k)},$$

*where $\widetilde{\lambda_*}$ is a degree $n-1$ polynomial in $k$, and $\widetilde{\lambda_i}$ is a degree $n-i$ polynomial in $k$ for all $i$.*

*Proof.* Apply Theorem 6.1 to each of the summands in the Boij-Söderberg decomposition of $S/J$, and add. □

*Proof of Theorem* 6.1. Since both Betti tables are pure, it suffices to compute the total Betti numbers of each directly.

Fix a homological degree $i$. Then, from the Herzog-Kühl relations, we have

$$\begin{aligned}\beta_{i,d_i+k}(P_{S/\mu I}) &= \beta_{i,d_i}(P_{S/I}) \\ &= \frac{(d_1)\cdots(d_n)}{d_i\,[(d_i-d_1)\cdots(d_i-d_{i-1})]\,[(d_{i+1}-d_i)\cdots(d_n-d_i)]}.\end{aligned}$$

Meanwhile, the same Betti number of the right-hand side is given by

$$\begin{aligned}&\beta_{i,d_i+k}\left(\sum_{j=1}^{n}\lambda_j P_{(0,d_1+k,\ldots,d_j+k)}\right) \\ &\quad=\sum_{j=i}^{n-1}\frac{(d_1)\cdots(d_{j-1})\ (k)}{(d_i+k)\,[(d_i-d_1)\cdots(d_i-d_{i-1})]\,[(d_{i+1}-d_i)\cdots(d_j-d_i)]} \\ &\qquad+\frac{(d_1)\cdots(d_{n-1})\ (d_n+k)}{(d_i+k)\,[(d_i-d_1)\cdots(d_i-d_{i-1})]\,[(d_{i+1}-d_i)\cdots(d_n-d_i)]}.\end{aligned}$$

After clearing denominators, we must prove the arithmetic identity

$$[(d_1)\cdots(d_n)]\,(d_i+k)=\sum_{j=i}^{n-1}A_j+B_n,$$

where

$$\begin{aligned}A_j &= [(d_1)\cdots(d_{j-1})]\,[(d_{j+1}-d_i)\cdots(d_n-d_i)]\,(d_i)(k) \\ B_n &= [(d_1)\cdots(d_{n-1})]\,(d_n+k)(d_i).\end{aligned}$$

Now, set

$$C_j=[(d_1)\cdots(d_j)]\,[(d_{j+1}-d_i)\cdots(d_n-d_i)]\,(k),$$

and observe that $A_i=C_i$, and $C_j+A_{j+1}=C_{j+1}$, so (inductively)

$$\sum_{j=i}^{n-1}A_j=C_{n-1}.$$

Finally, observe that

$$B_n + C_{n-1} = [(d_1) \cdots (d_{n-1})]\,[(d_n)(d_i + k)]\,,$$

as desired.

□

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