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FACTORING IDEALS AND STABILITY IN INTEGRAL DOMAINS

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ABSTRACT. In an integral domain R, a nonzero ideal is called a *weakly ES-stable ideal* if it can be factored into a product of an invertible ideal and an idempotent ideal of R; and R is called a *weakly ES-stable domain* if every nonzero ideal is a weakly ES-stable ideal. This paper studies the notion of weakly ES-stability in various contexts of integral domains such as Noetherian and Mori domains, valuation and Prüfer domains, pullbacks and more. In particular, we establish strong connections between this notion and well-known stability conditions, namely, Lipman, Sally-Vasconcelos and Eakin-Sathaye stabilities.

1. Introduction. Throughout, all rings are assumed to be integral domains. There are several closely related ideas of stability in the literature. Some of the first were those of L-stable ideals and L-stable domains due to Lipman [**26**]. Recall that an ideal I of a domain R said to be L-stable if the zero cohomology ring

$$R^I := \bigcup_{n\geq 0}(I^n : I^n)$$

of I coincides with its endomorphism ring $(I : I)$; and the domain R is said to be L-stable provided that every ideal is L-stable. Lipman used this notion to characterize Arf rings in the context of one-dimensional Noetherian rings.

Later, Sally and Vasconcelos [**33, 34**] introduced the notion of stable ideals (or SV-stable ideals) as ideals I that are projective in their endomorphism rings. (Note that, in the case of integral domains, this

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is equivalent to saying that I is invertible in its endomorphism ring $(I : I)$). Originally, L-stability and SV-stability were studied only in Noetherian rings. The notion of stability (i.e., SV-stability) was first considered beyond the Noetherian settings in **[1]** by Anderson, Huckaba, and Papick. In **[13]**, Eakin and Sathaye extended the results of Sally and Vasconcelos and defined the so-called ES-stable ideals and ES-stable domains. A nonzero ideal is an ES-stable ideal if $I^2 = JI$ for some invertible subideal J of I, and a domain R is ES-stable if every ideal is ES-stable.

Since then, different kinds of stability, such as local, finite, etc., have been defined and studied. The importance of the notion of stability (in the sense of Lipman and Sally-Vasconcelos) resides in its wide connection to other notions in commutative algebra in Noetherian and non-Noetherian settings, namely, the notions of the 2-generator property, the Warfield duality, Clifford regularity and the Ratliff-Rush closure. We refer the reader to **[27, 28, 29, 30]** for more details.

This paper is concerned with factorization of ideals of an integral domain into products of invertible fractional ideals and idempotent fractional ideals. Our objective is to show that the issue here is closely related to different versions of stability of ideals, which is a relativized version of invertibility. Precisely, we aim at answering the following two questions:

(1) Given an integral domain R, which ideals I of R can be factored as a product of an invertible ideal J and an idempotent ideal E?

(2) Which integral domains have the property that every ideal can be factored as a product of an invertible ideal and an idempotent ideal?

In Section 2, we define the notions of weakly ES-stable ideals and weakly ES-stable domains, and we collect preliminary results that are useful. We also put these notions in the stability perspective (see the diagram in Figure 1).

In Section 3, we prove that the notions of weakly ES-stable domains and Eakin-Sathaye stability coincide in the context of Noetherian domains (Theorem 3.1) and some particular classes of Mori domains (Theorem 3.3).

Section 4 deals with Prüfer and valuation domains. We start with a characterization of weakly ES-stable domain in the context of integrally closed domains. This leads to a new characterization of Prüfer domains, that is, a domain R is Prüfer if and only if R is integrally closed and every finitely generated ideal of R is a weakly ES-stable ideal (Theorem 4.1). Also, we prove that the notion of weakly ES-stable domains coincides with Eakin-Sathaye and Sally-Vasconcelos stabilities for strongly discrete Prüfer domains (Theorem 4.8).

Section 5 deals with pullback constructions in order to provide larger classes of weakly ES-stable domains.

2. General results. We begin this section with the following definition.

Definition 2.1. Let R be an integral domain.

(i) A nonzero ideal I of R is said to be a *weakly ES-stable ideal* if there is an invertible fractional ideal J and an idempotent fractional ideal E of R such that $I = JE$, and R is said to be a *weakly ES-stable domain* if every nonzero ideal of R is a weakly ES-stable ideal.

(ii) A nonzero ideal I of R is said to be an *almost weakly ES-stable ideal* if some power of I is a weakly ES-stable ideal, and R is said to be an *almost weakly ES-stable domain* (respectively, a finitely weakly ES-stable domain) if every ideal (respectively, every finitely generated ideal) of R is almost weakly (respectively, a weakly) ES-stable ideal.

Strongly stable ideals, i.e., I is principal in $(I : I)$, and ES-stable ideals are clearly weakly ES-stable ideals; however, stable ideals are not necessarily weakly ES-stable ideals. Indeed, it is well known that non-divisorial ideals of a valuation domain V are of the form aM where M is the maximal ideal of V and M is idempotent. Such ideals constitute a source of weakly ES-stable ideals that are neither stable nor ES-stable. The diagram in Figure 1 places the notion of weakly ES-stable ideals and weakly ES-stable domains, respectively, in the stability perspective.

The next proposition and remark illustrate the diagram in Figure 1 and show that the class of weakly ES-stable ideals properly stands between the classes of L-stable and ES-stable ideals.

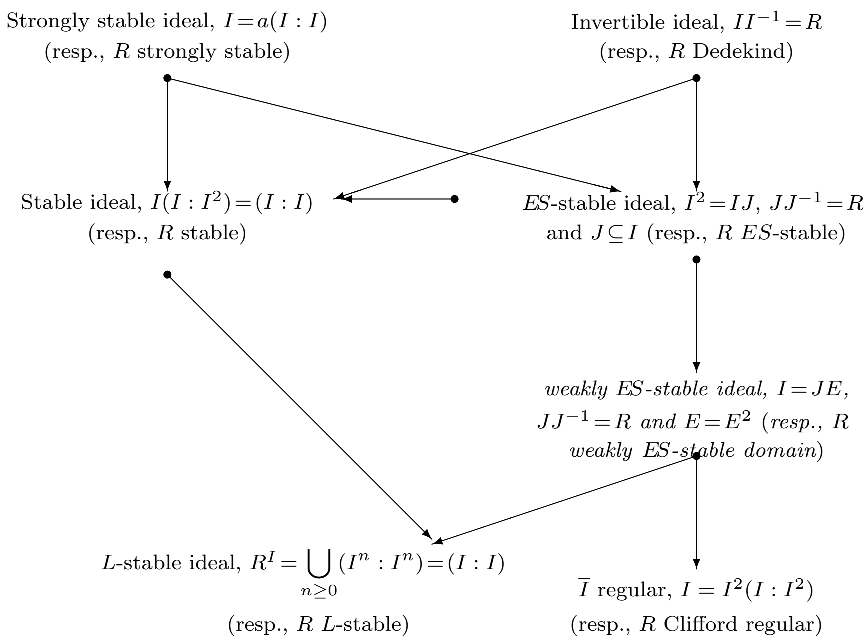

FIGURE 1.

Proposition 2.2. *Let R be an integral domain and I a nonzero ideal of R. Then:*

(i) *If I is a weakly ES-stable ideal, then I is L-stable.*

(ii) *I is a weakly ES-stable ideal if and only if $I^2 = JI$ for some invertible ideal J of R.*

(iii) *I is ES-stable if and only if $I = JE$ where J is invertible, $E = E^2$ and $J \subseteq I \subseteq E$.*

Proof.

(i) Assume that I is a weakly ES-stable ideal, and set $I = JE$ where $JJ^{-1} = R$ and $E = E^2$. If $x \in (I : I)$, then $xJE = xI \subseteq I = JE$.

Since $JJ^{-1} = R$, $xE = xJJ^{-1}E \subseteq JJ^{-1}E = E$ and so $x \in (E : E)$. Conversely, if $xE \subseteq E$, then $xI = xJE \subseteq JE = I$, and thus, $x \in (I : I)$. Hence, $(I : I) = (E : E)$. Now, let n be any positive integer, and let $x \in (I^n : I^n)$. Since E is idempotent, $E^n = E$, and thus, $xJ^nE = xI^n \subseteq I^n = J^nE$. Since J is invertible, we easily obtain $xE \subseteq E$, and thus, $x \in (E : E) = (I : I)$. Hence, $(I^n : I^n) = (I : I)$ for every positive integer n, and therefore, I is L-stable.

(ii) Assume that I is a weakly ES-stable ideal, and set $I = JE$ where $JJ^{-1} = R$ and $E = E^2$. Then $I^2 = J^2E^2 = J^2E = J(JE) = JI$. Conversely, if $I^2 = JI$ for some invertible ideal J of R, then $I = J(J^{-1}I)$ and clearly $J^{-1}I$ is idempotent, as desired.

(iii) Assume that I is ES-stable. Then $I^2 = JI$ for some invertible sub-ideal J of I. Set $E = J^{-1}I$. Then $I = JE$ and $E^2 = (J^{-1})^2I^2 = (J^{-1})^2JI = J^{-1}I = E$ since $JJ^{-1} = R$. Now, since $J \subseteq I$, $I^{-1} \subseteq J^{-1}$, and thus, $I \subseteq II^{-1} \subseteq IJ^{-1} = E$, therefore $J \subseteq I \subseteq E$ as desired. The converse is clear. $\square$

Remark 2.3.

(i) If I is a weakly ES-stable ideal and $I = JE$ with $JJ^{-1} = R$ and $E = E^2$, then the ideal J is not necessarily a sub-ideal of I. Thus, the definition of weakly ES-stable ideal can be viewed as a weaker condition of Eakin-Sathaye stability where J is supposed to be a sub-ideal of I.

(ii) An almost weakly ES-stable ideal is not necessarily an L-stable ideal. Indeed, let $\mathbb{Q}$ be the field of rational numbers, X an indeterminate over $\mathbb{Q}$, and let $R = \mathbb{Q} + X\mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]] = \mathbb{Q} + M$ be the PVD domain issued from the valuation domain

$$V = \mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]] = \mathbb{Q}(\sqrt{2}, \sqrt{3}) + X\mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]] = \mathbb{Q}(\sqrt{2}, \sqrt{3}) + M,$$

where $M = XV$. By [**7,** Theorem 2.1], R is a Noetherian local domain. Set $W = \mathbb{Q} + \sqrt{2}\mathbb{Q} + \sqrt{3}\mathbb{Q}$, and let I be the ideal of R given by $I = X(W + M)$. It is easy to check that, for every $n \geq 2$, $W^n = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $(W : W) = \mathbb{Q}$. Then $(I : I) = (W : W) + M = \mathbb{Q} + M = R$, and, for every $n \geq 2$,

$$I^n = X^n(W^n + M) = X^n(\mathbb{Q}(\sqrt{2}, \sqrt{3}) + M) = X^nV = M^n.$$

Thus, for every $n \geq 2$, I^n is strongly stable and a fortiori a weakly ES-stable ideal. Hence, I is an almost ES-stable ideal which is not a weakly ES-stable ideal since I is not strongly stable. Moreover, for every positive integer $n \geq 2$, $(I^n : I^n) = (M^n : M^n) = V$, and thus,

$$R^I := \bigcup_{n\geq 0} (I^n : I^n) = V.$$

However, $(I : I) = R$, and hence, I is not L-stable.

(iii) It is clear that, if R is a weakly ES-stable domain, then R_M is a weakly ES-stable domain for each maximal ideal M of R. However, the converse is not true. Indeed, let

$$V = \mathbb{Q}[[X]] = \mathbb{Q} + M$$

where $M = XV$ and $R = \mathbb{Z} + M$. By [**7,** Theorem 2.1], every maximal ideal of R is of the form $p\mathbb{Z} + M = pR$ for some positive prime integer p and $R_{pR} = \mathbb{Z}_p + M$. Let I be an ideal of R_{pR}. If $M \subsetneqq I$, then $I = A + M$ where A is an ideal of $\mathbb{Z}_p$. Thus, A is principal and so is I.

Assume that $I \subseteq M$. If I is an ideal of V, then I is principal in V and therefore, I is strongly stable. If I is not an ideal of V, then $I = a(W + M)$ where $\mathbb{Z}_p \subseteq W \subsetneqq \mathbb{Q}$. But, then, W is a fractional ideal of $\mathbb{Z}_p$ (since $\mathbb{Z}_p$ is a valuation domain and thus a conducive domain, [**12**]). Hence, $W = f\mathbb{Z}_p$, and thus, $I = afR_{pR}$. Therefore, R_{pR} is a weakly ES-stable domain (in fact a strongly stable domain). However, by Corollary 2.6, R is not a weakly ES-stable domain since R is not of finite character (M is contained in infinitely many maximal ideals of R).

The next lemma is crucial. It presents important information about weakly ES-stable ideals, and we often use it as a key for many proofs of our results.

Lemma 2.4. *Let R be an integral domain and I an* (*integral*) *ideal of R.*

(i) *If I is a weakly ES-stable ideal and $I = JE$ where $JJ^{-1} = R$ and $E = E^2$, then $(I : I) = (E : E)$ and $E = I(I : I^2)$.*

(ii) *If I is a finitely generated weakly ES-stable ideal, then $I_t \subsetneqq R$; and, if, R is a weakly ES-stable domain, then $A_t \subsetneqq R$ for every* (*integral*) *ideal A of R.*

(iii) *If R is a weakly ES-stable domain and T is an overring of R which is a fractional ideal of R, then T is a weakly ES-stable domain.*

(iv) *If R is a weakly ES-stable domain, then every finitely generated ideal is ES-stable* (*in particular R is finitely stable*).

Proof.

(i) $(I : I) = (E : E)$ follows from the proof of Proposition 2.2.

It suffices to prove that $E = I(I : I^2)$. Since $I^2 = JI$ and J is invertible, $J^{-1}I^2 = I$, and thus, $J^{-1} \subseteq (I : I^2)$. Hence, $E = J^{-1}I \subseteq I(I : I^2)$. Conversely, let $x \in (I : I^2)$. Then $xI^2 \subseteq I$ implies that $xJ^2E \subseteq JE$. Since J is invertible, $xJE \subseteq E$. Hence, $xJ \subseteq (E : E)$, and thus, $xI = xJE \subseteq E(E : E) = E$. Therefore, $I(I : I^2) \subseteq E$ and $I(I : I^2) = E$ as desired.

(ii) Assume that I is finitely generated, and suppose that $I_t = R$. Then $(I : I) = I^{-1} = R$. Set $I = JE$ where $JJ^{-1} = R$ and $E = E^2$. By (i), $(E : E) = (I : I) = R$. Since $E^2 = E$, $E \subseteq (E : E) = R$, and thus, $I = JE \subseteq J$. Hence, $J^{-1} \subseteq I^{-1} = R$, and thus, $R = JJ^{-1} \subseteq J$. Then, $I \subseteq IJ = I^2$, and thus, $I = I^2$, which is a contradiction since I is finitely generated ([**25,** Theorem 76]).

Now assume that R is a weakly ES-stable domain. If A is an integral ideal such that $A_t = R$, then there is a finitely generated sub-ideal B of A such that $B_t = B_v = R$, which is absurd since B is a finitely generated weakly ES-stable ideal. Hence, $A_t \subsetneq R$, as desired.

(iii) Let T be an overring of R which is a fractional ideal of R, and let I be a nonzero ideal of T. Then I is a (fractional) ideal of R, and thus, $I = JE$ where $JJ^{-1} = R$ and $E = E^2$. Set $A = JT$ and $F = ET$. Then, clearly, $A(T : A) = T$, $F = F^2$ and $I = AF$, as desired.

(iv) Let I be a finitely generated ideal, and set $I = JE$ where $JJ^{-1} = R$ and $E = E^2$. Set $T = (I : I)$. By (iii), T is a weakly ES-stable domain and, by (i), $E = I(T : I)$ is an idempotent (integral) ideal of T. Suppose that E is a proper ideal of T. Since $E = IJ^{-1}$ is a finitely generated (fractional) ideal of R, it is a fortiori a finitely generated ideal of T. On the other hand, $(T : E) = (E : E) = (I : I) = T$. Hence, $E_{t_T} = E_{v_T} = T$ (where t_T and v_T are the t- and v-operations with respect to T), which contradicts (ii) since T is a weakly ES-stable domain. Hence, $E = T$, and thus, $I = JT$. Therefore, $J \subseteq I$, and therefore, I is ES-stable by Proposition 2.2. $\square$

Corollary 2.5. *Let R be an integral domain and I a nonzero ideal of R. Then I is ES-stable if and only if I is stable and weakly ES-stable.*

Proof. Assume that I is both stable and weakly ES-stable, and set $I = JE$ where $JJ^{-1} = R$ and $E = E^2$. By Lemma 2.4, $E = I(I : I^2)$, and, since I is stable, $E = I(I : I^2) = (I : I)$. Thus, $I = J(I : I)$ and hence, $J \subseteq I \subseteq E$. Therefore, I is ES-stable, as desired. $\square$

Recall that an integral domain R is said to be Clifford regular if its class semigroup $S(R) = F(R)/P(R)$ is von Neumann regular, where $F(R)$ is the semigroup of all nonzero fractional ideals of R and $P(R)$ is its subgroup of principal fractional ideals (see [**8, 9**]). Clifford regularity has well-known links with stability conditions, for instance, see [**8, 9, 10, 24**].

Our next corollary shows that a weakly ES-stable domain must be Clifford regular. Recall that a domain is of finite character if every nonzero nonunit element is contained in a finitely many maximal ideals. Equivalently, for each nonzero proper ideal I of R, the set $\mathrm{Max}(R, I)$ of all maximal ideals of R containing I is finite. Clifford regular domains are of finite character [**10,** Theorem 4.7].

Corollary 2.6. *Let R be an integral domain. If R is a weakly ES-stable domain, then R is Clifford regular. In particular, R has finite character.*

Proof. Assume that R is a weakly ES-stable domain, and let I be an ideal of R. Then $I = JE$, where $JJ^{-1} = R$ and $E = E^2$. By Lemma 2.4, $E = I(I : I^2)$. Hence, $I^2(I : I^2) = IE = JE^2 = JE = I$. Thus, $\overline{I}$ is regular in $S(R)$ [**8,** Lemma 1.1], and therefore, R is Clifford regular. $\square$

Corollary 2.7. *Let R be an integral domain. The following statements are equivalent.*

(i) *$R[X]$ (respectively, $R[[X]]$) is a weakly ES-stable domain*;

(ii) $R[X]$ (*respectively,* $R[[X]]$) *is an almost weakly* ES*-stable domain*;

(iii) $R[X]$ (*respectively,* $R[[X]]$) *is an almost finitely weakly* ES*-stable domain*;

(iv) R *is a field.*

Proof. It suffices to prove (iii) $\Rightarrow$ (iv). Assume that $R[X]$ (respectively, $R[[X]]$) is an almost finitely weakly ES-stable domain, and suppose that R is not a field. Let d be a nonzero nonunit element of R, and let I be the ideal of $R[X]$ (respectively, $R[[X]]$) given by $I = (d, X)$. Then I is a proper finitely generated ideal and $I^{-1} = (I : I) = R[X]$ (respectively, $I^{-1} = (I : I) = R[[X]]$). Hence, $I^{-n} = (I^n : I^n) = R[X]$ (respectively, $I^{-n} = (I^n : I^n) = R[[X]]$) for every positive integer n. Thus, $(I^n)_t = R[X]$ (respectively, $(I^n)_t = R[[X]]$), which is a contradiction by Lemma 2.4 since some power of I is supposed to be a weakly ES-stable ideal. It follows that R is a field, as desired. $\square$

3. Noetherian-like settings. Our first theorem characterizes Noetherian weakly ES-stable domains in terms of stability. It turns out that, in the Noetherian context, the notion of weakly ES-stability and Eakin-Sathaye stability coincide.

Theorem 3.1. *Let R be a Noetherian domain and I an ideal of R. Then I is a weakly ES-stable ideal if and only if I is ES-stable. In particular, R is a weakly ES-stable domain if and only if R is ES-stable.*

Proof. Let I be a weakly ES-stable ideal of R, and set $I = JE$ where $JJ^{-1} = R$, $E = E^2$; in addition, set $T = (I : I)$. By Lemma 2.4, $E = I(T : I)$, E is a trace (integral) ideal of T which is idempotent. But, since T is Noetherian, necessarily $E = T$, hence, $I = JT$. Thus, $J \subseteq I$ and therefore I is ES-stable. $\square$

Recall that an integral domain R is said to be a strong Mori domain if R satisfies the *acc* on w-ideals. Noetherian domains are strong Mori domains, and strong Mori domains are Mori domains.

The next corollary shows that 'strong Mori' and 'Noetherian' coincide in class of weakly ES-stable domains.

Corollary 3.2. *A strong Mori which is a weakly ES-stable domain is Noetherian.*

Proof. Let M be a maximal ideal of R. By Lemma 2.4, M is a t-maximal ideal of R and, by [**14,** Theorem 1.9], R_M is Noetherian. But, since R_M is a weakly ES-stable domain, by Theorem 3.1, R_M is an ES-stable domain and so $\dim R_M = 1$. Thus, $\dim R = 1$, and again by [**14,** Corollary 1.10], R is Noetherian. $\square$

The next theorem shows that, for some classes of Mori domains, the notion of weakly ES-stable domains coincides with the Eakin-Sathaye stable domains.

Theorem 3.3. *Let R be a Mori domain such that $(I : I)$ is a Mori domain for each nonzero ideal I of R. Then, R is a weakly ES-stable domain if an only if R is ES-stable.*

Proof. Let I be a nonzero ideal of R, and set $I = JE$ where $JJ^{-1} = R$ and $E = E^2$. By hypothesis and Lemma 2.4, $T = (I : I) = (E : E)$ is a Mori domain and, since $E^2 = E$, $E \subseteq (E : E) = T$. Thus, E is an idempotent integral ideal of T. Since

$$(T : E)=((E : E) : E)=(E : E^2)=(E : E)=T, \quad E_{t_T} = E_{v_T} = T,$$

where t_T and v_T are the t- and v-operations with respect to T. By Lemma 2.4, $E = T$, and therefore, $I = JT$. Thus, $J \subseteq I$, and hence, I is ES-stable. $\square$

In [**31**], Olberding proved that there is a one-dimensional local "bad" stable domain R which is not Noeherian and, with normalization V, a DVR. In [**21,** Theorem 2.34], or [**19,** Theorem 2.17], Gabelli and Roitman proved that a one-dimensional stable domain must be a Mori domain. This result shows that the local one-dimensional domains that are stable and not Noetherian constructed by Olberding [**31**] are in fact Mori domains. Moreover, such domains are strongly stable by [**28,** Lemma 3.1]. It follows that a strongly stable Mori domain is not necessarily a Noetherian domain.

Theorem 3.4. *Let R be a Mori domain which is a weakly ES-stable domain. Then*:

(i) *Every v-ideal of R is ES-stable.*

(ii) *Every nonzero prime ideal of R is divisorial.*

(iii) *If R satisfies the acc on fractional overrings that are v-ideals, e.g., if $(R : \overline{R}) \neq (0)$, where $\overline{R}$ is the complete integral closure of R, then* $\dim R \leq 1$.

Proof.

(i) Similar to the proof of Theorem 3.3, $(I : I)$ is a Mori domain for every v-ideal I of R.

(ii) Assume that R is a weakly ES-stable domain, and let P be a nonzero prime ideal of R. By [**32,** Proposition 26.1], or [**5,** Theorem 3.1], P is divisorial if $\operatorname{ht} P = 1$; and, if $\operatorname{ht} P \geq 2$, then either $P^{-1} = R$ or P is strongly divisorial, i.e., $P = P_v = PP^{-1}$. But, since $P_v = I_v$ for some finitely generated subideal I of P and $I_t \subsetneqq R$ (Lemma 2.4), P is strongly divisorial.

(iii) Suppose that $\dim R \geq 2$. Without loss of generality, we may assume that R is local with maximal ideal M and $\dim R = \operatorname{ht} M \geq 2$. Note that every prime ideal P of R is divisorial and, by Theorem 3.3, P is strongly stable. Indeed, as in [**28**], set $R_0 = R$, $M_0 = M$, $R_1 = M^{-1} = (M : M)$ and $M = aR_1$ for some $a \in M$. By [**28,** Lemma 4.1], R_1 is integral over R, and thus, $\dim R_1 = \dim R \geq 2$. Moreover, R_1 is a Mori domain which is a weakly ES-stable domain. Since $R \subsetneqq R_1$ (as M is divisorial), and, by [**28,** Proposition 4.2], if R_1 has two or three maximal ideals, then each is principal. Since $M = aR_1$ and these maximal ideals are minimal over M, they must be of height 1 by [**3,** Proposition 3.4] (or [**5,** Theorem 3.7]). This contradicts the fact that $\dim R_1 \geq 2$. Hence, R_1 must be a local domain with maximal ideal M_1. Applying the same process (for the Mori local weakly ES-stable domain R_1), we obtain that

$$R_1 \subsetneqq R_2 = (R_1 : M_1) = (M_1 : M_1)$$

is a local Mori weakly ES-stable domain with maximal ideal M_2 and $\dim R_2 = \dim R_1 = \dim R \geq 2$. Thus, we construct a chain of proper local Mori overrings $\{(R_i, M_i)\}_i$ of R that are weakly ES-stable domains with dimension greater than 2. But, since, for each $i \geq 0$,

$$R_{i+1} = (R_i : M_i) = (R : MM_1 \cdots M_i),$$

all of these overrings are (fractional) divisorial ideals of R. Therefore, the chain must stabilizes, which yields a contradiction. $\square$

Recall that a domain R is seminormal if $x \in R$, for each $x \in qf(R)$ with $x^2, x^3 \in R$, equivalently $x \in R$ for every $x \in qf(R)$, $x^n \in R$ for $n \gg 0$.

The next result shows that a seminormal Mori domain R which is a weakly ES-stable domain is a one-dimensional domain.

Corollary 3.5. *Let R be a seminormal Mori domain. If R is a weakly ES-stable domain, then* $\dim R \leq 1$.

Proof. Without loss of generality, we may assume that R is local with maximal ideal M. If $\operatorname{ht} M \geq 2$, then $T = M^{-1} = (M : M)$ is a Mori domain which is a weakly ES-stable domain by Lemma 2.4. By Theorem 3.4, every prime ideal of T is divisorial, which contradicts [**6,** Lemma 2.5] since T must contain a nondivisorial prime ideal that contracts to M. $\square$

4. Valuation and Prüfer case. Our first result in this section is a new characterization of Prüfer domains in the context of weakly ES-stable domains. First, recall that an integral domain R is a $PVMD$ if every finitely generated ideal is t-invertible, i.e., $(II^{-1})_t = R$, equivalently R_M is a valuation domain for every t-maximal ideal M of R. Prüfer domains are exactly the $PVMD$s where every maximal ideal is t-maximal.

Theorem 4.1. *Let R be an integral domain. The following are equivalent.*

(i) *R is an integrally closed domain which is a finitely weakly ES-stable domain*;

(ii) *R is an integrally closed domain which is an almost finitely weakly ES-stable domain*;

(iii) *R is a Prüfer domain.*

Proof.

(i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are trivial.

(ii) $\Rightarrow$ (iii). Let I be a finitely generated ideal of R. Since R is integrally closed, $(I^s : I^s) = R$ for every positive integer s. Assume that I^n is a weakly ES-stable ideal. Then $I^{2n} = JI^n$ for some invertible ideal J of R. Further,

$$\begin{aligned}(R : I^n) &= ((I^n : I^n) : I^n) = (I^n : I^{2n}) \\ &= (I^n : JI^n) = ((I^n : I^n) : J) = (R : J) = J^{-1}.\end{aligned}$$

Hence, $(I^n)_t = (I^n)_v = J_v = J$. Thus, $(I^n)_t I^{-n} = JJ^{-1} = R$, and therefore, I^n is t-invertible. But since $I^n I^{-n} \subseteq II^{-1}$, I is t-invertible and so R is a $PVMD$. By Lemma 2.4, $M_t \subsetneqq R$ for every maximal ideal M of R. Thus, $M = M_t$, and hence, R is Prüfer as desired. $\square$

Recall that a domain R is a pseudo-Dedekind, respectively pseudo-principal, domain if every v-ideal is invertible, respectively principal. From the proof of Theorem 4.1, one can deduce that, if $(I : I) = R$ and I is a weakly ES-stable ideal, then I_v is invertible. Thus, a completely integrally closed domain which is a weakly ES-stable domain is pseudo-Dedekind. In [**17,** Theorem 7.4.6], it was proved that an integral domain R is quasi-Prüfer, i.e., the integral closure R' of R is a Prüfer domain, if and only if every (nonzero) finitely generated ideal is SV-prestable.

Our next corollary and example show that a finitely weakly ES-stable domain is quasi-Prüfer but that a quasi-Prüfer domain need not be a finitely weakly ES-stable domain.

Corollary 4.2. *Let R be an integral domain. If R is a finitely weakly ES-stable domain, then R is quasi-Prüfer.*

Proof. Let J be a finitely generated ideal of the integral closure R' of R, and set

$$J = \sum_{i=1}^{n} b_i R'.$$

Then

$$I = \sum_{i=1}^{n} b_i R$$

is a finitely generated fractional ideal of R, and thus, $I = AE$ where $AA^{-1} = R$ and $E = E^2$. Set $B = AR'$ and $F = ER'$. Clearly, $J = IR' = AER' = BF$ and $F^2 = F$. Moreover, $A^{-1} = (R : A) \subseteq (R' : AR') = (R' : B)$, and thus, $R = AA^{-1} \subseteq A(R' : B) \subseteq B(R' : B)$. Hence, $R' = B(R' : B)$, and therefore, J is a weakly ES-stable ideal of R'. By Theorem 4.1, R' is Prüfer as desired. □

The next example shows that a quasi-Prüfer domain need not be a finitely weakly ES-stable domain.

Example 4.3. Let $V = \mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]] = \mathbb{Q}(\sqrt{2}, \sqrt{3}) + M$ and $R = \mathbb{Q} + M$. Clearly $R' = V$ is a valuation domain. However, R is not a finitely weakly ES-stable domain since the ideal $I = X(\mathbb{Q} + \sqrt{2}\mathbb{Q} + \sqrt{3}\mathbb{Q} + M)$ is a finitely generated ideal of R which is not a weakly ES-stable ideal as it is shown in Remark 2.3.

Recall that a domain R is said to be conducive if $(R : T) \neq (0)$ for each overring T of R with $T \subsetneqq qf(R)$, equivalently, $(R : V) \neq (0)$ for some valuation overring V of R [**12,** Theorem 3.2]). Combined with Corollary 2.6, the next corollary shows that a conducive domain which is a weakly ES-stable domain must be semi-local (i.e., has only finitely many maximal ideals).

Corollary 4.4. *Let R be a conducive domain which is a weakly ES-stable domain. Then R is semi-local.*

Proof. Let R' be the integral closure of R. Notice that R' is a conducive domain, and since, for every $Q \in \operatorname{Max}(R)$, there is $N \in \operatorname{Max}(R')$ such that $Q = N \cap R$, it suffices to show that $\operatorname{Max}(R')$ is finite. By Corollary 4.2, R' is a Prüfer domain and by Lemma 2.4, R' is a weakly ES-stable domain (since $(R : R') \neq (0)$). Thus, without loss of generality, we may assume that R is a conducive Prüfer domain which is a weakly ES-stable domain. Now, let $M \in \operatorname{Max}(R)$ and set $Q = (R : R_M)$. We may assume that R is not local. Then, Q is a proper ideal, and, by [**12,** Lemma 2.10], Q is a prime ideal of both R and R_M. Let N be any maximal ideal of R with $M \neq N$, and let $a \in N \setminus M$. Then, for each $x \in Q$, $x/a \in QR_M = Q$. Thus, $x \in aQ \subseteq QN \subseteq N$. Therefore, $Q \subseteq N$ and, since $Q \subseteq M$, we obtain $Q \subseteq P$ for each maximal ideal P of R. Hence, $\operatorname{Max}(R) = \operatorname{Max}(R, Q)$

which is finite since R has finite character (Corollary 2.6). It follows that R is semi-local. $\square$

The next theorem characterizes weakly ES-stable ideals in a valuation domain. Such a characterization provides weakly ES-stable ideals that are not stable. Recall that a domain R has the trace property (or is a TP domain) if, for every ideal I of R, either $II^{-1} = R$ or II^{-1} is a prime ideal of R, see [**18**]. Valuation domains are TP-domains. Also recall that the maximal ideal of a valuation domain is either principal or idempotent.

Theorem 4.5. *Let V be a valuation domain, and I a nonzero ideal of V. Then, I is a weakly ES-stable ideal if and only if either I is strongly stable or $I = aP$ for some $0 \neq a \in qf(V)$ and an idempotent prime ideal P of V.*

Proof. Let I be a weakly ES-stable ideal of V, and set $I = JE$ where $JJ^{-1} = V$ and $E = E^2$. Necessarily, $J = aV$ for some $0 \neq a \in qf(V)$, and thus, $I = aE$. Let $P = Z(V, I)$ be the prime ideal of V consisting of all zero-divisors of V modulo I. Then $(I : I) = V_P$. Assume that I is not strongly stable. Since V_P is a TP-domain, $I(V_P : I) = QV_P = Q$ for some prime ideal Q of V with $Q \subseteq P$. Thus, $(V : I) \subseteq (V_P : I) = (Q : I) \subseteq (V : I)$, and therefore, $(V : I) = (V_P : I) = (Q : I)$. Hence, $I(V : I) = I(V_P : I) = Q$. By [**1,** Theorem 2.8], $V_P = (I : I) = V_{II^{-1}} = V_Q$, and thus, $Q = P$. Hence, P is a trace ideal of both V and V_P. Therefore, P, as the maximal ideal of V_P, is not divisorial; thus, it is idempotent. By Lemma 2.4,

$$E = I(I : I^2) = I((I : I) : I) = I(V_P : I) = P,$$

and hence, $I = aE = aP$ as desired. $\square$

Recall that a domain R is said to be divisorial if every nonzero ideal of R is a v-ideal; and R is totally divisorial provided that every overring of R is divisorial [**11**]. Also, a valuation (respectively, Prüfer) domain is strongly discrete if $P^2 \subsetneq P$ for every nonzero prime ideal P. A combination of [**11,** Proposition 7.6] and [**28,** Lemma 3.1] yields that a valuation domain V is strongly stable if and only if V is ES-stable, stable, strongly discrete and totally divisorial. On the other hand, it is well known that, in a valuation domain V, non-divisorial ideals of V

are of the form aM where M, the maximal ideal of V, is idempotent. Thus, non-divisorial ideals of V are weakly ES-stable.

Our next corollary concentrates on divisorial ideals.

Proposition 4.6. *Let V be a valuation domain. Then V is weakly ES-stable if and only if every divisorial ideal of V is either principal or of the form aP for some non-maximal idempotent prime ideal P.*

Proof. Let I be a divisorial ideal of V, and suppose that I is not principal. Set $II^{-1} = P$. Since I is weakly ES-stable, $I = aE$ for some idempotent fractional ideal E of R. By Lemma 2.4, $V_P = (I : I) = (E : E)$, and thus, E is an idempotent integral ideal of V_P. Hence, E is a prime ideal of V_P. Set $E = QV_P$ where Q is a prime ideal of V with $Q \subseteq P$. Then, $V_P = (E : E) = (QV_P : QV_P) = V_Q$, and thus, $Q = P$. Hence, $E = QV_P = PV_P = P$. Therefore, P is an idempotent prime ideal of V and $I = aE = aP$. Now, if $P = M$, then $M = M^2$, and thus, $M_v = V$. It follows that $aM = I = I_v = aM_v = aV$, a contradiction. Hence, P is not maximal as desired. $\square$

In the case of one-dimensional valuation domain V, if V is divisorial, then it is a DVR. If V is not divisorial, from Proposition 4.6, V is weakly ES-stable if and only if every divisorial ideal is principal.

We obtain the following corollary (the equivalence (ii) $\Leftrightarrow$ (iii) can be found in [**2,** page 327]).

Corollary 4.7. *Let V be a one-dimensional valuation domain. The following conditions are equivalent.*

(i) *V is a weakly ES-stable domain.*

(ii) *V is a pseudo-principal domain.*

(iii) *The value group of V is isomorphic to a complete subgroup of the real numbers.*

Before stating our next theorem, we recall that, if I is a nonzero (integral) ideal of a Prüfer domain R, then a representation of $\mathrm{End}(I)$

as a sub-intersection of R is given by

$$\mathrm{End}(I) = (I : I) = \left(\bigcap_{M \supseteq I} R_{G(M)} \right) \cap \mathcal{C}(I),$$

where $G(M) = Z(R_M, IR_M) \cap R$ is the unique prime ideal of R such that $(IR_M : IR_M) = R_{G(M)}$ for every maximal ideal M containing I; and

$$\mathcal{C}(I) = \bigcap_{I \nsubseteq M} R_M,$$

see **[17]**. Also we note that, if A and B are R-submodules of $qf(R)$ and I is an ideal of R, then $I(A \cap B) = IA \cap IB$. Moreover, if R has finite character, then $I\mathcal{C}(I) = \mathcal{C}(I)$. Finally, a Prüfer domain is strongly discrete if every nonzero prime ideal is not idempotent.

Theorem 4.8. *Let R be a strongly discrete Prüfer domain. Then, R is a weakly ES-stable domain if and only if R is ES-stable if and only if R is stable.*

Proof. Assume that R is a weakly ES-stable domain. By Corollary 2.6, R has finite character. Let I be a nonzero (integral) ideal of R, and set $\mathrm{Max}(R, I) = \{M_1, \ldots, M_n\}$. For every maximal ideal $M_i \in \mathrm{Max}(R, I)$, R_{M_i} is a strongly discrete valuation domain which is a weakly ES-stable domain. By Theorem 4.5, IR_{M_i} is strongly stable. Set $IR_{M_i} = a_i R_{G(M_i)}$ for some $a_i \in I$. Now, let

$$J = \sum_{i=1}^{n} a_i R.$$

Then $J \subseteq I$ and, for each $i \in \{1, \ldots, n\}$, $JR_{G(M_i)} = IR_{M_i}$. Two cases are then possible.

Case 1. $\mathrm{Max}(R, I) = \mathrm{Max}(R, J)$. Then $\mathcal{C}(I) = \mathcal{C}(J)$, and

$$\begin{aligned} I &= \left(\bigcap_{i=1}^{n} IR_{M_i} \right) \cap \mathcal{C}(I) = \bigcap_{i=1}^{n} JR_{G(M_i)} \cap \mathcal{C}(J) \\ &= \bigcap_{i=1}^{n} JR_{G(M_i)} \cap J\mathcal{C}(J) = J\left(\bigcap_{i=1}^{n} R_{G(M_i)} \cap \mathcal{C}(J) \right) \end{aligned}$$

$$= J\left(\bigcap_{i=1}^{n} R_{G(M_i)} \cap \mathcal{C}(I)\right) = J(I : I).$$

Hence, $I^2 = JI$, and therefore, I is an ES-stable ideal.

Case 2. $\mathrm{Max}(R, I) \subsetneqq \mathrm{Max}(R, J)$. Set $\mathrm{Max}(R, J) = \{M_1, \dots, M_n, M_{n+1}, \dots, M_s\}$, and let

$$a \in I \setminus \bigcup_{j=n+1}^{s} M_j.$$

Set $A = J + aR$. Then, $A \subseteq I$, and it is easy to check that $\mathrm{Max}(R, I) = \mathrm{Max}(R, A)$, and $IR_{M_i} = AR_{G(M_i)}$ for each $i \in \{1, \dots, n\}$. Thus, as in the first case, $I = A(I : I)$. Hence, $I^2 = AI$, and therefore, I is ES-stable as desired. $\square$

In [**28,** Theorem 3.3], Olberding proved that a domain is stable if and only if it is locally stable with finite character. A Prüfer domain that is locally weakly ES-stable need not be weakly ES-stable (Remark 2.3 (iii)). Similarly, if R is an almost Dedekind domain which is not Dedekind, clearly R_M is a DVR, and thus, R is locally strongly stable. However, R is not weakly ES-stable. Indeed, let M be a non-invertible maximal ideal of R. Then, $M^{-1} = (M : M) = R$. If M is weakly ES-stable, then $M^2 = JM$ for some invertible ideal J of R. Thus,

$$R = (R : M^2) = (R : JM) = ((R : M) : J) = (R : J),$$

and therefore, $J = J(R : J) = R$. Hence, $M^2 = M$, a contradiction.

It follows that an almost Dedekind is a weakly ES-stable domain if and only if it is a Dedekind domain. We are not able to prove or disprove whether a locally weakly ES-stable domain of finite character is weakly ES-stable or not.

5. Pullbacks. The purpose of this section is to investigate the transfer of the notion of weakly ES-stable domains to classical pullback constructions. Our work is motivated by an attempt to generate new families of weakly ES-stable domains.

First, let us fix the notation for the rest of this section and recall some useful properties of classical pullbacks. Let T be a domain, M a maximal ideal of T, $K = T/M$ its residue field and D a subring of K.

Let R be defined by the pullback diagram:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \underset{\phi}{\longrightarrow} & K = T/M. \end{array}$$

We assume that $R \subsetneq T$, and we refer to this diagram as a diagram of type $(\square)$; furthermore, if $qf(D) = T/M$, we refer to the diagram as a diagram of type $(\square^*)$. The case where $T = V$ is a valuation domain is crucial, and we refer to this case as a classical diagram of type $(\square)$. Recall that $(R : T) = M$ is a prime ideal of R as $R/M \simeq D$, and if T is local, every ideal of R is comparable (under inclusion) to M, and R is local if and only if T and D are local. For more details on general pullbacks, we refer the reader to **[15, 16, 20]** and **[7]** for classical "D+M" constructions.

Now we are ready to state the main theorem of this section.

Theorem 5.1.

(i) *For the diagram of type* $(\square)$*, if R is a weakly ES-stable domain, then T is a weakly ES-stable domain, D is a semi-local domain which is a weakly ES-stable domain and* $[K : qf(D)] \leq 2$.

(ii) *For the classical diagram of type* $(\square)^*$*, assume that D is conducive. Then, R is a weakly ES-stable domain if and only if T is a weakly ES-stable domain and D is a semi-local weakly ES-stable domain.*

The proof of this theorem involves the following preparatory lemmas.

Lemma 5.2. *For the diagram of type* $(\square)$ *assume that there is a (nonzero) D-submodule W of K such that* $(W : W^2) = 0$. *Then, R has an ideal which is not a weakly ES-stable ideal.*

Proof. Let W be a D-submodule of K such that $(W : W^2) = 0$, and let I be the ideal of R given by $I = a\phi^{-1}(W)$ for some nonzero element $a \in M$. Suppose that I is a weakly ES-stable ideal, and write $I = JE$ where $JJ^{-1} = R$ and $E = E^2$. Then,

$$(I : I^2) = (a\phi^{-1}(W) : a^2\phi^{-1}(W^2))$$

$$= a^{-1}\phi^{-1}(W : W^2)$$
$$= a^{-1}\phi^{-1}(0) = a^{-1}M.$$

By Lemma 2.4, $E = I(I : I^2) = a\phi^{-1}(W)a^{-1}M = M$. Thus, $M = M^2$, and so,

$$aM = IM = JEM = JM^2 = JM = I = a\phi^{-1}(W).$$

Hence, $M = \phi^{-1}(W)$ and so $W = 0$, which is absurd. It follows that I is not a weakly ES-stable ideal. $\square$

Lemma 5.3. *For the diagram of type* ($\square$) *assume that R is an almost weakly ES-stable domain* (*respectively, a weakly ES-stable domain*)*. Then, K is algebraic over k* (*respectively,* $[K : k] \leq 2$).

Proof. Assume that R is an almost weakly ES-stable domain, and suppose that K is not algebraic over $k = qf(D)$. Let $\lambda \in K$ be transcendental over k, and set $W = k + k\lambda$. Let $0 \neq a \in M$, and set $I = a\phi^{-1}(W)$.

Claim 1. $(W^s : W^s) = k$ for every positive integer s. Indeed, let $f \in (W^s : W^s)$. Since $1 \in W^s = k + k\lambda + \cdots + k\lambda^s$, $f \in W^s$. We write $f = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_s\lambda^s$ for some $\alpha_0, \ldots, \alpha_s \in k$. Since $\lambda \in W^s$,

$$\alpha_0\lambda + \cdots + \alpha_s\lambda^{s+1} = f\lambda \in W^s.$$

Hence, $\alpha_s = 0$, otherwise, λ would be algebraic over k, a contradiction. Thus,

$$f = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_{s-1}\lambda^{s-1}.$$

Again, since $\lambda^2 \in W^s$, $\alpha_0\lambda + \cdots + \alpha_{s-1}\lambda^{s+1} = f\lambda^2 \in W^s$; hence, $\alpha_{s-1} = 0$. Iterating this process, we obtain

$$\alpha_s = \alpha_{s-1} = \cdots = \alpha_1 = 0.$$

Hence, $f = \alpha_0 \in K$, and therefore, $(W^s : W^s) = k$. It follows that $(W^s : W^{2s}) = ((W^s : W^s) : W^s) = (k : W^s) = 0$, and by Lemma 5.2, I^s is not a weakly ES-stable ideal for all s, which is a contradiction. It follows that K is algebraic over k.

Assume that R is a weakly ES-stable domain.

Step 1. For every $\lambda \in K$, $[k(\lambda) : k] \leq 2$. Indeed, by the first part of the proof, K is algebraic over k. Suppose that there is a $\lambda \in K$ such that $[k(\lambda) : k] = n \geq 3$. Then $1, \lambda, \ldots, \lambda^{n-1}$ is a basis of $k(\lambda)$ as a k-vector space. Set

$$W = k + k\lambda + \cdots + k\lambda^{n-2} \quad \text{and} \quad I = a\phi^{-1}(W)$$

for some $0 \neq a \in M$.

Claim 2. $(W : W) = k$. Indeed, let $f \in (W : W) \subseteq W$, since $1 \in W$, and write

$$f = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_{n-2}\lambda^{n-2},$$

where $\alpha_i \in k$ for all i. Since $\lambda \in W$,

$$\alpha_0\lambda + \alpha_1\lambda^2 + \cdots + \alpha_{n-2}\lambda^{n-1} = f\lambda \in W.$$

Then $\alpha_{n-2} = 0$, and thus,

$$f = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_{n-3}\lambda^{n-3}.$$

Again, $\lambda^2 \in W$ implies that

$$\alpha_0\lambda^2 + \alpha_1\lambda^3 + \cdots + \alpha_{n-3}\lambda^{n-1} \in W.$$

Then $\alpha_{n-3} = 0$. Iterating this process, we obtain

$$\alpha_{n-2} = \alpha_{n-3} = \cdots = \alpha_1 = 0.$$

Hence, $f = \alpha_0 \in k$ as desired. Therefore, $(W : W^2) = ((W : W) : W) = (k : W) = 0$, and by Lemma 5.2, I is not a weakly ES-stable ideal, which is a contradiction. Thus $[k(\lambda) : k] \leq 2$ for every $\lambda \in K$.

Step 2. $[K : k] \leq 2$. We state the contradiction. Assume that $[K : k] \geq 3$, and let $1, \lambda, \mu$ be a free system of K as a k-vector space. Set $W = k + k\lambda + k\mu$ and $I = a\phi^{-1}(W)$ for some $0 \neq a \in M$.

Claim 3. $(W : W) = k$. Indeed, let $f \in (W : W) \subseteq W$, and write $f = \alpha_0 + \alpha_1\lambda + \alpha_2\mu$ for some α_0, α_1 and α_3 in k. By Step 1, $[k(\lambda) : k] = [k(\mu) : k] = 2$. Then,

$$\lambda^2 = \beta_0 + \beta_1\lambda \quad \text{and} \quad \mu^2 = \gamma_0 + \gamma_1\mu,$$

where β_0, β_1 and γ_0, γ_1 are in k with $\beta_0\gamma_0 \neq 0$ (for instance, if $\beta_0 = 0$, $\lambda^2 = \beta_1\lambda$ and so $\lambda = \beta_1 \in k$, which is absurd). Since $\lambda \in W$,

$$\alpha_0\lambda + \alpha_1\lambda^2 + \alpha_2\lambda\mu = f\lambda \in W.$$

Thus,

$$\beta_0\alpha_1 + (\alpha_0 + \alpha_1\beta_1)\lambda + \alpha_2\lambda\mu \in W,$$

and so, $\alpha_2\lambda\mu \in W$. Hence, $\alpha_2 = 0$ for, if not, we obtain $\lambda\mu \in W$. But, since

$$k(\lambda) = k + k\lambda \subseteq W \quad \text{and} \quad k(\mu) = k + k\mu \subseteq W,$$

$k(\lambda, \mu) \subseteq W$. Then, $4 = [k(\lambda,\mu) : k] \leq [W : k] = 3$, a contradiction. Therefore, $\alpha_2 = 0$, and hence, $f = \alpha_0 + \alpha_1\lambda$. Similarly, since $\mu \in W$,

$$\alpha_0\mu + \alpha_1\lambda\mu = f\mu \in W,$$

and thus, $\alpha_1 = 0$. Hence, $f = \alpha_0 \in k$, as desired. Thus, $(W : W^2) = ((W : W) : W) = (k : W) = 0$. Again, by Lemma 5.2, I is not a weakly ES-stable ideal, and this yields a contradiction. It follows that $[K : k] \leq 2$. $\square$

Proof of Theorem 5.1.

(i) Assume that R is a weakly ES-stable domain. By Lemma 2.4, T is a weakly ES-stable domain, and by Lemma 5.3, $[K : qf(D)] \leq 2$. By Corollary 2.6, R is of finite character, and since, for every maximal ideal q of D, $Q = \phi^{-1}(q)$ is a maximal ideal of R containing M and $\mathrm{Max}(R, M)$ is finite, D must be semi-local. It remains to prove that D is a weakly ES-stable domain. Let A be a nonzero ideal of D, and set $I = \phi^{-1}(A)$. Then, I is an ideal of R containing M, and thus, $I = JE$ where $JJ^{-1} = R$ and $E = E^2$. Since $M \subsetneqq I$, $IT = T$. In addition, since $E^2 = E$,

$$\begin{aligned} E \subseteq (E : E) = (I : I) &= (\phi^{-1}(A) : \phi^{-1}(A)) \\ &= \phi^{-1}(A :_K A) \subseteq \phi^{-1}(K) = T. \end{aligned}$$

By Lemma 2.4, $E = I(I : I^2)$, and thus, $I \subseteq E \subseteq T$. Therefore, $T = IT \subseteq ET \subseteq T$, and hence, $IT = ET = T$. Therefore, $JT = JET = IT = T$. Thus, $IT = JT = ET = T$, and hence, $M \subsetneq J$ and $M \subsetneq E$. Therefore, $J = \phi^{-1}(B)$, and $E = \phi^{-1}(F)$ for some nonzero fractional ideals B and F of D. Thus, B is invertible (since $JJ^{-1} = \phi^{-1}(B(D : B))$, $F = F^2$ and clearly $A = BF$ as desired.

(ii) Assume that D is conducive. If R is a weakly ES-stable domain, the conclusion follows from (i). Conversely, assume that V is a weakly ES-stable domain and D is a semi-local weakly ES-stable domain. Let I be a nonzero ideal of R. If $M \subsetneq I$, then $I = \phi^{-1}(A)$ for some nonzero ideal A of D. Since D is a weakly ES-stable domain, $A = BF$ where $B(D : B) = D$ and $F = F^2$. Now it is easy to see that $I = JE$ where

$J = \phi^{-}(B)$ is invertible, and $E = \phi^{-1}(F)$ is idempotent as desired. Assume that $I \subseteq M$. If I is an ideal of V, then I is a weakly ES-stable ideal of V and, a fortiori, a weakly ES-stable ideal of R. Finally, if I is not an ideal of V, then $I = a\phi^{-1}(W)$ where W is a D-submodule of K with $D \subseteq W \subsetneqq K$. But, since D is conducive, W is a fractional ideal of D, and thus, $W = BF$ where $B(D : W) = D$ and $F = F^2$. Set $J = a\phi^{-1}(B)$ and $E = \phi^{-1}(F)$. Clearly $JJ^{-1} = R$, $E = E^2$ and $I = JE$, as desired. $\square$

The next example shows that Theorem 5.1 (ii) cannot be extended to a classical diagram of type ($\square$).

Example 5.4. Let $V = \mathbb{Q}(i)[[X]] = \mathbb{Q}(i) + M$, where i is the complex number with $i^2 = -1$, and set $R = \mathbb{Z}_2 + M$. Let W be the $\mathbb{Z}_2$-submodule of $\mathbb{Q}(i)$ given by $W = \mathbb{Z}_2 + i\mathbb{Q}$. We claim that $(W : W^2) = (0)$. Indeed, $(W : W) = \mathbb{Z}_2$. To see this, let $f \in (W : W)$. Since $1 \in W$, $f \in W$. We write $f = a + ib$ for some $a \in \mathbb{Z}_2$ and $b \in \mathbb{Q}$. Suppose that $b \neq 0$, and let $0 \neq c \in \mathbb{Q} \setminus \mathbb{Z}_2$. Then $i(c/b) \in W$, and thus, $-c + a(c/b)i = f(i(c/b)) \in W$. Thus, $-c \in \mathbb{Z}_2$, which is a contradiction. Hence, $b = 0$, and so, $f = a \in \mathbb{Z}_2$. Thus, $(W : W) = \mathbb{Z}_2$, and hence,

$$\begin{aligned}(W : W^2) &= ((W : W) : W) = (\mathbb{Z}_2 : W) = (\mathbb{Z}_2 : \mathbb{Z}_2 + i\mathbb{Q}) \\ &= (\mathbb{Z}_2 : \mathbb{Z}_2) \cap (\mathbb{Z}_2 : i\mathbb{Q}) = \mathbb{Z}_2 \cap (0) = (0).\end{aligned}$$

By Lemma 5.2, R has an ideal I which is not a weakly ES-stable ideal, as desired.

The next example illustrates Lemma 5.2 and shows how to construct a classical pullback of type $(\square)^*$ which has an ideal of the form $I = a\phi^{-1}(W)$ where W is a D-submodule of K satisfying $(W : W^2) = (0)$. Thus, I is not a weakly ES-stable ideal.

Example 5.5. Let $V = \mathbb{Q}[[X]] = \mathbb{Q} + M$ and $R = \mathbb{Z} + M$, where $M = XV$ is the maximal ideal of V. Let W be the $\mathbb{Z}$-submodule of $\mathbb{Q}$ given by

$$W = \sum_{p \text{ prime}} \frac{1}{p}\mathbb{Z},$$

and consider the ideal $I = X(W+M)$. We claim that $(W:W)=\mathbb{Z}$ and $(\mathbb{Z}:W)=(0)$. Indeed, since $(1/p)\mathbb{Z}\subseteq W$ for every prime p, $(\mathbb{Z}:W)\subseteq(\mathbb{Z}:(1/p)\mathbb{Z})=p\mathbb{Z}$, and thus,

$$(\mathbb{Z}:W)\subseteq \bigcap_{p\,\text{prime}} p\mathbb{Z}=(0).$$

It follows that $(\mathbb{Z}:W)=(0)$. Clearly, $\mathbb{Z}\subseteq(W:W)$. Suppose that $\mathbb{Z}\subsetneqq(W:W)$, and let $f\in(W:W)\setminus\mathbb{Z}$. Since $1\in W$, $(W:W)\subseteq W$, and thus, $f\in W$. We write

$$f=\sum_{i=1}^{n}\frac{a_i}{p_i}$$

for some $a_1,\dots,a_n\in\mathbb{Z}$ and some prime integers $p_1,\dots,p_n$. Without loss of generality, we may assume that $a_i\notin p_i\mathbb{Z}$ for all i. For, if $a_{i_1}\in p_{i_1}\mathbb{Z},\dots,a_{i_s}\in p_{i_s}\mathbb{Z}$ for some $i_1,\dots,i_s\in\{1,\dots,n\}$, then

$$g=\sum_{j=1}^{s}\frac{a_{i_j}}{p_{i_j}}\in\mathbb{Z}\subseteq W,$$

and thus,

$$h=f-g=\sum_{i\neq i_j}\frac{a_i}{p_i}\in W$$

with $a_i\notin p_i\mathbb{Z}$ for all $i\notin\{i_1,\dots,i_s\}$.

Now set

$$\lambda=\prod_{i=1}^{n}p_i$$

and for each $i\in\{1,\dots,n\}$, set

$$\lambda_i=\prod_{j\neq i}p_j.$$

Then, $p_i\lambda_i=\lambda$, and thus, $f=a/\lambda$ where

$$a=\sum_{i=1}^{n}a_i\lambda_i.$$

Note that $a \notin p_i\mathbb{Z}$ for each i since $a_i\lambda_i \notin p_i\mathbb{Z}$. Since $1/p_1 \in W$, $1/p_1 f \in W$. We write

$$\frac{a}{p_1\lambda} = \frac{1}{p_1} f = \sum_{j=1}^{r} \frac{b_i}{q_j} = \frac{b}{\mu}$$

for some $b \in \mathbb{Z}$ and $\mu = \prod_{j=1}^{r} q_j$ with q_j distinct prime integers. Then,

$$a\mu = p_1 b\lambda \in \lambda\mathbb{Z} = \bigcap_{i=1}^{n} p_i\mathbb{Z},$$

and, since $a \notin p_i\mathbb{Z}$,

$$\prod_{j=1}^{r} q_j = \mu \in p_i\mathbb{Z} \quad \text{for each } i.$$

Necessarily $s \geq n$ and, for each $i \in \{1, \ldots, n\}$, $p_i = q_j$ for some $j \in \{1, \ldots, s\}$.

Suppose that $s > n$, and, without loss of generality, assume that $p_i = q_i$ for each $i = 1, \ldots, n$. Then

$$\mu = \lambda \prod_{j=n+1}^{s} q_j.$$

It then follows that

$$a\lambda \prod_{j=n+1}^{s} q_j = a\mu = p_1 b\lambda,$$

and thus,

$$a \prod_{j=n+1}^{s} q_j = p_1 b \in p_1\mathbb{Z}.$$

But, since $a \notin p_1\mathbb{Z}$,

$$\prod_{j=n+1}^{s} q_j \in p_1\mathbb{Z},$$

and thus, $q_1 = p_1 = q_j$ for some $j \in \{n+1, \ldots, s\}$, which is a contradiction. Hence, $s = n$, and therefore, $\mu = \lambda$. Thus, $a\lambda = a\mu = p_1 b\lambda$, and so, $a = p_1 b \in p_1\mathbb{Z}$, which is absurd. Hence, $(W : W) = \mathbb{Z}$.

Therefore,

$$(W : W^2) = ((W : W) : W) = (\mathbb{Z} : W) = 0,$$

and, by Lemma 5.2, $I = X(W + M)$ is not a weakly ES-stable ideal.

Corollary 5.6. *Let $A \subsetneq B$ be an extension of integral domains, X an indeterminate and $R = A + XB[X]$ (respectively, $R = A + XB[[X]]$). If R is a weakly ES-stable domain, then $B = K$ is a field, A is a semi-local domain which is a weakly ES-stable domain and $[K : qf(A)] \leq 2$.*

Proof. Since $XB[X] \subseteq (R : B[X])$, by Lemma 2.4, $B[X]$ is a weakly ES-stable domain, and, by Corollary 2.7, $B = K$ is a field. Now, by Theorem 5.1, A is a semi-local weakly ES-stable domain and $[K : qf(A)] \leq 2$. The proof is similar for $R = A + XB[[X]]$. □

Recall from **[23]** that R is a pseudo-valuation domain if R is local and shares its maximal ideal with a valuation overring V. In view of **[4,** Proposition 2.6], R is a pullback determined by the diagram of canonical homomorphisms:

$$\begin{array}{ccc} R = \phi^{-1}(k) & \twoheadrightarrow & k \\ \downarrow & & \downarrow \\ V & \underset{\varphi}{\twoheadrightarrow} & K := V/M, \end{array}$$

where M is the maximal ideal of V and k is a subfield of K. A fortiori, M is the maximal ideal of R with residual field k.

The next corollary characterizes PVD domains that are weakly ES-stable domains.

Corollary 5.7. *Let R be a PVD, V its associated valuation overring, M its maximal ideal, and set $K = V/M$ and $k = R/M$. Then, R is a weakly ES-stable domain if and only if V is a weakly ES-stable domain and $[K : k] \leq 2$.*

Proof.

$\Rightarrow$. This follows immediately from Theorem 5.1.

$\Leftarrow$. Let I be a nonzero ideal of R. If I is an ideal of V, then I is a weakly ES-stable ideal of V and, a fortiori, a weakly ES-stable ideal of R.

Assume that I is not an ideal of V. Then, as in [**7,** Theorem 2.1], $I = a\phi^{-1}(W)$ where $k \subseteq W \subsetneq K$. (Note here that if $a \in I$ and $\alpha \in V$ such that $a\alpha \notin I$, then it is easy to check that $IV = aV$. Next, set

$$W = \{\lambda \in K \mid \text{if } \lambda = \phi(x), \text{ then } xa \in I\},$$

and easily check that $I = a\phi^{-1}(W)$). Now, since $[K : k] \leq 2$, $W = k$, and thus, $I = aR$, as desired. $\square$

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