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HENSEL'S LEMMA AND THE INTERMEDIATE VALUE THEOREM OVER A NON-ARCHIMEDEAN FIELD

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ABSTRACT. This paper proves that all power series over a maximal ordered Cauchy complete non-Archimedean field satisfy the intermediate value theorem on every closed interval. Hensel's lemma for restricted power series is the main tool of the proof.

1. Introduction. It is well known that, over a complete Archimedean field, i.e., the field $\mathbb{R}$ of real numbers, the intermediate value theorem holds for every function continuous on a closed interval. The proof is strongly based on the Archimedean property, or better, on the dichotomy procedure, which is a consequence of it.

Over a non-Archimedean field $\mathbb{K}$ the theorem is false for the class of all continuous functions, see [**5,** Example 4.1], even in the event that $\mathbb{K}$ is maximal ordered and Cauchy complete. Nevertheless, it was proven long ago for polynomials and rational functions defined over a maximal ordered, not necessarily complete, non-Archimedean field, see [**2,** Section 2, Proposition 5].

In [**5**], we proved that, when $\mathbb{K}$ is maximal ordered, the theorem can be extended to any power series which is algebraic over the field $\mathbb{K}(X)$ of rational functions, with a direct proof that makes use of the theorem for polynomials.

In the present paper, we investigate the intermediate value theorem for an arbitrary power series over a non-Archimedean maximal ordered

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and complete field $\mathbb{K}$. In order to attain the result we use Hensel's lemma, in its strong version, for the ring of restricted power series. In fact, we apply it to the valuation ring of all elements of $\mathbb{K}$ which are not infinitely large with respect to $\mathbb{Q}$ (or to any other subfield $\mathbb{L}$ with which $\mathbb{K}$ is non-comparable). Since the maximal ideal of infinitesimals may contain no topologically nilpotent element, we need the stronger version of Hensel's lemma proved in **[22]**.

We want to point out that, if we consider separately the cases of the maximal ideal with or without topologically nilpotent elements, two proofs that are quite different can be given. With topologically nilpotent elements the Bourbaki version of Hensel's lemma for restricted power series is the right tool, while the theorem in the other case can be proved without use of the Hensel lemma; this approach is based on the construction of a Cauchy sequence of roots of the partial sums which converges to one root of the series.

Since such a proof fails when there are topologically nilpotent elements, we also show (in general, with or without nilpotents) that each root of odd order of a power series is the limit of a sequence of roots of the partial sums (the order, or multiplicity, being defined in [**5,** Theorem 3.11]). The even order behaves quite differently; in fact, it may occur, and sometimes does, that the root of the series is only the limit of a sequence of extremes of the partial sums.

In regards to the above proofs, we also want to emphasize the following fact: the approach when there are no topologically nilpotent elements shows that, for every change in sign, at least one root of the series is the limit of a suitable sequence of roots of the partial sums, while the general approach based on Hensel's lemma as stated in **[22]** shows that all the roots in the interval are the roots of a suitable polynomial which is a factor of the series but gives, to our knowledge, no information about their approximation by roots of the partial sums.

It is worth observing, in **[5]** we proved that *algebraic* power series satisfy the intermediate value theorem on a closed interval of a *maximal ordered* field, here we prove that the intermediate value theorem holds for *general* power series over a *maximal ordered and complete* field.

We show that, if the intermediate value theorem holds true in $\mathbb{K}$ for every power series, then $\mathbb{K}$ must be Cauchy complete.

As for the cardinality of the set of zeros, we show that, on the whole field $\mathbb{K}$, infinitely many roots may occur, but they shrink to finitely many on the subring of the elements that are not infinitely large with respect to a subfield with which $\mathbb{K}$ is non-comparable.

Standard corollaries of the intermediate value theorem are Rolle's theorem, the mean value theorem and the extreme value theorem. Here, we briefly review them for a general power series.

We want to recall that the theory of non-Archimedean ordered fields goes back to the 19th century and was introduced by Veronese and Levi-Civita, see [**11, 12**], and also [**1, 21**]. As for the intermediate value theorem on the Levi-Civita field of functions from $\mathbb{Q}$ to $\mathbb{R}$ with left-finite support, a proof is given in [**20**], while in [**10, 18**], a quite different point of view is considered.

2. Notation and general facts. Unless otherwise stated, $\mathbb{K}$ is a *non-Archimedean, maximal ordered, complete field* (for ordered fields and completions in general, we refer the reader to [**2, 9, 23**]). We recall that $\mathbb{K}$ is called *maximal ordered* if every ordered algebraic extension of $\mathbb{K}$ coincides with $\mathbb{K}$ ([**2,** Section 2, Definition 4]) and that $\mathbb{K}$ is maximal ordered if and only if every positive element of $\mathbb{K}$ is a square and moreover every odd degree polynomial over $\mathbb{K}$ has a root in $\mathbb{K}$ ([**2,** Section 2, Theorem 3]).

We assume throughout that the order topology has a countable basis for the neighborhoods of 0 (see [**8,** page 50, Chapter I] and [**13,** page 335, X]). An element $x \in \mathbb{K}$ is *topologically nilpotent* if $\lim_{n\to\infty} x^n = 0$, see [**7,** page 19].

The following cases may occur, see [**5,** Section 3]:

(1) there is a topologically nilpotent element $\epsilon \in \mathbb{K}$;
(2) there is a sequence $(\epsilon_0 > \epsilon_1 > \epsilon_2 > \cdots)$, converging to 0, such that, for all n, $\epsilon_n > 0$ and for all n and i, $\epsilon_n^i > \epsilon_{n+1}$. We always choose $\epsilon_0 = 1$.

In case (1), The sets $U_n = \{x \in \mathbb{K}, |x| < \epsilon^n\}$ give a basis for the neighborhoods of 0, while, in case (2), we need the sets $\overline{U}_n = \{x \in \mathbb{K}, |x| < \epsilon_n\}$.

For every non-Archimedean ordered field $\mathbb{L}$, $\overline{\mathbb{L}}$ denotes the ordered closure and $\widehat{\mathbb{L}}$ the Cauchy completion.

If $S(X) = \sum a_n X^n$ is a power series over $\mathbb{K}$, we denote the nth partial sum $\sum_{i=0}^{n} a_i X^i$ by $S_n(X)$.

If $\mathbb{L} \subset \mathbb{K}$ is a subfield such that $\mathbb{K}$ contains at least one element x larger than every $a \in \mathbb{L}$, we say that $\mathbb{K}$ is non-comparable with $\mathbb{L}$, or non-Archimedean over $\mathbb{L}$. For instance, $\mathbb{K}$ is non-comparable with $\mathbb{Q}$.

An element $x \in \mathbb{K}$ is called infinitely large with respect to $\mathbb{L}$ if $|x| > a$ for all $a \in \mathbb{L}^+$ (the set of all positive elements). An element $y \in \mathbb{K}$, $y \neq 0$, is called infinitely small (or infinitesimal) with respect to $\mathbb{L}$ if $|y| < a$ for all $a \in \mathbb{L}^+$. If x is infinitely large, then $1/x$ is infinitely small, and conversely. When $\mathbb{L} = \mathbb{Q}$, *we simply say that x is infinitely large and y is infinitely small or infinitesimal.* If $x \in \mathbb{K}$ is algebraic over the subfield $\mathbb{L}$, then x is neither infinitely large nor infinitely small with respect to $\mathbb{L}$, see [**2,** page 57, Exercise 14]. The same is true for every $y \in \widehat{L}$ (as a consequence of the definition of order in the completion, see [**23,** page 67]).

Given $\mathbb{L} \subset \mathbb{K}$ such that $\mathbb{K}$ is non-comparable with it, we set:

$$A_{\mathbb{L}} = \{x \in \mathbb{K}, x \text{ is not infinitely large with respect to } \mathbb{L}\},$$
$$M_{\mathbb{L}} = \{x \in \mathbb{K}, x \text{ is infinitely small with respect to } \mathbb{L}\}.$$

Then $A_{\mathbb{L}}$ is a subring of $\mathbb{K}$ [**2,** page 53, Exercise 1] and $M_{\mathbb{L}}$ is a maximal ideal of $A_{\mathbb{L}}$ [**2,** page 57, Exercise 11 b)]. Moreover, $A_{\mathbb{L}}$ is a valuation ring, since either x or $1/x$ belongs to $A_{\mathbb{L}}$ for all $x \in \mathbb{K}\backslash\{0\}$ [**4,** Chapter 6, Theorem 1].

Remark 2.1. Let us assume that there is a sequence (ϵ_n) converging to 0 but not a single ϵ such that $\lim_{n\to\infty} \epsilon^n = 0$.

(a) Given a subfield $\mathbb{L}$ over which $\mathbb{K}$ is non-Archimedean, we can assume that all the ϵ_n's (except $\epsilon_0 = 1$) are infinitesimal with respect to $\mathbb{L}$ (by possibly discarding finitely many of them).

It follows that every $\omega_n = 1/\epsilon_n$, $n \geq 1$, is infinitely large with respect to $\mathbb{L}$.

(b) It follows that $\epsilon_1\epsilon_n > \epsilon_{n+1}$, since $\epsilon_1 > \epsilon_n$ implies $\epsilon_1\epsilon_n > \epsilon_n^2 > \epsilon_{n+1}$ (this inequality actually follows from case (2) above).

(c) It is easy to see that each ϵ_{i+1} is transcendental over $\mathbb{L}(\epsilon_1, \epsilon_2, \dots, \epsilon_i) \subset \mathbb{K}$, since algebraic elements cannot be infinitesimal.

Remark 2.2. If there is a topologically nilpotent ϵ, then ϵ does not belong to every $\mathbb{L}$ with which $\mathbb{K}$ is non-comparable. In fact, $\epsilon \in \mathbb{L}$ implies $\epsilon^n \in \mathbb{L}$ for all n, in contradiction with the non-Archimedean assumption on $\mathbb{K}$ (choose $x \in \mathbb{K}$ infinitely small with respect to $\mathbb{L}$, hence less than every ϵ^n, and x must be 0).

A local ring (A, M) is called Henselian if the following property holds. Let $P(X) \in A[X]$ be a polynomial such that its canonical image $\overline{P}(X)$ into the quotient ring $(A/M)[X]$ is the product $\overline{Q}(X)\overline{T}(X)$ of a monic polynomial $\overline{Q}(X)$ and another polynomial $\overline{T}(X)$, the two factors being coprime. Then $P(X) = Q(X)T(X)$, where $Q(X)$ is a monic polynomial that lifts $\overline{Q}(X)$, and $T(X)$ is a polynomial that lifts $\overline{T}(X)$. Moreover, $P(X)$ and $Q(X)$ are uniquely determined and coprime.

Hensel's lemma states that a complete local ring is Henselian, see [**17,** page 103, Chapter V, Section 30]. There is a wide class of Henselian rings, see for instance, [**6**].

A monic polynomial $X^r + \cdots + c_1 X + c_0$ is an *N-polynomial* if $c_0 \in M$, $c_1 \notin M$. A local ring is *Henselian* if and only if every N-polynomial has a root in M ([**17,** page 179, Chapter V, Section 43] and also [**6,** Theorem 5.11]).

If $S(X)$ is defined over a topological ring A with a linear topology, i.e., with a basis of the neighborhoods of 0 formed with ideals, $S(X)$ is called *restricted* if $\lim_{n\to\infty} a_n = 0$, see [**7,** page 18].

Hensel's lemma can also be given for a restricted power series (see [**3,** Chapter 3, Section 4, Theorem 1] and [**7,** page 19, Theorem 3.7]).

Lemma 2.3 (Hensel's lemma for a restricted power series)**.** *Let A be a complete, separated ring with respect to a linear topology, M a closed ideal whose elements are topologically nilpotent, $S(X)$ a restricted power series such that its canonical image $\overline{S}(X)$ into the topological quotient ring $(A/M)\{X\}$ is the product $\overline{P}(X)\overline{T}(X)$ of a monic polynomial $\overline{P}(X)$ and a restricted series $\overline{T}(X)$, the two factors being coprime. Then $S(X) = P(X)T(X)$, where $P(X)$ is a monic polyno-*

mial that lifts $\overline{P}(X)$, and $T(X)$ is a restricted series that lifts $\overline{T}(X)$. Moreover, $P(X)$ and $T(X)$ are uniquely determined and coprime.

3. Non-comparable subfields and some topology. In this section, we investigate some topological properties of the valuation ring of non-infinitely large elements that follow from the ordering of the field $\mathbb{K}$, with special focus on topologically nilpotent elements.

Lemma 3.1. *Let Y be the set of all subfields $\mathbb{L} \subset \mathbb{K}$ with which $\mathbb{K}$ is non-comparable. If there is a topologically nilpotent element ϵ, then Y contains a maximal element, which is both maximal ordered and Cauchy complete.*

Proof. Y is not empty and can be ordered by inclusion. First of all we show that, if F is a subset of Y, then $\mathbb{L}' = \cup F$ belongs to Y. In order to prove this property let us recall (Remark 2.2 above) that no $\mathbb{L} \in Y$ can contain ϵ. As a consequence, $\epsilon \notin \mathbb{L}'$ so that $\mathbb{K}$ is non-comparable with $\mathbb{L}'$.

Therefore, by Zorn's lemma, Y has a maximal element $\mathbb{L}$. Such a subfield is both maximal ordered and complete, since the ordered closure and the completion of an ordered field are comparable with it. $\square$

Proposition 3.2. *If $\mathbb{K}$ contains a topologically nilpotent element ϵ and $\mathbb{L}$ is maximal as above, then every element of $M_{\mathbb{L}}$ is topologically nilpotent.*

Proof. Let x be in $M_{\mathbb{L}}$. Put $y = 1/x$, and observe that $|y| > |h|$ for all $h \in \mathbb{L}$ so that $y \notin \mathbb{L}$, which implies that y is transcendental over $\mathbb{L}$ (every algebraic element over $\mathbb{L}$ is comparable with $\mathbb{L}$). Let us assume that $y > 0$, and consider the field $\mathbb{L}(y) \subset \mathbb{K}$. Since $\mathbb{L}$ is maximal with the property that $\mathbb{K}$ is non-comparable with it, $\mathbb{L}(y)$ is comparable with $\mathbb{K}$, i.e., every $k \in \mathbb{K}$ is neither infinitely large nor infinitely small with respect to $\mathbb{L}(y)$, so that there is a rational function $P(y)/Q(y)$ which is larger than $\omega = 1/\epsilon$. However, in the unique ordering of $\mathbb{L}(y)$, see [**16,** Section 3], every rational function is less than some power of y^m, so that $\omega < y^m$ for some $m \geq 1$. Therefore,

$$\lim_{n\to\infty} x^n = \lim_{n\to\infty} \frac{1}{y^m} = 0. \qquad \square$$

Remark 3.3. Observe that $\mathbb{K}$ is a topological field and every subring is a topological ring.

Let $\mathbb{L}$ be any subfield with which $\mathbb{K}$ is non-comparable. In all cases, with or without topologically nilpotent elements, the ring $A_{\mathbb{L}}$, equipped with the topology induced by the ordering of $\mathbb{K}$, is a *topological ring*, i.e., the functions $(x, y) \to (x+y), (x, y) \to xy$ and $x \to -x$ are continuous. Moreover, it is *Hausdorff* since, if $|x| \leq \epsilon^n$, respectively ϵ_n, for all n, then $x = 0$, see [**7,** page 2].

Lemma 3.4. *For every subfield $\mathbb{L}$ such that $\mathbb{K}$ is non-comparable with it, the topology of $A_{\mathbb{L}}$ is linear, i.e., there is a basis of the neighborhoods of* 0 *whose elements are ideals of $A_{\mathbb{L}}$.*

Proof.

Case 1. There is a sequence of infinitesimal elements (respectively, ϵ_n, $n \in \mathbb{N}$) and no nilpotent element. Set $U_n = \{x \in A_{\mathbb{L}}, |x| < \epsilon_n\}$, $V_n = \epsilon_n A_{\mathbb{L}}$. It is enough to show that

$$U_{n+1} \subset V_{n+1} \subset U_n \quad \text{for all } n.$$

Step 1. $U_{n+1} \subset V_{n+1}$. Assume that $x \in U_{n+1}$. This means that $a = x/\epsilon_{n+1} \in A_{\mathbb{L}}$; thus, $x \in V_{n+1}$.

Step 2. $V_{n+1} \subset U_n$. Let x be in V_{n+1}, i.e., $x = a\epsilon_{n+1}$ for some $a \in A_{\mathbb{L}}$, and set $\omega_1 = 1/\epsilon_1$. Since $|a| < \omega_1$ for all $a \in A_{\mathbb{L}}$, see Remark 2.1 (a), it follows that $|x| < \epsilon_{n+1}\omega_1 < \epsilon_n$ by our choice of the infinitesimals, which proves the claim.

Case 2. There is a topologically nilpotent element ϵ.

The proof above works if we replace ϵ_n by ϵ^n for all $n \in \mathbb{N}$ (and use Remark 2.2). $\square$

Proposition 3.5. *For every subfield $\mathbb{L}$ that is non-comparable with $\mathbb{K}$, the following hold true*:

(i) *$A_{\mathbb{L}}$ is closed in $\mathbb{K}$ and Cauchy complete.*

(ii) *$M_{\mathbb{L}}$ is closed in $A_{\mathbb{L}}$.*

Proof.

(i) It is enough to prove that $A_{\mathbb{L}}$ is complete. Let (c_n) be any Cauchy sequence with $c_n \in A_{\mathbb{L}}$ for all n. Hence, there is a $c \in \mathbb{K}$ such that $c = \lim_{n\to\infty} c_n$. This implies that, for every infinitesimal h, the open interval $]c-h, c+h[$ contains at least one element c_N, i.e., $c - c_N = \eta$, with η infinitely small and thus belonging to $M_{\mathbb{L}} \subset A_{\mathbb{L}}$. We obtain that $c - c_N = \eta \in A_{\mathbb{L}}$, i.e., that $c = c_N + \eta \in A_{\mathbb{L}}$.

(ii) Choose $d \in A_{\mathbb{L}}$ such that every open neighborhood of infinitesimal radius h of $d \in A_{\mathbb{L}}$ contains some $x \in M_{\mathbb{L}}$, $x \neq d$. This means that $|x - d| < h$, i.e., $x - d \in M_{\mathbb{L}}$; thus, $d \in M_{\mathbb{L}}$. □

We will consider the special case $\mathbb{L} = \mathbb{Q}$, since we know that $\mathbb{Q}$ is a subfield with which $\mathbb{K}$ is non-comparable. The ring $A = \{x \in \mathbb{K}, x$ is not infinitely large over $\mathbb{Q}\}$ is a valuation ring, as we have already seen, and $M = \{x \in A, x$ is infinitely small over $\mathbb{Q}\}$ is its maximal ideal.

4. Changing the interval and transforming $S(X)$ into a restricted power series over A. We want to show that every power series converging in some closed interval can be transformed into a restricted power series over the ring of non-infinitely large elements, the simple tools being a linear change of variable and the multiplication by a suitable element in $\mathbb{K}$. The series so acquires a few good properties which are useful in what follows.

Proposition 4.1. *Let $S(X)$ be a power series defined over a set $D_S \subset \mathbb{K}$ containing the closed interval $[a, b]$. Then there is a linear change of variable $X = hZ + k$ such that*

(i) *$[a, b]$ is mapped one-to one onto $[1, 2]$, and $X = a$ corresponds to $Z = 1$, $X = b$ corresponds to $Z = 2$,*

(ii) *$S(hZ + k) = T(Z)$ is a power series whose domain contains $[1, 2]$.* *Moreover, there are $d \in \mathbb{K}$ and $N \in \mathbb{N}$, such that:*

(iii) *$dT(Z)$ is a restricted power series over the ring A of non-infinitely large elements and $da_N = 1, da_{N+h} \in M$ for all $h \geq 1$.*

Proof.

(i) and (ii) We set $X = (b-a)Y$, obtaining a series in the variable Y, say

$$U(Y) = \sum a_n (b-a)^n Y^n.$$

$U(Y)$ is convergent at least on $[a/(b-a), b/(b-a)]$.

Now, we set $\overline{k} = (2a-b)/(b-a)$ and operate the translation $Y = Z + \overline{k}$. This translation is allowed if $U(\overline{k})$ is a converging series, see [**5**], Theorem 3.7. This holds true since $U(Y)$ is convergent both at $a/(b-a)$ and at $b/(b-a)$, thus also at $2a/(b-a)$ and at $2a/(b-a) - b/(b-a)$, see [**5,** Theorem 3.3]. Hence,

$$T(Z) = S((b-a)Z + (2a-b)) = U(Y)$$

is convergent on $[1, 2]$, see [**5,** Theorem 3.7], and $S(a) = T(1), S(b) = T(2)$, so that (ii) is fulfilled with $h = b - a, k = 2a - b$.

(iii) Set $T(Z) = \sum t_n Z^n$. We have $\lim_{n\to\infty} t_n = 0$ because the series is convergent at $Z = 1$ (see [**5,** General facts, Theorem 2.1] and [**13,** page 335, XII]), and thus, only finitely many coefficients lie outside of A, since A contains all of the infinitely small elements.

Now let $|t_h| = a$ be the largest among all of the absolute values $|t_n|$. Then, $b_n = t_n/a$ is not infinitely large, and thus,

$$H(X) = \frac{S(X)}{a} = \sum b_n X^n$$

is a power series over A such that $\lim_{n\to\infty} b_n = 0$ (it is convergent at 1). Therefore, $H(X) \in A\{X\}$ is a ring of restricted power series over A. Observe that $|b_h| = 1$ implies $H(X) \notin M\{X\}$ (restricted series with all coefficients in M). In this event, there is the largest integer, say N, such that $b_N \notin M$. As a consequence, b_N is invertible in A, and we can consider the following series:

$$V(X) = \frac{H(X)}{b_N} = \sum c_n X^n,$$

which is still restricted over A and has $c_N = 1, c_n \in M$ for all $n > N$. Therefore, $d = 1/ab_N$. $\square$

Remark 4.2.

(i) If $[a, b]$ is any interval, we can transform $S(X)$ into another series $T(Z)$ converging in $[1, 2]$ and then we can replace $T(Z)$ by $dT(Z)$. Observe that $T(1)T(2) < 0$ if and only if $S(a)S(b) < 0$.

(ii) It is worth pointing out that there is a one-to-one correspondence between the zeros of $S(X)$ in $[a, b]$ and the zeros of $T(Z)$ in $[1, 2]$.

(iii) Obviously, the above proof works if $[1, 2]$ *is replaced by any interval whose endpoints are neither infinitely large nor infinitely small.*

5. Hensel's lemma for restricted power series. In the present section, $\mathbb{K}$ is *maximal ordered and complete, with the exception of the following Proposition* 5.1*, which holds true for a maximal ordered field* $\mathbb{K}$. We choose a subfield $\mathbb{L} \subset \mathbb{K}$ such that $\mathbb{K}$ is not comparable with it, for instance, $\mathbb{L} = \mathbb{Q}$. We want to show that Hensel's lemma for restricted power series [**7,** page 19] holds on the local ring A of elements which are non-infinitely large with respect to $\mathbb{L}$, even if there is no topologically nilpotent element in the maximal ideal M. By [**22,** Theorem 5], it is enough to show that (A, M) is a Henselian pair.

Proposition 5.1. *Let* (A, M) *be a valuation ring of a maximal ordered, not necessarily complete, field* $\mathbb{K}$*. Then* (A, M) *is a Henselian pair.*

Proof. Since A is a local ring, it is enough to prove that every N-polynomial $P(X) = X^r + c_{r-1}X^{r-1} + \cdots + c_1X + c_0 \in A[X]$, i.e., with $c_0 \in M, c_1 \notin M$, has a root in M, see Section 2 and [**22,** Section 1].

First we observe that, if the polynomial has degree $r = 1$, then $P(X) = X + c_0$ has root $-c_0 \in M$.

Then, we consider the case of a degree 2 polynomial. If $P(X) = X^2 + 2bX + c$ is any N-polynomial, then it has two roots in $\mathbb{K}(i)$ (algebraic closure of $\mathbb{K}$, see [**2,** Section 2, Theorem 3]): $a = -b + \sqrt{b^2 - c}$, $a' = -b - \sqrt{b^2 - c}$. Since $b \notin M$, $c \in M$, $b^2 - c$ is positive, the two roots actually belong to $\mathbb{K}$ (which is maximal ordered), and hence, also to A, which is integrally closed as a valuation ring [**4,** Chapter 6, Section 1, Corollary 1]. Therefore, there is no irreducible degree 2 N-polynomial.

Now we point out that, if an N-polynomial $p(X) = X^m + \cdots + p_1X + p_0$ is the product of two factors $q(X) = X^s + \cdots + q_1X + q_0$, $r(X) = X^h + \cdots + r_1X + r_0$, then one and only one between the two factors is an N-polynomial. In fact, we have $q_0r_0 = p_0 \in M$ so that one factor must belong to M, say q_0. In this event, $q_0r_1 + q_1r_0 = p_1 \notin M$; thus, neither q_1 nor r_0 can belong to M. Therefore, $q(X)$ is an N-polynomial while $r(X)$ is not. This implies, in particular, that a degree two N-polynomial has exactly one root in M.

Now assume that $r \geq 3$. Since $\mathbb{K}$ is maximal ordered, $P(X)$ is the product of linear factors, say $P_1(X), \ldots, P_h(X)$, and irreducible second degree factors, say $Q_1(X), \ldots, Q_s(X)$, see [**2,** Section 2, Proposition 9]. Since A is integrally closed, all factors have coefficients in A, see [**3,** Chapter 5, Section 1, Proposition 11]. Therefore, we obtain that one and only one among the linear factors which is an N-polynomial, since the second degree irreducible factors cannot be N-polynomials. Such a factor has the required root. $\square$

Corollary 5.2. *A satisfies Hensel's lemma for restricted power series.*

Proof. This is [**22,** Theorem 5] since A is complete and Hausdorff with respect to a linear topology, M is closed and (A, M) is a Henselian pair. $\square$

Corollary 5.3. *Let* $S(X) = \sum_{n=0}^{\infty} a_nX^n$ *be a restricted power series over* A *such that the partial sum* $S_N(X)$ *is a monic polynomial for some* N*, and moreover,* $a_{N+h} \in M$ *for all* $h \geq 1$*. Then,* $S(X) = P(X)B(X)$*, where* $P(X)$ *is a monic polynomial such that* $P(X) = S_N(X) \bmod M$ *and* $B(X) \in 1 + M\{X\}$ *is a restricted power series.*

Proof. The proof is essentially [**19,** Theorem 10]. In fact, the proof of this theorem only makes use of Hensel's lemma for restricted power series, applied to the decomposition $\pmod{M} : \overline{S}(X) = \overline{S}_N(X) \cdot \overline{1}$. Corollary 5.3 states that such a lemma holds without topologically nilpotent elements. $\square$

Corollary 5.4. *Let* $S(X) = \sum_{n=0}^{\infty} a_nX^n$ *be a power series defined in the closed interval* $[a, b]$*. Then,* $S(X)$ *has only finitely many zeros.*

Proof. Due to Proposition 4.1, we may assume that the partial sum $S_N(X)$ is a monic polynomial for some N, while $a_{N+h} \in M$ for all $h \geq 1$. By Corollary 5.3, we obtain $S(X) = P(X)B(X)$, where $P(X)$ is a polynomial and $B(X) \in 1 + M\{X\}$ cannot have roots; therefore, $S(X)$ vanishes where a polynomial vanishes. □

Remark 5.5. In the above results, any field $\mathbb{L}$ such that $\mathbb{K}$ is non-comparable with it works. We obtain Corollary 5.3 with A is the ring of elements that are not infinitely large with respect to $\mathbb{L}$.

6. The intermediate value theorem.

Theorem 6.1. *Let $S(X)$ be a power series over $\mathbb{K}$ converging at least in $[a, b]$ and such that $S(a)S(b) < 0$. Then there is $c \in]a, b[$ such that $S(c) = 0$.*

Proof. Due to Proposition 4.1, we may assume that

(i) $a = 1, b = 2$;

(ii) $S(X)$ is a power series over the local ring (A, M), where $A = A_{\mathbb{L}}$ is the ring of elements that are not infinitely large over any subfield $\mathbb{L} \subset \mathbb{K}$, with which $\mathbb{K}$ is non-comparable, while $M = M_{\mathbb{L}}$ is the maximal ideal of all elements that are infinitely small over $\mathbb{L}$;

(iii) $S(X)$ is restricted because it is convergent at $X = 1$, which implies $\lim_{n\to\infty} a_n = 0$;

(iv) $S(X)$ has a coefficient $a_N = 1$ with the property that $a_m \in M$ for all $m > N$.

Therefore, by Corollary 5.3, $S(X) = P(X)B(X)$, where $P(X)$ is a monic polynomial over A such that $\overline{P}(X) = \overline{S}(X), B(X) = \sum b_n X^n$ is the restricted power series over A belonging to $1 + M\{X\}$. If $x \in [a, b]$, then $B(x) > 0$, since $B(x) = 1 + m$ for a suitable infinitesimal $m \in M$.

Now assume that $S(a)S(b) < 0$. Since $Q(x) > 0$ everywhere in the interval, we must have $P(a)P(b) < 0$, and so, by the intermediate value theorem for polynomials, there is a $c \in]a, b[$ such that $P(c) = 0$. This implies $S(c) = 0$. □

Remark 6.2. In the proof above, any $\mathbb{L}$ such that $\mathbb{K}$ is non-comparable with it can work, in particular, $\mathbb{L} = \mathbb{Q}$. When $\mathbb{K}$ contains a topologically nilpotent element ϵ, the proof of the intermediate value theorem may be based upon Hensel's lemma as stated in [**7,** page 19], provided that we choose a maximal subfield $\mathbb{L}$ with which $\mathbb{K}$ is non-comparable. It is, in fact, enough to observe that Hensel's lemma for restricted power series holds true, since A is equipped with a linear topology, see Lemma 3.4, and it, along with the maximal ideal M (Lemma 3.5), is Hausdorff and complete, while every element of M is topologically nilpotent (Lemma 3.2). Therefore, the proof based upon the decomposition $\mathrm{mod}\, M$ works.

7. An alternative proof without topologically nilpotent elements and without Hensel's lemma. $\mathbb{K}$ is, as usual, maximal ordered and complete. We assume that $\mathbb{K}$ contains no nilpotent element but has a countable basis for the neighborhoods of 0, i.e., there is a sequence $B = (\epsilon_0 = 1 > \epsilon_1 > \epsilon_2 > \cdots)$, where ϵ_i, $i > 0$, is infinitesimal (see Notation and [**15,** page 704]). We also recall (see end of Section 3) that A is the ring of all elements of $\mathbb{K}$ that are not infinitely large with respect to $\mathbb{Q}$.

From now on, for all $i \in \mathbb{N}^+ = \mathbb{N} - \backslash\{0\}$, we set:

$$A_i = \{x \in \mathbb{K}, |x| < \epsilon_i^{-1/n} \quad \text{for all } n \in \mathbb{N}^+\},$$
$$M_i = \{x \in \mathbb{K}, |x| < \epsilon_i^{1/n} \quad \text{for some } n \in \mathbb{N}^+\}.$$

It is easy to see that A_i is a ring for all $i \in \mathbb{N}^+$. Indeed, if $x, y \in A_i$, then $|xy| < \epsilon_i^{-1/n-1/m}$ for all n, m, and thus, $|xy| < \epsilon_i^{-1/p}$ for all p, since, given any $p \geq 1$, there are n, m such that $1/n + 1/m < 1/p$. Moreover, $x + y \in A_i$. In fact, we can assume that $0 < x \leq y$, obtaining that $x + y \leq 2y$. Now we can apply the property of the product already proved.

The ring A_i is, for every i, a valuation ring since, if $x \notin A_i$, then $1/x \in M_i \subset A_i$. Moreover, M_i is its maximal ideal.

Observe that the following inclusions hold true:

$$A_i \subset A_{i+1}, M_{i+1} \subset M_i \quad \text{for all } i \in \mathbb{N}^+.$$

As a consequence, for all $i \geq 1$ and for all $h \geq 1$, $M_{i+h} = M_{i+h} \cap A_i$ is a prime ideal of A_i.

Since $\cup_{i=0}^{\infty} A_i = \mathbb{K}$, for every closed interval $[a, b] \subset \mathbb{K}$, there is an $i \in \mathbb{N}$ such that $[a, b] \subset A_i$.

Moreover, the ideals M_j's define the topology of $\mathbb{K}$ and also of every A_i. In fact, set $V_i = \{x \in \mathbb{K}, |x| < \epsilon_i\}$. Then, it is easy to see that $V_{i+1} \subset M_{i+1} \subset V_i$.

Lemma 7.1. *The following hold true*:

1. *for all* $i \in \mathbb{N}^+$, A_i *is complete and* M_i *is closed*;
2. *for all* $i \in \mathbb{N}^+$, A_i/M_i *is maximal ordered and complete.*

Proof.

1. Assume that $\lim_{n\to\infty} a_n = a$, where $a \in \mathbb{K}$, $a_n \in A_i$, and choose any $r \in \mathbb{N}$. In order to show that $|a| < \epsilon_i^{-1/r}$ for all $r \in \mathbb{N}^+$, it is sufficient to select n such that $|a - a_n| < \epsilon_i$. In fact, we obtain:

$$|a| \leq |a - a_n| + |a_n| < \epsilon_i + |a_n| < \epsilon_i^{-1/r}$$

(A_i is a ring and contains the sum of its elements ϵ_i and $|a_n|$).

In a similar manner, we show that M_i is closed.

2. The order in the quotient field is defined as follows: $x + M_i > 0$ if and only if $x > 0$, $x \notin M_i$. Now, observe that every positive $x + M_i$ is a square. In fact, $x = y^2$, where $y \in \mathbb{K}$, and thus, $y \in A_i$ since it is integral over the valuation ring A_i. Therefore, $x + M_i = (y + M_i)^2$. Moreover, let $P(X) = a_0 + \cdots + X^n \in A_i[X]$ be a monic polynomial of odd degree that lifts the monic polynomial $\overline{P}(X) = \overline{a}_0 + \cdots + X^n \in (A_i/M_i)[X]$. Then, $P(X)$ has a root z belonging to $\mathbb{K}$, and thus, to A_i. Hence, $\overline{z}$ is the required root. Thus, A_i/M_i is maximal ordered, see Section 2.

Since A_i is complete and M_i is closed, the quotient is complete. □

Remark 7.2. Due to Proposition 4.1 and Remark 4.2, in order to prove the intermediate value theorem, we can assume that $a = 1$, $b = 2$ and that $S(X) \in A\{X\}$. As a consequence, $S(X) \in A_i\{X\}$ for all $i \geq 1$.

Remark 7.3. If $S(a)S(b) < 0$, then $S_n(a)S_n(b) < 0$ for n large enough, as a consequence of the definition of the partial sums: $S(x) = \lim_{n\to\infty} S_n(x)$ for all $x \in [a,b]$. It follows that, for n large enough, $S_n(x)$ has a root in $[a,b]$ because of the intermediate value theorem for polynomials [**2,** Section 2, Proposition 5].

Remark 7.4. If $]a,b[$ contains a sequence $C = (c_n)$ such that $S_n(c_n) = 0$ for all $n \in \mathbb{N}$, $n \geq n_0$ (for some n_0) and, moreover, C contains a subsequence which is Cauchy, then there is a $c \in\ ]a,b[$ such that $S(c) = 0$. In fact, the subsequence converges to a limit c and continuity implies that $S(c) = 0$.

Lemma 7.5. *Let $A \subset \mathbb{K}$ be a subring, and let I be a prime ideal of A. Moreover, let*

$$\mathcal{F} = \left\{ P_n(X) = \sum_{j=0}^{k(n)} a_{nj} X^j,\ n \in N \right\}$$

any subset of $A[X]$ such that:

(i) *$a_{nj} - a_{n'j} \in I$ for all n, n' and j;*
(ii) *each $P_n(X)$ has at least one root in $\mathbb{K}$.*

Then there are a subsequence

$$\mathcal{F}' = (P_{n_h}, h \in \mathbb{N}) \subset \mathcal{F}$$

and a sequence (c_h) of elements of A such that $P_{n_h}(c_h) = 0$, and moreover, $c_h - c_k \in I$ for all h, k.

Proof. We consider the (unique) polynomial $\overline{P}(X) = \overline{P_n}(X)\,(\text{mod } I) \in (A/I)[X]$. Each root c_n of $P_n(X)$ (existing by hypothesis) equals, mod I, a root of $\overline{P}(X)$. Since $\overline{P}(X)$ has all its coefficients in the quotient field of the integral domain A/I, it has finitely many roots so that there is at least one root, say $\overline{c}$, which is the image of infinitely many roots. These roots form the required sequence (c_h) and the P_{n_h}'s (of which they are roots) form the corresponding subset $\mathcal{F}'$. $\square$

Theorem 7.6. *Let $S(X)$ be a power series converging on the closed interval $[a,b]$. If $S(a)S(b) < 0$, then there is an $\overline{n} \in N$ such that for all $n \geq \overline{n}$, each partial sum has at least one root c_n in $[a,b]$,* *and moreover, there is a subsequence of* $C = (c_n)$ *which is a Cauchy sequence converging to a root* c *of* $S(X)$*.*

Proof. We have already seen by Remark 7.3 that there is an $\overline{n}$ such that $S_n(a)S_n(b) < 0$ for all $n \geq \overline{n}$, and thus, there exists a $c_n \in \,]a, b[$ such that $S_n(c_n) = 0$.

We want to show that there is a subsequence (c_{n_h}) of $C = (c_n)$ which is a Cauchy sequence. Recall that $S(X) \in A_1\{X\}$ and that C is a sequence in A_1, see Remarks 7.2 and 7.3.

First, we observe that, since $S(X)$ is restricted and $M_1, M_2, \ldots$ form a basis of the neighborhoods of 0, there is an increasing sequence $(n_h) = (n_0, n_1, \cdots)$ of natural numbers, such that

(i) $n_h \geq \overline{n}$ for all h,
(ii) for all $n \geq n_0$, $a_n \in M_1$ (observe that $S(X) \in A_1\{X\}$),
(iii) for all $h \geq 1$, if $n \geq n_h$, then $\overline{S}_n(X) = \overline{S}(X) \in (A_1/M_{1+h})[X]$.

Now we show that, for each $h \geq 0$, there is a Cauchy subsequence $C^{(h)} = (c_n^{(h)}) \subset C$ such that:

- two elements of $C^{(h)}$ differ by an element of M_{1+h},
- $C^{(h)} \subset C^{(h-1)}$.

We proceed with a recursive construction. Define F_0 as the set of all partial sums $\{S_n(X)\}, n \geq n_0$, and set $C^{(0)} = (c_n, n \geq n_0)$. As for $C^{(1)}$, it is built as follows.

Since $\overline{S_n}(X) = \overline{S}(X) \in (A_1/M_2)[X]$ for all $n \geq n_1$ and M_2 a prime ideal in A_1, there is, by Lemma 7.5, a countable subset $F_1 \subset F_0$ of partial sums, each element of which has at least one root chosen in the countable set $C^{(0)}$ lying on the same root $\mod M_2$. Therefore, $C^{(1)}$ is the set of such roots.

Now assume that $C^{(m)}$ is defined, for every $m \leq h$, in such a way that the two properties above are satisfied. Then, define $C^{(h+1)}$, $h \geq 1$, as a subset of $C^{(h)}$. As we have just seen, for all $n \geq n_{h+1}$, $S(X)$ and $S_n(X)$ lie on the same polynomial in $(A_i/M_{2+h})[X]$. Hence, a suitable countable subset $C^{(h+1)} \subset C^{(h)}$ contains only roots differing from one another by an element of M_{2+h}.

Therefore, we have a sequence $C^{(0)}, C^{(1)}, \ldots$ of countable sets such that $C^{(h+1)} \subset C^{(h)}$ for all $h \geq 0$, and such that two elements of $C^{(h)}$ differ by an element of M_{1+h}.

Now, we choose in each countable set $C^{(h)}$ its first element $\bar{c}_h$, so obtaining the required Cauchy sequence. Such a sequence converges, in the complete ring A_1, to some element c, which is a root of $S(X)$, see Remark 7.4.

Remark 7.7. The proof of Theorem 7.6 is based upon the following fact. There is a basis of the neighborhoods of 0 whose elements are prime ideals of the ring A, see Lemma 7.5. This condition is not satisfied if $\mathbb{K}$ contains a topologically nilpotent element. $\square$

8. The set of zeros of a power series: Accumulation and cardinality. The following theorem and corollary hold both in the Archimedean and in the non-Archimedean cases.

Theorem 8.1. *Let $S(X)$ be a power series such that $S(a)S(b) < 0$. Then, at least one among the zeros of $S(X)$ in $[a, b]$ is an accumulation point for the set $Z = \cup Z_n$, where $Z_n = \{z \in [a, b], S_n(z) = 0\}$. Therefore, at least one zero is the limit of a sequence of zeros of partial sums.*

Proof. Let $(c_1 < c_2 < \cdots < c_k)$ be the finitely many zeros of $S(X)$ in the interval.

Assume $S(a) < 0$, $S(b) > 0$ and that c_1 is not an accumulation point. Then, there is an interval $I = [c_1 - r, c_1 + r]$ containing no element of the set Z, no zeros of $S(X)$ except c_1 and neither a nor b. This implies that the partial sum $S_n(X)$ has no root in I and that $S_n(x)$ is either > 0 or < 0 for all $x \in I$. Since $S(x) = \lim_{n\to\infty} S_n(x)$ for all $x \in I$, there is an n_0 such that $n > n_0$ implies $S_n(x) < 0$ for all $x \in I$, $x < c_1$; as a consequence, $S_n(x) < 0$ for all $x \in I$.

We conclude that, if c_1 is not an accumulation point, then there is a suitable $r > 0$ such that $S(c_1 + r) < 0$. Now shrink $[a, b]$ to the subinterval $[c_1 + r, b]$ so that the smallest zero of $S(X)$ becomes c_2. We repeat the argument and assume that c_2 is not an accumulation point. After finitely many steps, we find that, if no zero is an accumulation point, then there is an s such that $S(c_k - s) < 0$ and $S(c_k + s) < 0$. But, $S(b) > 0$ implies that there is a zero between $c_k + s$ and b, which is a contradiction.

If c is an accumulation point of a sequence, then it is obvious that it is the limit of a suitable subsequence. $\square$

Remark 8.2. By [**5,** Theorem 3.11], it is easy to define the concept of order (or multiplicity) of a power series at $c \in \mathbb{K}$: if $S(X) = (X - c)^s q(X)$, where $q(c) \neq 0$, s is the order. Therefore, we may use the terms *odd order* and *even order*. It is clear that $S(X)$ has odd order at c if and only if there is an $\alpha > 0$ such that $S(a)S(b) < 0$ for all $a \in (c - \alpha, c)$ and $b \in (c, c + \alpha)$.

Observe that, by using [**5,** Theorem 3.11], it is easy to prove that, if a given zero is of even order, then it is either a local minimum or a local maximum for the series.

Corollary 8.3. *Every zero of $S(X)$ whose order is odd is an accumulation point for the set Z.*

Proof. If c is a zero whose order is $2r + 1$, there is an open neighborhood, and thus, by possibly shrinking it, also a closed neighborhood $J = [c - \delta, c + \delta]$ where $S(x) < 0$ for all $x < c$, $S(x) > 0$ for all $x > c$ (or conversely). Therefore, c is the only zero of $S(X)$ in J. Now apply the proof of the above theorem. $\square$

Theorem 8.4. *Let c be a v of even order of $S(X)$ (thus, it is a local extreme). Then, c is an accumulation point of local extremes of the partial sums, c and the extremes having the same type.*

Proof. We know that c is a zero of odd order of the derivative $S'(X)$; as a consequence, it is an accumulation point for the set of zeros of the partial sums $(S'(X))_n = S'_{n+1}(X)$, each of which is a local extreme of $S_{n+1}(X)$.

Now, we observe that $S(X) = (X - c)^{2p} T(X)$ where $2p$ is the even order of the zero and $T(c) \neq 0$, see [**5,** Theorem 3.11]. Therefore, see

[**5,** Theorem 3.7],

$$\frac{S(X)}{(X-c)^{2p}} = S^{(2p)}(c) + r(X-c),$$

where $\lim_{X\to c} r(X-c) = 0$. It follows that the sign of $S(X)$ around c is the same as the sign of $S^{(2p)}(c)$, which implies that c is a maximum (minimum) if and only if $T(c) < 0$ (> 0) or if and only if $S^{(2p)}(c) < 0$ (> 0). In order to see that a maximum is an accumulation point of maxima and a minimum of minima, it is now sufficient to observe that, for n large enough, $(S^{(2p)}(c))_n = S^{(2p)}_{n-2p}(c)$ has the same sign as $S^{(2p)}(c)$ since $S^{(2p)}(c) = \lim_{n\to\infty} S^{(2p)}_n(c)$. □

The next example shows that a double zero of a series is approximated by extremes but not always by roots of the partial sums. As usual, ϵ is topologically nilpotent in $\mathbb{K} = \mathbb{Q}[\epsilon]$.

Example 8.5. Let

$$F(X) = \sum_{n=0}^{\infty} b_n X^n$$

be any power series, c any element of $\mathbb{K}$, and set:

$$T(X) = (x-c)^2 F(X).$$

Then, a straightforward computation on the partial sums $T_n(X), F_n(X)$ shows that $T_n(X) = (x-c)^2 F_{n-1}(X) + x^n(c^2 b_n - x b_{n-1})$.

Now, choose $c = 1$, $F(X) = 2 - \sum_{n=1}^{\infty} \epsilon^{n+1} X^n = 2 - (\epsilon^2 X)/(1-\epsilon X)$. Then both $F(X)$ and $T(X)$ converge between $1/2$ and 2, and moreover, $F(x) > 1$ for every x such that $1 \le x \le 2$, since it is the difference between 2 and an infinitesimal element. Therefore, $F_n(x) > 1/2$ for n large enough. Hence, we obtain the following inequality:

$$T_n(x) \ge \frac{(x-1)^2}{2} + x^n(-\epsilon^{n+1} + x\epsilon^n),$$

where also the second term is strictly positive. Therefore, $T_n(X)$ has no root converging to 1.

Remark 8.6. The set of zeros belonging to A of the power series $S(X)$ is finite, see Corollary 5.4. However, there is a power series whose domain is $\mathbb{K}$ and has infinitely many zeros, as the next two examples show.

Example 8.7. Let $\mathbb{K}$ be a maximal ordered, complete field having a topologically nilpotent element ϵ (the sets $U_n = \{x \in \mathbb{K}, |x| < \epsilon^n, n \in \mathbb{N}\}$ form a basis for the neighborhoods of 0). Set

$$S(X) = \sum_{n=0}^{\infty} (-1)^n \epsilon^{n^2} X^n.$$

Since $\lim_{n\to\infty} (\epsilon^{n^2})^{1/n} = 0$, the domain D_S of $S(X)$ is the whole $\mathbb{K}$, see [**5,** Notation] and [**14,** page 137, (IV)]. We want to show that:

$$S(\epsilon^m) < 0 \quad \text{if } m = -4l - 2,\ l \in \mathbb{N}$$

and

$$S(\epsilon^m) > 0 \quad \text{if } m = -4l,\ l \in \mathbb{N}.$$

To this end, we observe that

$$S(\epsilon^{-4l}) = \sum (-1)^n \epsilon^{n^2 - 4ln},$$

where ϵ^{n^2-4ln} is finite for $n = 0$ and $n = 4l$, infinitesimal for $n > 4l$ and infinitely large for $0 < n < 4l$. Since the maximum of $\phi(n) = 4ln - n^2$ is attained at $n = 2l$, the largest term of the series is ϵ^{-4l^2} with positive sign, and it forces the series to attain a positive value.

An analogous argument shows that

$$S(\epsilon^{-4l-2}) = \sum (-1)^n \epsilon^{n^2 - 4ln - 2n}$$

contains finitely many non-infinitesimal terms among which $-\epsilon^{-4l^2-4l-1}$ is infinitely large and most negative, so that it forces the series to attain a negative value. As a consequence, $S(X)$ vanishes infinitely many times, with one root at least between ϵ^{-4l-2} and ϵ^{-4l} for all $l \in \mathbb{N}^+$, due to the intermediate value theorem.

Example 8.8. Let $\mathbb{K}$ be a maximal ordered, complete field having no topologically nilpotent element, but a decreasing sequence of infinitesimal elements $(1 = \epsilon_0 > \epsilon_1 > \epsilon_2 > \cdots)$ such that the sets

$$(U_n = \{x \in \mathbb{K}, |x| < \epsilon_n, n \in \mathbb{N}\}$$

form a basis for the neighborhoods of 0. We assume (Section 2) that $\epsilon_n^i > \epsilon_{n+1}$ for all $n \in \mathbb{N}$ and for all $i \in \mathbb{N}$. As a consequence, the following holds: $\epsilon_i^n > m\epsilon_{i+1}$ for all $n, i, m \in \mathbb{N}$.

Now set

$$S(X) = \sum_{n=0}^{\infty} (-1)^n \epsilon_n X^n.$$

The domain D_S of $S(X)$ contains at least $X = 1$; thus, it is the whole $\mathbb{K}$, see [**14,** page 137, (IV)] and [**5,** Notation].

We now compute $S(\epsilon_h^{-1})$:

$$\begin{aligned} S(\epsilon_h^{-1}) = 1 - \epsilon_1\epsilon_h^{-1} + \epsilon_2\epsilon_h^{-2} - \epsilon_3\epsilon_h^{-3} + \cdots \\ + (-1)^h \epsilon_h \epsilon_h^{-h} + (-1)^{h+1}\epsilon_{h+1}\epsilon_h^{-(h+1)} \\ + (-1)^{h+2}\epsilon_{h+2}\epsilon_h^{-(h+2)} \cdots . \end{aligned}$$

Observe that

(i) $\epsilon_{h+1}\epsilon_h^{-(h+2)}$ is infinitesimal with respect to $\mathbb{Q}$ since $\epsilon_{h+1}\epsilon_h^{-(h+2)} < 1/n$ for all n is equivalent to $n\epsilon_{h+1} < \epsilon_h^{h+2}$.

(ii) $\epsilon_{h+1}\epsilon_h^{-(h+1)} > \epsilon_{h+2}\epsilon_h^{-(h+2)}$ since this is equivalent to $\epsilon_{h+1}\epsilon_h > \epsilon_{h+2}$; moreover, we know that $\epsilon_h > \epsilon_{h+1}$, and hence, $\epsilon_{h+1}\epsilon_h > \epsilon_{h+1}^2 > \epsilon_{h+2}$.

As a consequence (if $u \in \mathbb{N}$ and $u \geq 1$), we obtain

$$\left|(-1)^{h+1}\epsilon_{h+1}\epsilon_h^{-(h+1)} + (-1)^{h+2}\epsilon_{h+2}\epsilon_h^{-(h+2)} + \cdots + (-1)^{h+u}\epsilon_{h+u}\epsilon_h^{-(h+u)}\right| < u\epsilon_{h+1}\epsilon_h^{-(h+1)}.$$

From the inequality $u < \epsilon_h^{-1}$, for all $u \in N$, we obtain $R_{h+1} < \epsilon_{h+1}\epsilon_h^{-(h+2)}$ where R_{h+1} is the remainder of order $(h+1)$ of the series. Therefore, R_{h+1} is infinitesimal.

Now we consider the first $h+1$ terms of the series, where $h \geq 2$:

$$S_h(\epsilon_h^{-1}) = 1 - \epsilon_1\epsilon_h^{-1} + \epsilon_2\epsilon_h^{-2} + \cdots + (-1)^h\epsilon_h\epsilon_h^{-h}.$$

All terms, except the first, are infinitely large and

$$m\epsilon_i\epsilon_h^{-i} < \epsilon_h^{1-h}, i < h, \quad \text{for all } m \in N,$$

since this is equivalent to $m\epsilon_i < \epsilon_h^{h+i-1}$. It follows that

$$|S_{h-1}(\epsilon_h^{-1})| < \epsilon_h \epsilon_h^{-h}.$$

Hence, we see that the sign of the series coincides with the sign of $(-1)^h \epsilon_h \epsilon_h^{-h}$. Therefore, we have:

$$S(\epsilon_h^{-1}) > 0, \text{ if } h \text{ is even} \qquad S(\epsilon_h^{-1}) < 0, \text{ if } h \text{ is odd} \quad (> 1).$$

As a consequence, $S(X)$ vanishes infinitely many times due to the intermediate value theorem.

9. The mean value, Rolle's and extreme value theorems. Once the intermediate value theorem is established, the mean value theorem, Rolle's theorem and the extreme value theorem are formal consequences of it, see [**5,** Section 4]. We give a brief sketch below, since the proofs given in [**5,** Section 4] hold in the present, more general case.

In what follows, $S(X)$ is a power series defined on a closed interval $[a,b]$ having coefficients in a non-Archimedean, maximal ordered, Cauchy complete field $\mathbb{K}$.

(1) Rolle's theorem. Assume that $S(X)$ vanishes both at a and at b. Then the equation $S'(X) = 0$ has at least one root $x \in \mathbb{K}$, $a < x < b$.
(2) The mean value theorem. There is a $c \in \,]a,b[$ such that $S(b) - S(a) = (b-a)S'(c)$.
(3) Monotonic functions.
    (a) $S(X)$ is strictly increasing (decreasing) at $x \in [a,b]$ if $S'(x) > 0$ ($S'(x) < 0$), where the derivative at a (b) is the right-hand derivative (left-hand derivative).
    (b) If $S(X)$ is increasing (decreasing) at $x \in [a,b]$, then $S'(x) \geq 0$ ($S'(x) \leq 0$).
    (c) $S(X)$ is increasing (decreasing) in $[a,b]$ if and only if $S'(x) \geq 0$ ($S'(x) \leq 0$) for all $x, a \leq x \leq b$.
    (d) At a local maximum or a local minimum $x \in \,]a,b[$, the derivative $S'(x)$ vanishes.

    (The arguments for proving (a), (b) and (d) do not depend upon the intermediate value theorem; rather, they depend upon the definition of the derivative as a limit, while (c) requires the mean value theorem which follows from the intermediate value theorem).

(4) Absolute maxima and minima.
   (a) $S(X)$ is bounded above and below.
   (b) $S(X)$ attains both its absolute maximum value and its absolute minimum value.

Remark 9.1. In order to prove (4) (as in [**5,** Section 4]) we need the following property: the set of zeros of $S'(X)$ is finite, see Corollary 5.4.

10. The intermediate value theorem and completeness. In this section, we consider an *arbitrary ordered field* $\mathbb{K}$. If $S(X)$ is a power series with coefficients in $\mathbb{K}$ whose partial sums form a Cauchy sequence at $x \in \mathbb{K}$, then $S(X)$ converges at x to $S(x) \in \widehat{\mathbb{K}}$, so that $S(X)$ can be considered as a function $\mathbb{K} \to \widehat{\mathbb{K}}$.

We say that $\mathbb{K}$ has the *intermediate value property for power series* (IVPPS) if the following property holds true:

(IVPPS) *For every power series $S(X)$ with coefficients in $\mathbb{K}$, if $S(a)S(b) < 0$ for some $a, b \in \mathbb{K}$, $a < b$, then there is a $c \in \mathbb{K}$, $a < c < b$, such that $S(c) = 0$.*

In order to prove the main result of this section, i.e., that (IVPPS) implies completeness, we need the next preliminary result.

Theorem 10.1. *Assume that $c \in \widehat{\mathbb{K}}$. Then there is a power series $T(X)$ with coefficients in $\mathbb{K}$ such that*:

(i) $T(c) = 0$.
(ii) $T'(c) \neq 0$; *hence, $T(X)$ is strictly monotone at c.*

Proof. Assume first that c belongs to $\mathbb{K}$. Then $T(X) = X - c$ fulfills both conditions.

Now, assume that c is in $\widehat{\mathbb{K}} - \mathbb{K}$. We want to show that it is enough to prove the statement for every c such that $0 < c < 1$. In fact, if $c < 0$, we replace c by $-c$. If we prove that there is a $T(X) = \sum_0^\infty t_n X^n$ satisfying (i) and (ii) at $-c$, then $\overline{T}(X) = \sum_0^\infty (-1)^n t_n X^n$ satisfies (i) and (ii) at c. We should only observe that $T'(-c) = -\overline{T}'(c)$ so that, if $T(X)$ is increasing, then $\overline{T}(X)$ is decreasing and conversely. If $c > 1$, we replace c by c/a where $a \in \mathbb{K}$, $a > c$ (this is allowed because $\widehat{\mathbb{K}}$ is comparable with $\mathbb{K}$). If we prove that there is a

$$T(X) = \sum_0^\infty t_n X^n$$

satisfying (i) and (ii) at c/a, then

$$\overline{T}(X) = \sum_0^\infty \frac{1}{a^n} t_n X^n$$

satisfies (i) and (ii) at c (again use the derivative, observing that the type of monotonicity is preserved). Therefore, we can assume that $0 < c < 1$, which implies that $\lim_{n\to\infty} d_n c^n = 0$ whenever $\lim_{n\to\infty} d_n = 0$.

We consider two cases.

Case A. There is a nilpotent element $\epsilon \in \mathbb{K}$. Accordingly, choose $a_n \in \mathbb{K}, n \in \mathbb{N} \setminus \{0\}$ recursively as follows. When $n = 1$, we choose as a_1 any positive element in $\mathbb{K}$ such that $1/c - \epsilon < a_1 < 1/c$.

For $n > 1$, assume that

$$S_{n-1} = \sum_1^{n-1} a_i c^i$$

has been chosen in such a way that

$$1 - \epsilon^{n-1} c^{n-1} < S_{n-1} < 1$$

(observe that $1 - \epsilon c < S_1 < 1$, i.e., in the case $n = 2$ the inequalities hold true). Then, select $a_n \in \mathbb{K}, a_n > 0$ such that $1 - \epsilon^n c^n < S_n < 1$, i.e., such that

$$\frac{1 - \epsilon^n c^n - S_{n-1}}{c^n} < a_n < \frac{1 - S_{n-1}}{c^n}.$$

Then a sequence of positive coefficients $(a_1, a_2, \ldots)$ is defined, and thus, is the series

$$S(X) = \sum_{n=1}^\infty a_n X^n.$$

Moreover, for all $n \geq 1$, we obtain

$$1 - \epsilon^n c^n < S_n < 1.$$

Therefore, S_n converges to 1, and the series

$$T(X) = S(X) - 1 = \sum_1^\infty a_n X^n - 1$$

vanishes exactly at c. Since $a_n > 0$ for all $n \geq 1$, $T(X)$ is increasing at c and thus negative when $x < c$, positive when $x > c$.

Case B. There is no nilpotent element, and the topology is defined by the sequence $(\epsilon_0 = 1 > \epsilon_1 > e_2 > \cdots)$ (see Notation). By replacing ϵ^n by ϵ_n, the proof holds. □

Corollary 10.2. *Let* $\mathbb{K}$ *be an ordered field with* (IVPPS). *Then* $\mathbb{K}$ *is both maximal ordered and complete.*

Proof. First, observe that the intermediate value property for polynomials is enough to ensure that $\mathbb{K}$ is maximally ordered, see [**2,** Section 2, Theorem 3].

We now discuss completeness. Let c be any element of $\widehat{\mathbb{K}}$. By Theorem 10.1, there is a power series $T(X) \in \mathbb{K}[[X]]$, monotonic at c and such that $T(c) = 0$. This means that, in a suitable interval $[a, b] \subset \mathbb{K}$ containing c, $T(a)T(b) < 0$ and $T(X)$ vanishes only at c. Therefore, by (IVPPS), c is forced to belong to $\mathbb{K}$, and we obtain: $\mathbb{K} = \widehat{\mathbb{K}}$. □

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