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CASTELNUOVO-MUMFORD REGULARITY OF SYMBOLIC POWERS OF TWO-DIMENSIONAL SQUARE-FREE MONOMIAL IDEALS

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ABSTRACT. Let I be a square-free monomial ideal of a polynomial ring R such that $\dim(R/I) = 2$. We give explicit formulas for computing the a_i-invariants $a_i(R/I^{(n)})$, $i = 1, 2$, and the Castelnuovo-Mumford regularity $\operatorname{reg}(R/I^{(n)})$ for all n. The values of these functions depend on the structure of an associated graph. It turns out that these functions are linear functions of n for all $n \geq 2$.

Introduction. Let I be a square-free monomial ideal of a polynomial ring $R = k[x_1, \ldots, x_r]$ over a field k. Then I can be considered as a Stanley-Reisner ideal associated to a simplicial complex. In recent years, the study of powers I^n and symbolic powers $I^{(n)}$ has attracted the attention of many authors (see, e.g., **[3, 5, 9, 12]**). In the two-dimensional case, the associated simplicial complex is a graph G, and we may write a two-dimensional square-free monomial ideal in the form:

$$I_G = \bigcap_{\{i,j\} \in E(G)} P_{ij} \bigcap_{i \in V_0(G)} P_i,$$

where $E(G)$ is the edge set of G, $V_0(G)$ the set of isolated vertices, $P_{ij} = (\{x_1, \ldots, x_r\} \setminus \{x_i, x_j\})$, and $P_i = (\{x_1, \ldots, x_r\} \setminus \{x_i\})$. Some algebraic properties of I_G^n and $I_G^{(n)}$ can be characterized in terms of G (see, e.g., **[8, 7]**). In this paper, we are interested in computing the Castelnuovo-Mumford regularity. Let us recall this notion. Let J be a proper homogeneous ideal of R. Set

$$a_i(R/J) = \sup\{t \mid H^i_{\mathfrak{m}}(R/J)_t \neq 0\},$$

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where $H^i_{\mathfrak{m}}(R/J)$ is the local cohomology module with the support $\mathfrak{m} = (x_1, \ldots, x_r)$. The Castelnuovo-Mumford regularity of R/J is defined by

$$\mathrm{reg}(R/J) = \max\{a_i(R/J) + i \mid 0 \leqslant i \leqslant \dim R/J\}.$$

Let $J^{(n)}$ be the nth symbolic powers of J. It is well known that $\mathrm{reg}(R/J^n)$ is a linear function of n for $n \gg 0$ (see [**1,** Theorem 1.1] or [**6,** Theorem 5]). Concerning $\mathrm{reg}(R/J^{(n)})$, it was shown in some cases that this function is bounded by a linear function of n (see [**4,** Section 2]). Moreover, when $J = I$ is a square-free monomial ideal, in [**5,** Theorem 4.1 and Theorem 4.9] we proved that $a_i(R/I^{(n)})$ and $\mathrm{reg}(R/I^{(n)})$ are quasi-linear functions of n for $n \gg 0$. But it is still not known whether they are linear functions of n for $n \gg 0$. Therefore, we start to investigate this problem when $\dim R/I = 2$, i.e., when $I = I_G$ for a graph G. The main purpose of this note is to give explicit formulas for computing $a_i(R/I_G^{(n)})$, $i = 1, 2$ and $\mathrm{reg}(R/I_G^{(n)})$ (see Theorem 2.3, Theorem 2.8 and Theorem 2.9). It turns out that all these functions are linear functions of n for $n \geq 2$. The proofs of these results are based on Takayama's formula for computing local cohomology modules of monomial ideals (see Lemma 1.1) and a formula for computing simplicial complexes associated to symbolic powers of square-free monomial ideals (see Lemma 1.3), which extends a result given in [**8**].

The paper is divided into two sections. In Section 1, we recall Takayama's formula, a generalized version of Hochster's formula, to compute local cohomology modules of monomial ideals and then give some descriptions of associated simplicial complexes. In Section 2, we prove the main results.

1. Auxiliary results. A simplicial complex Δ on the finite set $[r] := \{1, \ldots, r\}$ is a collection of subsets of $[r]$ such that $F \in \Delta$ whenever $F \subseteq F'$ for some $F' \in \Delta$. Notice that we do not impose the condition that $\{i\} \in \Delta$ for all $i \in [r]$. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. The Stanley-Reisner ideal of Δ is the following ideal of $R := k[x_1, \ldots, x_r]$:

$$I_\Delta := (x_{i_1} \cdots x_{i_s} \mid \{i_1, \ldots, i_s\} \notin \Delta) = \bigcap_{F \in \mathcal{F}(\Delta)} P_F,$$

where P_F is the prime ideal of R generated by all variables x_i with $i \notin F$. It is a square-free monomial ideal. Conversely, if I is a square-free monomial ideal, then it is the Stanley-Reisner ideal associated to the following simplicial complex

$$\Delta(I) = \{\{i_1, \ldots, i_s\} |\ x_{i_1} \cdots x_{i_s} \notin I\}.$$

If I is an arbitrary monomial ideal we set $\Delta(I) = \Delta(\sqrt{I})$. For a subset F of $[r]$, let $R_F := R[x_i^{-1} \mid i \in F]$ and for $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$, and let $x^\alpha = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$. We define the co-support of α to be the set $CS_\alpha := \{i \mid \alpha_i < 0\}$. Set

$$\Delta_\alpha(I) = \{F \subseteq [r] \setminus CS_\alpha \mid x^\alpha \notin IR_{F \cup CS_\alpha}\}.$$

We set $\widetilde{H}_i(\emptyset; k) = 0$ for all i, $\widetilde{H}_i(\{\emptyset\}; k) = 0$ for all $i \neq -1$, and $\widetilde{H}_{-1}(\{\emptyset\}; k) = k$. Thanks to [**2,** Lemma 1.1], we may formulate Takayama's formula as follows.

Lemma 1.1. ([**11,** Theorem 2.2]).

$$\dim_k H^i_{\mathfrak{m}}(R/I)_\alpha = \begin{cases} \dim_k \widetilde{H}_{i-|CS_\alpha|-1}(\Delta_\alpha(I); k) & \textit{if } CS_\alpha \in \Delta(I), \\ 0 & \textit{otherwise.} \end{cases}$$

It was then shown in [**8,** Lemma 1.3] that $\Delta_\alpha(I)$ is a subcomplex of $\Delta(I)$. For a face $F \in \Delta$, the link of F is defined by

$$\mathrm{lk}_\Delta(F) = \{G \subseteq [r] \setminus F |\ F \cup G \in \Delta\}.$$

The next lemma gives a more precise description of $\Delta_\alpha(I)$ and will be useful in its computation.

Lemma 1.2. *Assume that $CS_\alpha \in \Delta(I)$ for some $\alpha \in \mathbb{Z}^r$. Then*

$$\Delta_\alpha(I) = \{F \in \mathrm{lk}_{\Delta(I)}(CS_\alpha) \mid x^\alpha \notin IR_{F \cup CS_\alpha}\}.$$

Proof. Let $F \subseteq [r] \setminus CS_\alpha$. Note that, if $F \cup CS_\alpha \notin \Delta(I)$, then $\sqrt{I} R_{F \cup CS_\alpha} = R_{F \cup CS_\alpha}$, which yields $IR_{F \cup CS_\alpha} = R_{F \cup CS_\alpha}$ and $F \notin \Delta_\alpha(I)$. So, if $F \in \Delta_\alpha(I)$, we must have $F \cup CS_\alpha \in \Delta(I)$, i.e., $F \in \mathrm{lk}_{\Delta(I)}(CS_\alpha)$. □

The nth symbolic power $I_\Delta^{(n)}$ is defined by

$$I_\Delta^{(n)} = \bigcap_{F \in \mathcal{F}(\Delta)} P_F^n.$$

The following lemma extends [**8,** Lemma 2.1] and plays a crucial role in studying properties of $\Delta_\alpha(I_G^{(n)})$.

Lemma 1.3. *Assume that $CS_\alpha \in \Delta$ for some $\alpha \in \mathbb{Z}^r$. Then*

$$\mathcal{F}(\Delta_\alpha(I_\Delta^{(n)})) = \Big\{ F \in \mathcal{F}(\mathrm{lk}_\Delta(CS_\alpha)) \mid \sum_{i \notin F \cup CS_\alpha} \alpha_i \leqslant n-1 \Big\}.$$

Proof. By Lemma 1.2, it follows that a facet $F \in \Delta_\alpha(I_\Delta^{(n)})$ has the form $F = F' \setminus CS_\alpha$, where F' is a facet of Δ containing CS_α and $x^\alpha \notin I_\Delta^{(n)} R_{F'}$. Since $I_\Delta^{(n)} R_{F'} = (x_i | \ i \notin F')^n$, the last condition is equivalent to $x^{\alpha'} \notin (x_i \mid i \notin F')^n$, where $x^{\alpha'} = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_s}^{\alpha_{i_s}}$ if we set $[r] \setminus F' = \{i_1, \dots, i_s\}$. Clearly, this condition is equivalent to $\sum_{i \notin F \cup CS_\alpha} \alpha_i \leq n-1$. □

Notation 1.4. Put $|\alpha| = \alpha_1 + \cdots + \alpha_r$.

Lemma 1.5. *If $\Delta_\alpha(I_\Delta^{(n)}) = \{\emptyset\}$, then $CS_\alpha \in \mathcal{F}(\Delta)$ and $|\alpha| \leq n - 1 - |CS_\alpha|$. Moreover, if $F \in \mathcal{F}(\Delta)$, then*

$$\max\{|\beta| \mid CS_\beta = F \quad \textit{and} \quad \Delta_\beta(I_\Delta^{(n)}) = \{\emptyset\}\} = n - 1 - |F|.$$

Proof. From Lemma 1.3, it immediately follows that $CS_\alpha \in \mathcal{F}(\Delta)$. Since $\alpha_i \leq -1$ for all $i \in CS_\alpha$, and $\emptyset \in \Delta_\alpha(I_\Delta^{(n)})$, again by Lemma 1.3, we have

$$|\alpha| = \sum_{i \in CS_\alpha} \alpha_i + \sum_{j \notin CS_\alpha} \alpha_j \leq -|CS_\alpha| + n - 1.$$

Now let $F \in \mathcal{F}(\Delta)$. Without loss of generality, we may assume that $F = \{1, \dots, s\}$. Let $\beta = (-1, \dots, -1, n-1, 0, \dots, 0)$ (s entries of -1). Then $CS_\beta = F$, $|\beta| = n - 1 - s$, and one can use Lemma 1.3 to verify that $\Delta_\beta(I_\Delta^{(n)}) = \{\emptyset\}$. Hence, the second statement follows from the first one. □

A graph G is an undirected simple graph with the vertex set $V(G) \subseteq [r]$ having no loops. The set of isolated vertices is denoted by $V_0(G)$, which can be empty. The set of edges of G is denoted by $E(G)$ and is assumed not to be empty. We always consider G as the simplicial complex Δ of dimension one, such that $\mathcal{F}(\Delta) = E(G) \cup \{\{i\} \mid i \in V_0(G)\}$. If there is no confusion, we will use the same notation G to denote this simplicial complex. Recall that a connected graph without cycles is called a *tree*, and a disjoint union of trees is called a *forest*. The following result is probably known, but we could not find a reference. We provide a proof for the sake of completeness.

Lemma 1.6. *Let G be a graph considered as a simplicial complex of dimension one. Then $\widetilde{H}_1(G,k) = 0$ if and only if G is a forest.*

Proof. Let $G_1, \ldots, G_s$ be the connected components of G. Then the reduced Euler characteristic $\widetilde{\chi}(G)$ of G can be computed in two ways (see, e.g., **[10,** Definition 3.2]):

$$\widetilde{\chi}(G) = -1 + |V(G)| - |E(G)| = \dim_k \widetilde{H}_0(G;k) - \dim_k \widetilde{H}_1(G;k).$$

Since $\dim_k \widetilde{H}_0(G;k) = s-1$, we deduce that

$$\dim_k \widetilde{H}_1(G;k) = |E(G)| + s - |V(G)| = \sum_{i=1}^{s}(|E(G_i)| + 1 - |V(G_i)|).$$

As each G_i is a connected graph, we have $|E(G_i)| + 1 \geq |V(G_i)|$, and the equality holds if and only if G_i is a tree. Thus, $\dim_k \widetilde{H}_1(G;k) = 0$ if and only if all $G_1, \ldots, G_s$ are trees, that means G is a forest, as required. $\square$

2. Castelnuovo-Mumford regularity of symbolic powers. Since we are considering graphs with possibly isolated vertices, any square-free monomial ideal of dimension two can be seen as I_G for some graph G. Since $I_G^{(n)}$ has no $\mathfrak{m}$-primary component, $H^0_{\mathfrak{m}}(R/I_G^{(n)}) = 0$. Hence,

$$\operatorname{reg}(R/I_G^{(n)}) = \max\{a_1(R/I_G^{(n)}) + 1,\ a_2(R/I_G^{(n)}) + 2\}.$$

So, in order to compute $\operatorname{reg}(R/I_G^{(n)})$, we have to compute $a_1(R/I_G^{(n)})$ and $a_2(R/I_G^{(n)})$. The computation of $a_1(R/I_G^{(n)})$ in the unmixed case was implicitly done in **[7, 8]**. We formulate these results below. Since

$\dim(R/I_G^{(n)}) = 2$, it follows that $a_1(R/I_G^{(n)}) = -\infty$ if and only if the ring $R/I_G^{(n)}$ is Cohen-Macaulay.

We recall some notions from graph theory. The distance between two vertices i and j is the minimal length of paths which connect them. The maximal distance between two vertices of G is called the diameter of G and is denoted by $\operatorname{diam}(G)$. If G is not connected, we set $\operatorname{diam}(G) = \infty$. A graph is called a *matroid* if any two of its disjoint edges are contained in a cycle of length 4.

Lemma 2.1. *The ring $R/I_G^{(n)}$ is a Cohen-Macaulay ring if and only if G is connected and one of the following conditions is satisfied*:

(i) $n = 1$,
(ii) $\operatorname{diam}(G) = 2$ *and either* $n = 2$ *or* G *is a matroid.*

Proof. It is well known that the Cohen-Macaulayness of $R/I_G^{(n)}$ implies the connectedness of G. This also immediately follows from Lemma 1.1 by setting $\alpha = (0, \dots, 0)$ and $i = 1$. Hence, we may assume from the beginning that G is connected. In particular, G has no isolated vertex, and the statement follows from [**8,** Theorem 2.3 and Theorem 2.4]. □

Lemma 2.2. *Assume that G has no isolated vertex and $R/I_G^{(n)}$ is not a Cohen-Macaulay ring. Then $a_1(R/I_G^{(n)}) = 2n - 2$.*

Proof. By [**7,** Lemma 3.2(1)], $a_1(R/I_G^{(n)}) \leq 2n-2$. In order to show the reverse inequality, we distinguish three cases.

If $n = 1$, then, by Lemma 2.1, G is not connected. Hence, by [**7,** Lemma 3.2 (2)], $[H_{\mathfrak{m}}^1(R/I_G)]_0 \neq 0$.

If $n = 2$, then, by Lemma 2.1, $\operatorname{diam}(G) \geq 3$. Hence, by [**7,** Corollary 3.4], $[H_{\mathfrak{m}}^1(R/I_G^{(2)})]_2 \neq 0$.

Assume $n \geq 3$. By Lemma 2.1, G is not a matroid. Hence, by [**7,** Lemma 3.5], $[H_{\mathfrak{m}}^1(R/I_G^{(n)})]_{2n-2} \neq 0$.

Summing up, in all cases, $[H_{\mathfrak{m}}^1(R/I_G^{(n)})]_{2n-2} \neq 0$, which yields $a_1(R/I_G^{(n)}) \geq 2n - 2$, as required. □

Theorem 2.3. *Assume that $R/I_G^{(n)}$ is not a Cohen-Macaulay ring. Then $a_1(R/I_G^{(n)}) = 2n-2$.*

Proof. By Lemma 2.2, it suffices to assume that G has an isolated vertex, say 1. Since $E(G) \neq \emptyset$, we may assume that $\{2,3\} \in E(G)$. Let $\beta = (n-1, n-1, 0, \ldots, 0)$. We have $CS_\beta = \emptyset$, and, by Lemma 1.3, $\{1\}, \{2,3\} \in \Delta_\beta(I_G^{(n)})$. Hence, $\Delta_\beta(I_G^{(n)})$ is disconnected and, by Lemma 1.1,

$$\dim_k [H^1_{\mathfrak{m}}(R/I_G^{(n)})]_\beta = \dim_k \widetilde{H}_0(\Delta_\beta(I_G^{(n)}); k) \neq 0,$$

which implies $a_1(R/I_G^{(n)}) \geq |\beta| = 2n-2$.

We now show that $a_1(R/I_G^{(n)}) \leq 2n-2$. Let $\alpha \in \mathbb{Z}^r$ such that $a_1(R/I_G^{(n)}) = |\alpha|$ and

$$\dim_k [H^1_{\mathfrak{m}}(R/I_G^{(n)})]_\alpha = \dim_k \widetilde{H}_{-|CS_\alpha|}(\Delta_\alpha(I_G^{(n)}); k) \neq 0. \tag{2.1}$$

Hence, $|CS_\alpha| \leq 1$. If $|CS_\alpha| = 1$, the above inequality implies that $\Delta_\alpha(I_G^{(n)}) = \{\emptyset\}$. By Lemma 1.5, $|\alpha| \leq n-2$, a contradiction. Hence, $CS_\alpha = \emptyset$. In this case, by Lemma 1.2, $\Delta_\alpha(I_G^{(n)})$ is a subgraph of G and, by (2.1), it must be disconnected. We may assume that $\{1, i_1\}, \{2, i_2\}$ are facets of $\Delta_\alpha(I_G^{(n)})$ such that $i_1 \neq 2$, $i_2 \neq 1$ and $i_1 \neq i_2$ (but it may happen that $i_1 = 1$ and/or $i_2 = 2$). Then, by Lemma 1.3, $|\alpha| \leq \sum_{j \neq 1, i_1} \alpha_j + \sum_{j \neq 2, i_2} \alpha_j \leq 2n-2$, as required. □

We now compute $a_2(R/I_G^{(n)})$. For that, we need some preparation lemmas. Recall that the *girth* of G, denoted by $\mathrm{girth}(G)$, is the smallest length of cycles of G. If G contains no cycle (equivalently, G is a forest) we set $\mathrm{girth}(G) = \infty$. Thus, if $\mathrm{girth}(G)$ is finite, then $3 \leq \mathrm{girth}(G) \leq |V(G)|$.

From now on, let $\alpha \in \mathbb{Z}^r$ such that $[H^2_{\mathfrak{m}}(R/I_G^{(n)})]_\alpha \neq 0$. By Lemma 1.1,

$$\dim_k [H^2_{\mathfrak{m}}(R/I_G^{(n)})]_\alpha = \dim_k \widetilde{H}_{1-|CS_\alpha|}(\Delta_\alpha(I_G^{(n)}); k) \neq 0, \tag{2.2}$$

and CS_α is a face of the simplicial complex G. Hence, we must have $|CS_\alpha| \leq 2$. We distinguish three cases.

Lemma 2.4. *Assume that* $|CS_\alpha| = 0$, *i.e.,* $\alpha \in \mathbb{N}^r$. *Then* $3 \le s := \mathrm{girth}(G) \le r$, *and*

$$|\alpha| \le \left[\frac{s(n-1)}{s-2}\right].$$

Proof. Since $CS_\alpha = \emptyset$, by Lemma 1.2, $\Delta_\alpha(I_G^{(n)})$ is a subgraph of G. Since $\widetilde{H}_1(\Delta_\alpha(I_G^{(n)}); k) \neq 0$, by Lemma 1.6, $\Delta_\alpha(I_G^{(n)})$ must contain a cycle, say $C = (1, 2, \ldots, t)$, where $t \ge s = \mathrm{girth}(G)$. In particular, s is finite and $3 \le s \le r$. By Lemma 1.3, for all $l = 1, \ldots, t-1$, we have $\sum_{i \neq l, l+1} \alpha_i \le n - 1$ and $\sum_{i \neq t,1} \alpha_i \le n - 1$. Hence,

$$(t-2)|\alpha| \le \sum_{l=1}^{t-1} \sum_{\substack{i \neq l \\ l+1}} \alpha_i + \sum_{i \neq t,1} \alpha_i \le t(n-1),$$

which yields $|\alpha| \le [t(n-1)/(t-2)] \le [s(n-1)/(s-2)]$. □

Lemma 2.5. *Assume that* $|CS_\alpha| = 1$. *Then* $|\alpha| \le 2n - 3$.

Proof. We may assume that $CS_\alpha = \{r\}$. By Lemma 1.2, $\Delta_\alpha(I_G^{(n)}) \subseteq \mathrm{lk}_G(CS_\alpha)$, so it is $\emptyset$, or $\{\emptyset\}$, or a set of points. By (2.2), $\dim_k \widetilde{H}_0(\Delta_\alpha(I_G^{(n)}); k) \neq 0$. Therefore, $\Delta_\alpha(I_G^{(n)})$ must contain at least two points, say $1, 2$, and we must have $r \ge 3$. Since $\alpha_r \le -1$ and $\alpha \ge 0$ for $i \le r - 1$, by Lemma 1.3, we get

$$|\alpha| \le \sum_{i \neq 1, r} \alpha_i + \sum_{i \neq 2, r} \alpha_i + \alpha_r \le 2(n-1) - 1 = 2n - 3. \qquad \square$$

Lemma 2.6. *Assume that* $|CS_\alpha| = 2$. *Then* $|\alpha| \le n - 3$.

Proof. Since $\mathrm{lk}_{CS_\alpha}(G) = \{\emptyset\}$, by Lemma 1.2, $\Delta_\alpha(I_G^{(n)})$ is either $\emptyset$ or equal to $\{\emptyset\}$. By (2.2), we must have $\widetilde{H}_{-1}(\Delta_\alpha(I_G^{(n)}); k) \neq 0$. Therefore, $\Delta_\alpha(I_G^{(n)}) = \{\emptyset\}$. By Lemma 1.5, $|\alpha| \le n - 3$. □

Lemma 2.7. *Assume that* G *contains a vertex of degree at least* 2. *Then* $a_2(R/I_G^{(n)}) \ge 2n - 3$.

Proof. We may assume that $\{1,2\},\{1,3\} \in E(G)$. Let $\beta = (-1, n-1, n-1, 0, \dots, 0)$. Then $CS_\beta = \{1\}$, $\mathrm{lk}_G(CS_\beta) \supseteq \{2,3\}$. By Lemma 1.3, one can check that $\Delta_\beta(I_G^{(n)}) = \{\emptyset, \{2\}, \{3\}\}$. Hence, by Lemma 1.1,

$$\dim_k [H^2_{\mathfrak{m}}(R/I_G^{(n)})]_\beta = \dim_k \widetilde{H}_0(\Delta_\beta(I_G^{(n)}); k) = 1,$$

which implies $a_2(R/I_G^{(n)}) \geq |\beta| = 2n-3$. □

We are now able to compute $a_2(R/I_G^{(n)})$:

Theorem 2.8. *For all $n \geqslant 1$, we have*

(i) *If* $\mathrm{girth}(G) = 3$, *then* $a_2(R/I_G^{(n)}) = 3n-3$.
(ii) *If* $\mathrm{girth}(G) = 4$, *then* $a_2(R/I_G^{(n)}) = 2n-2$.
(iii) *If* $\infty > \mathrm{girth}(G) \geqslant 5$, *then* $a_2(R/I_G) = 0$ *and* $a_2(R/I_G^{(n)}) = 2n-3$ *for all* $n \geqslant 2$.
(iv) *If G is a forest with some vertex of degree at least 2, then* $a_2(R/I_G^{(n)}) = 2n-3$.
(v) *If G consists of $t \geq 1$ disjoint edges and possibly isolated vertices, then*

$$a_2(R/I_G^{(n)}) = \begin{cases} -2 & \text{if } r = 2, \\ n-3 & \text{if } r > 2, \end{cases}$$

where r is the number of variables of R.

Proof. Let $m := a_2(R/I_G^{(n)})$, and let α be chosen as in (2.2) such that $m = |\alpha|$. Let $s = \mathrm{girth}(G)$. In the case $s < \infty$, we may assume that $C = (1, 2, \dots, s)$ is a cycle of G. We distinguish four cases.

Case 1. $s = 3$. By Lemmas 2.4, 2.5 and 2.6, $m \leq 3n-3$. Let $\beta = (n-1, n-1, n-1, 0, \dots, 0)$. Using Lemma 1.3, one can immediately check that $\Delta_\beta(I_G^{(n)})$ is a subgraph of G and contains C. By Lemma 1.6, $\widetilde{H}_1(\Delta_\beta(I_G^{(n)}); k) \neq 0$. Then, by Lemma 1.1, $[H^2_{\mathfrak{m}}(R/I_G^{(n)})]_\beta \neq 0$, whence $m \geq |\beta| = 3n-3$. Hence, $m = 3n-3$.

Case 2. $s = 4$. Again, by Lemmas 2.4, 2.5 and 2.6, $m \leq 2n-2$. Let $\beta = (n-1, 0, n-1, 0, \dots, 0)$. With a similar argument as in Case 1, we get $m = 2n-2$.

Case 3. $5 \leq s < \infty$. If $n = 1$, then again by Lemmas 2.4, 2.5 and 2.6, $m \leq 0$. Using a similar argument as in Case 1 applied to

$\beta = (0, \dots, 0)$, we get $m = 0$. If $n \geq 2$, then $[s(n-1)/(s-2)] \leq 2n-3$. Again by Lemmas 2.4, 2.5 and 2.6, $m \leq 2n-3$. Using Lemma 2.7, we then get $m = 2n-3$.

Case 4. $s = \infty$, that means G is a forest. If G contains a vertex of degree at least 2, then combining Lemmas 2.5, 2.6 and 2.7, we get $m = 2n-3$. Otherwise, G consists of t disjoint edges, where $t \geq 1$, and possibly some isolated vertices. If $r = 2$, then $t = 1$ and $I_G^{(n)} = I_G = 0$. It is clear that $a_2(R/I_G^{(n)}) = -2$. Let $r \geq 3$. By Lemma 2.4, we must have $|CS_\alpha| = 1, 2$. Assume that $|CS_\alpha| = 1$. Since at most one vertex is joined to the vertex of CS_α, $\Delta_\alpha(I_G^{(n)})$ must be $\emptyset$ or $\{\emptyset\}$ or consists of exactly one point. In all cases, by Lemma 1.1

$$\dim_k [H_{\mathfrak{m}}^2(R/I_G^{(n)})]_\alpha = \dim_k \widetilde{H}_0(\Delta_\alpha(I_G^{(n)}); k) = 0,$$

a contradiction. Hence, $|CS_\alpha| = 2$. By Lemma 2.6, $m = |\alpha| \leq n-3$.

On the other hand, in this case we may assume that $\{1, 2\} \in E(G)$. Let $\beta = (-1, -1, n-1, 0, \dots, 0)$. Then $CS_\beta = \{1, 2\}$, $\mathrm{lk}_G(CS_\beta) = \{\emptyset\}$. By Lemma 1.3, one can check that $\Delta_\beta(I_G^{(n)}) = \{\emptyset\}$. Hence, by Lemma 1.1,

$$\dim_k [H_{\mathfrak{m}}^2(R/I_G^{(n)})]_\beta = \dim_k \widetilde{H}_{-1}(\Delta_\beta(I_G^{(n)}); k) = 1,$$

which implies $m = a_2(R/I_G^{(n)}) \geq |\beta| = n-3$, whence $m = n-3$. □

Finally, we can state and prove the main result on the Castelnuovo-Mumford regularity. One can see that, as $a_2(R/I_G^{(n)})$, the function $\mathrm{reg}(R/I_G^{(n)})$ mainly depends on the girth of G.

Theorem 2.9. *For all $n \geqslant 1$, we have*

- (i) *If* $\mathrm{girth}(G) = 3$, *then* $\mathrm{reg}(R/I_G^{(n)}) = 3n-1$.
- (ii) *If* $\mathrm{girth}(G) = 4$, *then* $\mathrm{reg}(R/I_G^{(n)}) = 2n$.
- (iii) *If* $\infty > \mathrm{girth}(G) \geqslant 5$, *then* $\mathrm{reg}(R/I_G) = 2$ *and* $\mathrm{reg}(R/I_G^{(n)}) = 2n-1$ *for all* $n \geqslant 2$.
- (iv) *If G is a forest with at least two edges or at least one isolated vertex, then* $\mathrm{reg}(R/I_G^{(n)}) = 2n-1$.

(v) *If G consists of exactly one edge, then* $\mathrm{reg}(R/I_G^{(n)}) = 0$ *if* $r = 2$ *and* $\mathrm{reg}(R/I_G^{(n)}) = n - 1$ *for all* $r \geq 3$, *where* r *is the number of variables of* R.

Proof. By Lemma 2.1 and Theorem 2.3, $a_1(R/I_G^{(n)}) + 1 \leq 2n - 1$. Since

$$\mathrm{reg}(R/I_G^{(n)}) = \max\{a_1(R/I_G^{(n)}) + 1,\ a_2(R/I_G^{(n)}) + 2\},$$

using Theorem 2.8 above one immediately get the statements in the first three cases and also in the case when G is a forest with a vertex of degree at least 2.

So, it is left to consider the case G being a forest and all its vertices having degree one or zero. In particular, all edges of G must be disjoint. Recall that G has at least one edge. If G is a forest consisting of at least two disjoint edges or at least one isolated vertex, then G is disconnected. By Lemma 2.1 and Theorem 2.3, $a_1(R/I_G^{(n)}) + 1 = 2n - 1$, while by Theorem 2.8 (v), $a_2(R/I_G^{(n)}) + 2 = n - 1$. Hence, $\mathrm{reg}(R/I_G^{(n)}) = 2n - 1$. In the last case, when G consists of exactly one edge, then $R/I_G^{(n)}$ is a Cohen-Macaulay ring. Therefore, $\mathrm{reg}(R/I_G^{(n)}) = a_2(R/I_G^{(n)}) + 2$, and the statement follows from Theorem 2.8 (v). $\square$

From Lemma 2.1 and Theorem 2.3, it is clear that $a_1(R/I_G^{(n)})$ is a linear function for all $n \geq 1$, and, from Theorem 2.8 and Theorem 2.9, $a_2(R/I_G^{(n)})$ and $\mathrm{reg}(R/I_G^{(n)})$ are linear functions for all $n \geq 2$.

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