JOURNAL OF COMMUTATIVE ALGEBRA
Volume 7, Number 3, Fall 2015

SEMIDUALIZING MODULES AND RINGS OF INVARIANTS

WILLIAM SANDERS

ABSTRACT. We show there exist no nontrivial semidualizing modules for nonmodular rings of invariants of order p^n with p a prime.

1. Introduction. This paper is concerned with the existence of nontrivial semidualizing modules. Recall

Definition 1.1. A finitely generated S-module C is semidualizing if the map $S \to \mathrm{Hom}_S(C,C)$ given by $s \mapsto (x \mapsto sx)$ is an isomorphism and $\mathrm{Ext}_S^{i>0}(C,C)=0$.

This is equivalent to saying S is totally C reflexive. Examples always include S and the dualizing module, if it exists; thus, we call *trivial* these semidualizing modules. Semidualizing modules were first discovered by Foxby [**5**]. They were later rediscovered by various other authors including Vasconcelos, who called them spherical modules, and Golod, who referred to them as suitable modules. In [**15**], Vasconcelos asks if there exist only a finite number of nonisomorphic semidualizing modules. This question is answered in the affirmative in [**3**] for equicharacteristic Cohen-Macaulay algebras, and in [**10**] for the semilocal case. Since their discovery, semidualizing modules have been the focus of much research. See, for example, [**1, 7, 9, 10, 12, 13, 15**].

It is natural to ask which rings have only trivial semidualizing modules. In [**9**], Jorgensen, Leuschke and Sather-Wagstaff give a very nice characterization of rings with a dualizing module and only trivial semidualizing modules. However, this characterization is somewhat abstract, and it is difficult to tell whether the conditions hold for a

2010 AMS *Mathematics subject classification.* Primary 13A50, 13C13.

Keywords and phrases. Invariant theory, semidualizing module, rational singularity.

Received by the editors on September 7, 2013.

DOI:10.1216/JCA-2015-7-3-411

particular ring. Also, in **[12]**, Sather-Wagstaff proves results relating the existence of nontrivial semidualizing modules to Bass numbers. In this paper, we pose the following question:

Question 1.2. *If a ring S has a nice, e.g., rational, singularity, then does S have only trivial semidualizing modules?*

The evidence suggests the answer is yes. In **[4]**, Celikbas and Dao show that only trivial semidualizing modules exist over Veronese subrings, which have a quotient singularity and hence a rational singularity. Furthermore, Sather-Wagstaff shows in **[11]** that only trivial semidualizing modules exist for determinantal rings, which also have a rational singularity. It is proven in **[13,** Example 4.2.14] that all Cohen-Macaulay rings with minimal multiplicity have no nontrivial semidualizing modules. Since rational singularity and dimension 2 imply minimal multiplicity, all rings with rational singularity and dimension 2 have no nontrivial semidualizing modules. The following example shows that there are dimension 3 rings with rational singularity that do not have minimal multiplicity.

Example 1.3. Let

$$S = k[[x,y,z]]^{(3)} = k[[x^3,y^3,z^3,x^2y,x^2z,y^2x,y^2z,z^2x,z^2y,xyz]],$$

which is the third Veronese subring in three variables. For the multiplicity of S to be minimal, it must equal $\operatorname{edim} S - \dim S + 1 = 10-3+1 = 8$. However, setting $\overline{S} = S/(x^3,y^3,z^3)S$, $e(S) = e(\overline{S}) = \lambda(\overline{S})$ where λ is length. Since

$$\overline{S} = k \oplus kx^2y \oplus kx^2z \oplus ky^2x \oplus ky^2z \oplus kz^2x \oplus kz^2y \oplus kxyz \oplus kx^2y^2z^2,$$

we thus have $e(S) = 9$.

In this paper, we add to the evidence that suggests that the answer to Question 1.2 is "yes" by investigating the case where S is a ring of invariants, a large class of rings with rational singularity. The following theorem is the main result of this paper.

Theorem. *If S is a power series ring over a field k in finitely many variables and G is a cyclic group of order p^l acting on S with* $\operatorname{char} k \neq p$*, then S^G has only trivial semidualizing modules.*

Our approach to the proof of this result, relying on Lemma 2.1, is different than those of the results in [**4, 11**]. In each of those papers, the key technique involves counting the number of generators, whereas we use Lemma 2.1. See Section 2 for a further explanation.

Section 2 gives preliminary results concerning rings of invariants and semidualizing modules and also gives a sketch of the proof. Section 3 proves a key technical theorem about when a ring has only trivial semidualizing modules, and then Section 4 uses this result to prove our main theorem.

All rings considered in this paper will be Noetherian and commutative.

2. Preliminaries.

In this section, let S be a Noetherian ring. The proof relies upon the following lemma from [**9**].

Lemma 2.1. *If C is a semidualizing S-module and D is a dualizing module for S, then the homomorphism $\eta : C \otimes \operatorname{Hom}_S(C, D) \to D$ given by $x \otimes \varphi \mapsto \varphi(x)$ is an isomorphism.*

The map η being an isomorphism is a strong condition since D is torsionless and since tensor products often have torsion elements. We will exploit this map using the following lemma from [**11,** Fact 2.4] and [**6,** Theorem 3.1].

Lemma 2.2. *If C is a semidualizing S-module and S is a normal domain, then C is reflexive and hence an element of the class group.*

Therefore, when S is normal, $\operatorname{Hom}(C, D)$ is the element of the class group associated with $C^{-1} \circ D$, and all three modules involved in Lemma 2.1 are elements of the class group. In Theorem 3.2, with strong assumptions on S, we show that $A \otimes B$ has torsion for any elements A and B in the class group of S which are not isomorphic to S. The construction of a torsion element is easy; however, it requires considerable work to show that this element is not zero in the tensor product. With this setup, because of Lemma 2.1 and since D does not have torsion, nontrivial semidualizing modules cannot exist. The proof also requires the following lemma which is easily proven in [**13,** Proposition 2.2.1].

Lemma 2.3. *If $R \to S$ is a faithfully flat extension, then C is a semidualizing R-module if and only if $C \otimes S$ is a semidualizing S-module.*

For the remainder of this paper, let R be a polynomial ring in finitely many variables over an algebraically closed field k, and let G be a finite group acting linearly on R. We shall assume that the characteristic of k does not divide the order of the group. To prove the main result, Section 4 shows that, when $|G| = p^l$ for some prime, R^G satisfies the assumptions of Theorem 3.2. In order to do this, we need the following definition and lemma.

Definition 2.4. Given a character $\chi : G \to k^\times$, we denote by R_χ the set of relative invariants, namely, the polynomials $f \in R$ such that $gf = \chi(g)f$.

Note that R_χ is an R^G-module. The following lemma is from [**2,** Theorem 3.9.2].

Lemma 2.5. *The ring R^G is a normal domain whose class group is the subgroup $H \subseteq \mathrm{Hom}(G, k^\times)$ which consists of the characters that contain all the pseudoreflections in their kernel. Furthermore, for any $\chi \in H$, the relative invariants $R_{\chi^{-1}}$ form the reflexive module corresponding to the element χ.*

3. Class groups. In this section, let S be a Noetherian ring. We say that an element μ in an S-module M is indivisible if no nonunit $a \in S$ and $\nu \in M$ exists such that $\mu = a\nu$.

Lemma 3.1. *Suppose S is a k-algebra, with k a field and M and N are S-modules. Furthermore, suppose $f \in M$ and $g \in N$ are indivisible, and $\gamma \in M$ and $\rho \in N$ are not unit multiples of f and g, respectively.*

If there exist k-bases E, F, X of M, N, S, respectively, with $f, \gamma \in E$ and $g, \rho \in F$ such that, for every $\xi \in X$, $\varepsilon \in E$ and $\eta \in F$, $\xi\varepsilon$ is a k-linear multiple of an element in E and $\xi\eta$ is a k-linear multiple of an element of F, then $f \otimes g - \gamma \otimes \rho$ is not zero in $M \otimes_S N$.

Proof. Suppose that such bases E, F and X exist. Let $\mathfrak{F}$ denote the free abelian group functor. Recall that, for any modules U and V over a ring R, we construct $U \otimes_R V$ by quotienting $\mathfrak{F}(U \cup V)$ by the submodule, which we will call $K_{U,V}(R)$, generated by the relations of the form:

$$(v_1, u_1 + u_2) - (v_1, u_1) - (v_1, u_2)$$
$$(v_1 + v_2, u_1) - (v_1, u_1) - (v_2, u_1)$$
$$(\lambda v_1, u_1) - (v_1, \lambda u_1)$$

with $v_i \in U$, $u_i \in V$ and $\lambda \in R$. Hence, $M \otimes_S N \cong \mathfrak{F}(M \cup N)/K_{M,N}(S)$ and $M \otimes_k N \cong \mathfrak{F}(M \cup N)/K_{M,N}(k)$. Notice that, since $k \subseteq S$, $K_{M,N}(k) \subseteq K_{M,N}(S)$. So $M \otimes_S N$ is a quotient of $M \otimes_k N$. Specifically, we have the following isomorphism

$$\frac{M \otimes_k N}{K_{M,N}(S)/K_{M,N}(k)} \cong \frac{\mathfrak{F}(M \cup N)/K_{M,N}(k)}{K_{M,N}(S)/K_{M,N}(k)} \cong \frac{\mathfrak{F}(M \cup N)}{K_{M,N}(S)} \cong M \otimes_S N.$$

We claim that every element of $K_{M,N}(S)/K_{M,N}(k) \subseteq M \otimes_k N$ is of the form

$$\sum_{s=1}^{r} \lambda_s(\mu_s \tau_s \otimes \nu_s) - \lambda_s(\mu_s \otimes \tau_s \nu_s)$$

with $\lambda_i \in k$, $\mu_i \in E$, $\nu_i \in F$, $\tau_i \in X \backslash k$ and $\lambda_i \in k$. Take $z \in K(S)/K(k)$. Since the generators of $K(S)$ of the form $(v_1, u_1 + u_2) - (v_1, u_1) - (v_1, u_2)$ and $(v_1 + v_2, u_1) - (v_1, u_1) - (v_2, u_1)$ are in $K(k)$, we may write

$$z = \sum_i (m_i t_i \otimes n_i - m_i \otimes t_i n_i)$$

with $m_i \in M$, $n_i \in N$ and $t_i \in S$. However, since E, F, X are bases of M, N, X respectively, we may also write

$$m_i = \sum_j \alpha_{i,j} \mu_{i,j}, \qquad n_i = \sum_l \beta_{i,l} \nu_{i,l}, \qquad t_i = \sum_k \kappa_{i,k} \tau_{i,k},$$

with each $\lambda_s \in k$, $\mu_s \in E$, $\nu_s \in F$, $\tau_s \in X\backslash k$ and $\lambda_s \in k$. So we have

$$\begin{aligned}
z &= \sum_i (m_i t_i \otimes n_i - m_i \otimes t_i n_i) \\
&= \sum_i \Bigg(\Big(\sum_j \alpha_{i,j}\mu_{i,j} \Big) \Big(\sum_k \kappa_{i,k}\tau_{i,k} \Big) \otimes \Big(\sum_l \beta_{i,l}\nu_{i,l} \Big) \\
&\quad - \Big(\sum_j \alpha_{i,j}\mu_{i,j} \Big) \otimes \Big(\sum_k \kappa_{i,k}\tau_{i,k} \Big) \Big(\sum_l \beta_{i,l}\nu_{i,l} \Big) \Bigg) \\
&= \sum_{i,j,k,l} (\alpha_{i,j}\mu_{i,j}\kappa_{i,k}\tau_{i,k} \otimes \beta_{i,l}\nu_{i,l} - \alpha_{i,j}\mu_{i,j} \otimes \kappa_{i,k}\tau_{i,k}\beta_{i,l}\nu_{i,l}) \\
&= \sum_{i,j,k,l} \alpha_{i,j}\beta_{i,l}\kappa_{i,k} (\mu_{i,j}\tau_{i,k} \otimes \nu_{i,l} - \mu_{i,j} \otimes \tau_{i,k}\nu_{i,l}) \\
&= \sum_{i,j,k,l} \alpha_{i,j}\beta_{i,l}\kappa_{i,k}(\mu_{i,j}\tau_{i,k} \otimes \nu_{i,l}) - \alpha_{i,j}\beta_{i,l}\kappa_{i,k}(\mu_{i,j} \otimes \tau_{i,k}\nu_{i,l}).
\end{aligned}$$

Lastly, if $\tau_{i,k}$ is in k, then $\mu_{i,j}\tau_{i,k} \otimes \nu_{i,l} - \mu_{i,j} \otimes \tau_{i,k}\nu_{i,l}$ is already zero in $M \otimes_k N$. Therefore, setting $\lambda_{i,j,k,l} = \alpha_{i,j}\beta_{i,l}\kappa_{i,k} \in k$, the claim is shown.

Now suppose $f \otimes g - \gamma \otimes \rho$ is zero in $M \otimes_S N$. Then in $M \otimes_k N$, we may write

$$f \otimes g - \gamma \otimes \rho = \sum_{s=1}^{r} \lambda_s(\mu_s\tau_s \otimes \nu_s) - \lambda_s(\mu_s \otimes \tau_s\nu_s)$$

with $\lambda_s \in k$, $\mu_s \in E$, $\nu_s \in F$, $\tau_s \in X\backslash k$ and $\lambda_s \in k$. Now $Z = \{a \otimes b \mid a \in E, b \in F\}$ is a k-basis of $M \otimes_k N$. Since $f, \gamma \in E$ and $g, \rho \in F$, $f \otimes g$ and $\gamma \otimes \rho$ are in Z. By assumption, each $\mu_s\tau_s \otimes \nu_s$ and $\mu_s \otimes \tau_s\nu_s$ is a linear multiple of an element in Z. Thus, $f \otimes g$ must be a linear multiple of either $\mu_s\tau_s \otimes \nu_s$ or $\mu_s \otimes \tau_s\nu_s$ for some s. But, since f and g are indivisible and, for all s, neither $\mu_s\tau_s$ nor $\tau_s\nu_s$ is indivisible, this is a contradiction. Therefore, $f \otimes g - \gamma \otimes \rho$ cannot be zero in $M \otimes_S N$. □

Take a ring S with class group L with operation $\circ$. Let $T = \bigoplus_{A \in L} A$. We can give this S-module an L-graded S-algebra structure. For any $A, B \in L$, recall that $A \circ B = \operatorname{Hom}(\operatorname{Hom}(A \otimes_S B, S), S) \in L$. We will define the multiplication on the homogenous elements of T with the natural map $\varphi_{A,B} : A \otimes_S B \to \operatorname{Hom}(\operatorname{Hom}(A \otimes_S B, S), S)$ by setting

$ab = \varphi_{A,B}(a \otimes b)$, for any $a \in A$ and $b \in B$. We can extend this multiplication linearly to the nonhomogenous elements of T. Since S is contained in T, this algebra is unital, and, because $\mathrm{Hom}(\mathrm{Hom}(A \otimes_S B, S), S) \cong \mathrm{Hom}(\mathrm{Hom}(B \otimes_S A, S), S)$, it is commutative as well. This construction is similar to an algebra considered in **[14]**.

Theorem 3.2. *Let S be a Noetherian k-algebra, with k a field. Suppose L is finite and cyclic with generator Λ. Also suppose that the L-grading on T can be refined to a grading Γ such that every Γ-homogenous component is one dimensional. If there exists a Γ-homogenous element $x \in \Lambda \subseteq T$ such that $x^n \in \Lambda^n \subseteq T$ is indivisible (as an element of an S-module) for all $n \in \mathbb{N}$ strictly less than $|\Lambda|$, then for any $A, B \in L$ where neither A nor B is isomorphic to S, the module $A \otimes_S B$ has torsion.*

Proof. Since Λ generates L, there exist a and b such that $\Lambda^a = A$ and $\Lambda^b = B$. Then there exist $a, b \in \mathbb{N}$ such that $x^a \in A$ and $x^b \in B$. Since neither A nor B is isomorphic to S, a and b are both strictly less than $|L|$, and so x^a and x^b are indivisible. We may assume without loss of generality that $a \geq b$.

Let Q be a minimal homogenous generating set of B which contains x^b. We may assume every element in Q is indivisible, since, by the Noetherian condition, we can replace any divisible element by an indivisible one. Since B is not isomorphic to S and is torsionless, we know that Q has another element y besides x^b. Besides being indivisible and homogeneous, y is also not a unit multiple of x^b.

Set $z = x^a \otimes y - yx^{a-b} \otimes x^b$. We show that z is a torsion element. Since x^{a-b} is in Λ^{a-b} and y is in $B = \Lambda^b$, yx^{a-b} is Λ^a, which is A. Thus, z is in $A \otimes_S B$. Furthermore, for any $f \in (A \circ B)^{-1}$, we have $x^a yf, x^{a+b} f \in S$. Thus, we have

$$(x^a yf)z = x^{2a} yf \otimes y - yx^{a-b} \otimes x^{a+b} yf = x^{2a} yf \otimes y - x^{2a} yf \otimes y = 0.$$

Thus, to show that z is a torsion element, it suffices to show that z is not zero in $A \otimes_S B$.

Note that, by construction, x^a and y are indivisible, and since y and x^b are not unit multiples of each other, neither are x^a and yx^{a-b}. Also yx^{a-b} is homogenous since x^{a-b} is. We can choose Γ-homogenous bases E and F of A and B, respectively, such that $x^a, y \in E$ and

$x^a, yx^{a-b} \in F$. Similarly we can choose a Γ-homogenous basis X of S. Since every Γ-homogenous component of T is one-dimensional, for every $\xi \in X$ and $\varepsilon \in E$ and $\eta \in F$, $\xi\varepsilon$ is a linear multiple of an element in E and $\xi\eta$ is a linear multiple of an element of F. Thus, z meets the hypotheses of the previous proposition. Therefore, z is not zero in $A \otimes_S B$. □

Corollary 3.3. *Assume the set up of the last theorem and that S has a dualizing module. Then S has no nontrivial semidualizing modules.*

Proof. Let C be a semidualizing module for S. Then $C \otimes \mathrm{Hom}(C, D) \cong D$, where D is a dualizing module. However, $\mathrm{Hom}(C, D) \cong C^{-1} \circ D$ is also an element of the class group. Thus, by the previous theorem, since D is torsionless, either C or $\mathrm{Hom}(C, D)$ is isomorphic to S. Therefore, C is isomorphic to S or D. □

4. Semidualizing modules of rings of invariants. Let R be the polynomial ring in d variables over k. We can apply the previous results to the semidualizing modules over rings of invariants for a certain cyclic group, but first we need a lemma.

Lemma 4.1. *Assume k is an algebraically closed field. If G is a finite cyclic group acting linearly on R generated by g whose order is not divisible by the characteristic of k, then there exist algebraically independent $x_1, \ldots, x_d \in R$ such that $R = k[x_1, \ldots, x_d]$ and $gx_i = \zeta^{\eta_i} x_i$ with $\zeta \in k$ a primitive $|G|$th root of unity.*

Proof. By putting g in the Jordan canonical form, it is an easy exercise to see that g is diagonalizable since $|G|$ and $\mathrm{char}\, k$ are coprime. Thus, we may choose an eigenbasis, $x_1, \ldots, x_d$, of R_1. So, $gx_i = \xi_i x_i$ with $\xi_i \in k$. Since $g^{|G|}$ should act as the identity, each ξ_i must be a $|G|$th root of unity, and so we may write $\xi = \zeta^{\eta_i}$ where ζ is some fixed primitive $|G|$th root of unity. Also, R is isomorphic to the symmetric algebra of R_1 which is a polynomial ring in the variables $x_1, \ldots, x_d$. Hence, $x_1, \ldots, x_d$ are algebraically independent. □

To apply the results of Section 3, we will observe that, in this case,

$$T = \bigoplus_{\chi \in L} R_{\chi^{-1}} \subseteq R,$$

where L is the class group of R^G. The desired grading Γ of T will be the monomial grading with respect to the variables $x_1, \ldots, x_d$ defined in the previous lemma. Before we proceed, however, we need to show that this grading is a refinement of L, to which end, the following lemma suffices.

Lemma 4.2. *If G consists of diagonal matrices, then for any character $\chi : G \to k^\times$, the set of all monomials in R_χ is a k-basis.*

Proof. Let X be the set of all monomials of R_χ. Since any distinct monomials are linearly independent, X is linearly independent. Take any $g \in G$. Then, for each i, $gx_i = \lambda_i x_i$ with $\lambda_i \in k$. So, for any $\underline{x}^{\underline{\alpha}} = x_1^{a_1} \cdots x_d^{a_d}$ in R, we have

$$g\underline{x}^{\underline{\alpha}} = gx_1^{a_1} \cdots x_n^{a_d} = (\lambda_1 x_1)^{a_1} \cdots (\lambda_d x_d)^{a_d} = \underline{\lambda} x_1^{a_1} \cdots x_d^{a_d} = \underline{\lambda x}^{\underline{\alpha}}$$

with $\underline{\lambda} = \lambda_1^{a_1} \cdots \lambda_d^{a_d}$. Take any $f \in R_\chi$. We may write $f = \kappa_1 \underline{x}^{\underline{\alpha}_1} + \cdots + \kappa_m \underline{x}^{\underline{\alpha}_m}$. On the one hand, we know that

$$\begin{aligned} gf &= g(\kappa_1 \underline{x}^{\underline{\alpha}_1} + \cdots + \kappa_m \underline{x}^{\underline{\alpha}_m}) \\ &= g\kappa_1 \underline{x}^{\underline{\alpha}_1} + \cdots + g\kappa_m \underline{x}^{\underline{\alpha}_m} \\ &= \kappa_1 \underline{\lambda}_1 \underline{x}^{\underline{\alpha}_1} + \cdots + \kappa_m \underline{\lambda}_m \underline{x}^{\underline{\alpha}_m} \end{aligned}$$

with $\underline{\lambda}_i = \lambda_1^{a_{1i}} \cdots \lambda_d^{a_{di}}$. By virtue of f being in R_χ, we also know that

$$gf = \chi(g) f = \chi(g)\kappa_1 \underline{x}^{\underline{\alpha}_1} + \cdots + \chi(g)\kappa_m \underline{x}^{\underline{\alpha}_m}.$$

However, since monomials are linearly independent, this means that, for each i, $\kappa_i \underline{\lambda}_i = \chi(g)\kappa_i$, and so $\underline{\lambda}_i = \chi(g)$. Therefore, for each i, $\underline{x}^{\underline{\alpha}_i}$ is in R_χ and thus also in X. Hence, X spans R_χ and is a basis. $\square$

Proposition 4.3. *Suppose S is a power series ring over a field k in d variables and G is a cyclic group of order n acting on R with* char k *not dividing n. If g generates G and has a primitive nth root of unity as an eigenvalue, then S^G has only trivial semidualizing modules.*

Proof. By Lemma 2.3, since $\overline{k}\otimes S^G$ is a faithfully flat extension of S^G, C is a semidualizing S^G-module if and only if $\overline{k}\otimes_{S^G} C$ is a semidualizing $\overline{k}\otimes S^G$-module. Thus, if there are no nontrivial semidualizing modules for $\overline{k}\otimes S^G$, then there are none for S^G. So, we may assume that k is algebraically closed.

Since G is cyclic and is generated by g, a character in $\operatorname{Hom}(G, k^\times)$ is completely determined by the image of g. However, g can only be sent to an nth root of unity. Since k is algebraically closed, and since $\operatorname{char} k$ does not divide n, there are n distinct nth roots of unity, which form a cyclic group. Therefore, G is isomorphic to $\operatorname{Hom}(G, k^\times)$. Since the class group of R^G is a subgroup of $\operatorname{Hom}(G, k^\times)$, this means the class group must be cyclic.

By the previous lemma, we may write $R = k[x_1, \ldots, x_d]$ where $gx_i = \zeta^{\eta_i} x_i$ with $\zeta \in k$ a primitive $|G|$th root on unity. The assumption tells us that we may assume that $\eta_1 = 0$. Define $\chi : G \to k^\times$ by $g \mapsto \zeta^{-1}$. Since ζ^{-1} is a primitive $|G|$th root of unity, χ generates $\operatorname{Hom}(G, k^\times)$. So, for some $\lambda \in \mathbb{N}$, χ^λ generates the class group L. Assume that λ is as small as possible. Note that $gx_1^\lambda = (\zeta x_1)^\lambda = \zeta^\lambda x_1^\lambda$, and so $x_1^\lambda \in R_{\chi^{-\lambda}}$, the reflexive module corresponding to χ^λ. Since we have chosen λ to be as small as possible, $|\chi^\lambda| = n/\lambda$. Thus, for each $1 \le \nu < |\chi^{-\lambda}| = n/\lambda$, $\lambda\nu$ is strictly less than n. Since the smallest power of x_1 that is invariant is n, this means that $(x_1^\lambda)^\nu$ is indivisible. Therefore, using the monomial grading, the conditions of Corollary 3.3 and Theorem 3.2 are satisfied, and thus R^G has no nontrivial semidualizing modules. Since S^G is the completion of R^G, and completion is faithfully flat, we are done by Lemma 2.3. $\square$

We can recover the nonmodular case of [**4,** Corollary 3.21].

Corollary 4.4. *There exist no semidualizing modules over nonmodular Veronese subrings.*

Proof. Let g be a $d \times d$ diagonal matrix whose entries are all ζ_n, a primitive nth root of unity. Then the n-Veronese subring in d variables is $R = k[[x_1, \ldots, x_d]]^G$ where G is the group generated by g. Since the order of G is n, the result follows from the previous proposition. $\square$

We now come to our main theorem.

Theorem 4.5. *If S is a power series ring over a field k in finitely many variables and G is a cyclic group of order p^l acting on S with* $\operatorname{char} k \neq p$*, then S^G has only trivial semidualizing modules.*

Proof. By Lemma 2.1, we may write $R = k[x_1, \ldots, x_d]$, where $gx_i = \zeta^{\eta_i} x_i$ with $\zeta \in k$ a primitive $|G|$th root on unity. We may assume that ζ^{η_1} has the greatest order of all the ζ^{η_i} and set $z = |\zeta^{\eta_1}|$. Since $|\zeta^{\eta_i}|$ is a power of p less than z, we have $|\zeta^{\eta_i}|$ divides z for each i, and so $(\zeta^{\eta_i})^z = 1$. Thus, viewing g as a diagonal matrix with entries ζ^{η_i}, g^z is the identity, and so $n \leq z$. But, z has to be less than n, giving us equality. Hence, ζ^{η_1} is a primitive nth root of unity. However, since our choice of ζ is arbitrary, we may assume that $\eta_1 = 1$. In short, we have $gx_1 = \zeta x_1$. The result follows from the previous proposition. $\square$

The proofs of Theorem 4.5 and Proposition 4.3 show that Theorem 3.2 applies to the class of rings under consideration. Thus, we actually have the following result, which resolves in the affirmative a special case of Conjecture 1.3 in [**8**].

Corollary 4.6. *Assume the set up of the previous theorem, and let D be a dualizing module for S. If M is a reflexive module of rank 1 and $M \otimes_S \operatorname{Hom}_S(M, D)$ is torsion free, then M is isomorphic to either S or D.*

Proof. Since M and $\operatorname{Hom}_S(M, D)$ are both elements of the class group, and since Theorem 3.2 applies, either M or $\operatorname{Hom}(M, D)$ is isomorphic to S. The latter case implies that $M \cong D$. $\square$

Acknowledgments. The author thanks Sean Sather-Wagstaff and Hailong Dao for their useful insights. The author would also like to thank the referee for their very helpful comments.

REFERENCES

1. Tokuji Araya and Ryo Takahashi, *A generalization of a theorem of Foxby*, Arch. Math. (Basel) **93** (2009), 123-127.

2. D.J. Benson, *Polynomial invariants of finite groups*, Lond. Math. Soc. Lect. Note Series **190**, Cambridge University Press, Cambridge, 1993.

3. Lars Winther Christensen and Sean Sather-Wagstaff, *A Cohen-Macaulay algebra has only finitely many semidualizing modules*, Math. Proc. Cambr. Phil. Soc. **145** (2008), 601-603.

4. Hailong Dao and Olgur Celikbas, *Necessary conditions for the depth formula over cohen-macaulay local rings*, 2011, `arXiv:1008.2573`.

5. Hans-Bjørn Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1973), 267-284.

6. A. Gerko, *On the structure of the set of semidualizing complexes*, Illinois J. Math. **48** (2004), 965-976.

7. A.A. Gerko, *Homological dimensions and semidualizing complexes*, Sov. Mat. Pril. **30** (2005), 3–30.

8. Shiro Goto, Ryo Takahashi, Naoki Taniguchi and Hoang Le Truong, *Huneke-wiegand conjecture and change of rings*, 2013, `arXiv:1305.4238v2`.

9. David A. Jorgensen, Graham J. Leuschke and Sean Sather-Wagstaff, *Presentations of rings with non-trivial semidualizing modules*, Coll. Math. **63** (2012), 165-180.

10. Saeed Nasseh and Sean Sather-Wagstaff, *A local ring has only finitely many semidualizing complexes up to shift-isomorphism*, 2012, `arXiv:1201.0037`.

11. Sean Sather-Wagstaff, *Semidualizing modules and the divisor class group*, Illinois J. Math. **51** (2007), 255-285.

12. ______, *Bass numbers and semidualizing complexes, Commutative algebra and its applications, Walter de Gruyter*, Berlin, 2009, 349-381.

13. ______, *Semidualizing modules*, 2009, `http://www.ndsu.edu/pubweb/ssatherw/DOCS/han.pdf`.

14. Masataka Tomari and Keiichi Watanabe, *Normal Z_r-graded rings and normal cyclic covers*, Manuscr. Math. **76** (1992), 325-340.

15. Wolmer V. Vasconcelos, *Divisor theory in module categories*, North-Holland Publishing Co., Amsterdam, 1974.

Norwegian University of Science and Technology, NO-7491 Trondheim, Norway

Email address: **william.sanders@math.ntnu.no**