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STANLEY CONJECTURE ON MONOMIAL IDEALS OF MIXED PRODUCTS

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ABSTRACT. It is proved that the Stanley conjecture holds for monomial ideals of mixed products, i.e., if I is an ideal of mixed products in a polynomial ring S over a field, then $\mathrm{sdepth}_S(I) \geq \mathrm{depth}_S(I)$ and $\mathrm{sdepth}_S(S/I) \geq \mathrm{depth}_S(S/I)$.

1. Introduction. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K and M a finitely generated $\mathbb{Z}^n$-graded S-module. A Stanley decomposition $\mathcal{D}$ of M is a finite direct sum of K-spaces:

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i],$$

where $m_i \in M$ is homogeneous and $Z_i \subseteq \{x_1, \dots, x_n\}$, $i = 1, \dots, r$, and its Stanley depth, $\mathrm{sdepth}_S(\mathcal{D})$, is defined as $\min\{|Z_i| \mid i = 1, \dots, r\}$. By definition, the Stanley depth of M is the following

$$\mathrm{sdepth}_S(M) = \max\{\mathrm{sdepth}_S(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

Stanley **[13]** conjectured that $\mathrm{sdepth}_S(M) \geq \mathrm{depth}_S(M)$. There has been much research on this conjecture, especially when M has the form S/I or I with I a square-free monomial ideal of S, cf., **[8, 9, 10, 14]**. In this paper, we consider the case where I is an ideal of mixed products.

Let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$, $S_1 = K[x_1, \dots, x_n]$ and $S_2 = K[y_1, \dots, y_m]$ be polynomial rings over K. Let $I \subset I' \subset S_1$ and

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$J \subset J' \subset S_2$ be non zero monomial ideals. We call a *generalized mixed product of ideals* the ideal defined as

$$L = (I'J + IJ')S.$$

If $I = I_q$ and $I' = I_r$ are square-free Veronese monomial ideals in S_1, and $J = J_s$ and $J' = J_t$ are square-free Veronese monomial ideals in S_2, i.e., they are generated by all the square-free monomials of degree q, r, s and t, respectively, we recover the class of ideals of mixed products, as defined in **[12]**. In particular, if the ideal L has the form I_rJ_r or $I_rJ_{r-1} + I_{r-1}J_r$, L is the generalized graph ideal of a complete bipartite graph whose vertices are $x_1, \dots, x_n, y_1, \dots, y_n$ (**[12]**). From the geometric point of view we can look at L as the ideal I_Δ where Δ is a simplicial complex and I_Δ is its Stanley-Reisner ideal. In particular, for $L = I_r + J_r$, as before, the structure of the simplicial complex Δ is not difficult to understand, being the join $\Delta_1 * \Delta_2$ of two disjoint simplicial complexes consisting of the $(r-1)$-skeletons of two $(r-1)$-simplices on the sets of vertices $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$, respectively, with $I_r = I_{\Delta_1}$ and $J_r = I_{\Delta_2}$. We refer to **[1, 15]** for basic properties of Stanley-Reisner ideals.

Section 2 consists of background material on depth and Stanley depth of ideals $IS + JS, IS \cap JS, S/(IS + JS)$ and $S/(IS \cap JS)$. In Section 3, we give lower bounds for the Stanley depth of ideals of generalized mixed products $L = (I'J + IJ')S$ and of S/L, under some hypotheses. These assumptions are not strong, since they are verified for large classes of ideals (for example, in the Veronese square-free case). In Section 4, the Stanley depth conjecture is verified for the class of ideals of mixed products as defined in **[5, 12]**.

2. Preliminaries. Let $S = K[x_1, \dots, x_n, y_1, \dots, y_m]$, $S_1 = K[x_1, \dots, x_n]$ and $S_2 = K[y_1, \dots, y_m]$. For $1 \leq q \leq n$ (respectively, $1 \leq p \leq m$), denote the ideal of S_1 (respectively, S_2) generated by all the square-free monomials of degree q (respectively, p) by I_q (respectively, J_p).

Lemma 2.1. (**[5,** Theorems 3.4 and 3.7]). *Let* $1 \leq q < s, 1 \leq t < r$. *Then*

(1) $\operatorname{depth}_S(S/I_qS) = m + q - 1$, $\operatorname{depth}(S/J_tS) = n + t - 1$;
(2) $\operatorname{depth}_S(S/(I_qJ_t)S) = q + t - 1$;

(3) $\operatorname{depth}_S(S/(I_q + J_t)S) = q + t - 2$;
(4) $\operatorname{depth}_S(S/(I_qJ_r + I_s)S) = q + r - 1$;
(5) $\operatorname{depth}_S(S/(I_qJ_r + I_sJ_t)S) = \min\{q + r, s + t\} - 1$.

The following lemma is a result observed in the proof of [**3,** Theorem 3.1].

Lemma 2.2. *Let H be a multigraded S_1-module and L a multigraded S_2-module. Then, as multigraded S-modules,*

$$\operatorname{sdepth}_S(H \otimes_K L) \geq \operatorname{sdepth}_{S_1}(H) + \operatorname{sdepth}_{S_2}(L).$$

For the Stanley depth, we have the following

Lemma 2.3. *Let $I \subset S_1$ and $J \subset S_2$ be monomial ideals. Then*

(1) ([**2,** Theorem 1.3]). $\operatorname{sdepth}_S(IS + JS) \geq \min\{\operatorname{sdepth}_S(IS), \operatorname{sdepth}_{S_2}(J) + \operatorname{sdepth}_{S_1}(S_1/I)\}$;
(2) ([**7,** Lemma 1.2]). $\operatorname{sdepth}_S(IS \cap JS) \geq \operatorname{sdepth}_{S_1}(I) + \operatorname{sdepth}_{S_2}(J)$;
(3) ([**11,** Theorem 3.1]). $\operatorname{sdepth}_S(S/(IS + JS)) \geq \operatorname{sdepth}_{S_1}(S_1/I) + \operatorname{sdepth}_{S_2}(S_2/J)$;
(4) ([**2,** Theorem 1.3]). $\operatorname{sdepth}_S(S/(IS \cap JS)) \geq \min\{\operatorname{sdepth}_S(S/IS), \operatorname{sdepth}_{S_1}(I) + \operatorname{sdepth}_{S_2}(S_2/J)\}$;
(5) $\operatorname{sdepth}_S(IS/(IS \cap JS)) \geq \operatorname{sdepth}_{S_1}(I) + \operatorname{sdepth}_{S_2}(S_2/J)$.

Proof. For item (5), note that $IS \cap JS = (IJ)S$ and $IS/(IS \cap JS) \cong I \otimes_K S_2/J$ as K-spaces. Then the result follows from Lemma 2.2. $\square$

From [**4,** Proposition 3.1], we have

Lemma 2.4.

(1) $\operatorname{sdepth}_{S_1}(S_1/I_q) = q - 1$;
(2) *Let $I = (u_1, \ldots, u_r)$ be a square-free monomial ideal of S_1. Then*

$$\operatorname{sdepth}_{S_1}(I) \geq \min\{\deg(u_i) \mid i = 1, \ldots, r\}.$$

Then, by [**4,** Lemma 3.6], $\operatorname{sdepth}_S(S/I_qS) = m + q - 1$ and $\operatorname{sdepth}_S(IS) = m + \operatorname{sdepth}_{S_1}(I)$, especially $\operatorname{sdepth}_S(I_qS) \geq m + q$.

3. Stanley depths of generalized mixed products of ideals. Let $I \subset I' \subset S_1$ and $J \subset J' \subset S_2$ be nonzero monomial ideals. We call the following ideal of S as a generalized mixed product of ideals:

$$(I'J + IJ')S.$$

For the generalized mixed products of ideals, the following four propositions estimate their Stanley depths.

Proposition 3.1.

(1) $\mathrm{sdepth}_S((IJ)S) \geq \mathrm{sdepth}_{S_1}(I) + \mathrm{sdepth}_{S_2}(J)$;
(2) $\mathrm{sdepth}_S(S/(IJ)S) \geq \mathrm{sdepth}_{S_2}(S_2/J) + \min\{\mathrm{sdepth}_{S_1}(I), \mathrm{sdepth}_{S_1}(S_1/I) + 1\}$.

Proof.

(1) Since $(IJ)S = IS \cap JS$, it follows from Lemma 2.3 (2).
(2) Firstly, note that $\mathrm{sdepth}_{S_2}(S_2/J) \neq m$; otherwise, $S_2/J = uS_2$ for some monomial u. If $u = 1$, then $J = 0$, and if $u \neq 1$, then $1 \in J$, a contradiction. Then, by Lemma 2.3 (4), we have

$$\begin{aligned}
&\mathrm{sdepth}_S(S/(IJ)S) = \mathrm{sdepth}_S(S/(IS \cap JS)) \\
&\geq \min\{\mathrm{sdepth}_{S_1}(S_1/I) + m, \\
&\qquad \mathrm{sdepth}_{S_2}(S_2/J) + \mathrm{sdepth}_{S_1}(I)\} \\
&\geq \min\{\mathrm{sdepth}_{S_1}(S_1/I) + \mathrm{sdepth}_{S_2}(S_2/J) + 1, \\
&\qquad \mathrm{sdepth}_{S_2}(S_2/J) + \mathrm{sdepth}_{S_1}(I)\} \\
&= \mathrm{sdepth}_{S_2}(S_2/J) + \min\{\mathrm{sdepth}_{S_1}(I), \\
&\quad \mathrm{sdepth}_{S_1}(S_1/I) + 1\}.
\end{aligned}$$

$\square$

Proposition 3.2. $\mathrm{sdepth}_S((I+J)S) \geq \mathrm{sdepth}_{S_2}(J) + \min\{\mathrm{sdepth}_{S_1}(I), \mathrm{sdepth}_{S_1}(S_1/I)\}$.

Proof. By virtue of Lemma 2.3 (1), one gets

$$\begin{aligned}
&\mathrm{sdepth}_S((I+J)S) \geq \min\{\mathrm{sdepth}_{S_1}(I) + m, \mathrm{sdepth}_{S_2}(J) \\
&\quad + \mathrm{sdepth}_{S_1}(S_1/I)\} \\
&\geq \min\{\mathrm{sdepth}_{S_1}(I) + \mathrm{sdepth}_{S_2}(J), \mathrm{sdepth}_{S_2}(J) \\
&\quad + \mathrm{sdepth}_{S_1}(S_1/I)\}
\end{aligned}$$

$$= \operatorname{sdepth}_{S_2}(J)+\min\{\operatorname{sdepth}_{S_1}(I), \operatorname{sdepth}_{S_1}(S_1/I)\}. \qquad \square$$

Proposition 3.3.

(1) *Suppose that* $\operatorname{sdepth}_{S_2}(J) > \operatorname{sdepth}_{S_2}(S_2/J)$. *Then*

$$\begin{aligned}\operatorname{sdepth}_S((I'J+I)S) &\geq \operatorname{sdepth}_{S_2}(S_2/J)\\ &\quad + \min\{\operatorname{sdepth}_{S_1}(I')+1, \operatorname{sdepth}_{S_1}(I)\}.\end{aligned}$$

(2) *Suppose that* $\operatorname{sdepth}_{S_1}(S_1/I) > \operatorname{sdepth}_{S_1}(S_1/I')$. *Then*

$$\begin{aligned}\operatorname{sdepth}_S(S/(I'J+I)S) &\geq \operatorname{sdepth}_{S_1}(S_1/I')\\ &\quad + \min\{\operatorname{sdepth}_{S_2}(J), \operatorname{sdepth}_{S_2}(S_2/J)+1\}.\end{aligned}$$

Proof.

(1) As K-spaces, we have the following decompositions

$$\begin{aligned}(I'J+I)S &\cong (I'J)S \oplus \frac{(I'J+I)S}{(I'J)S}\\ &\cong (I'J)S \oplus \frac{IS}{(I'J)S \cap IS}\\ &= (I'J)S \oplus \frac{IS}{IS \cap JS \cap I'S}\\ &= (I'J)S \oplus \frac{IS}{IS \cap JS}.\end{aligned}$$

It follows from Lemma 2.3 (5) that

$$\begin{aligned}&\operatorname{sdepth}_S((I'J+I)S)\\ &\geq \min\{\operatorname{sdepth}_S((I'J)S), \operatorname{sdepth}_S(IS/(IS\cap JS))\}\\ &\geq \min\{\operatorname{sdepth}_{S_1}(I')+\operatorname{sdepth}_{S_2}(J), \operatorname{sdepth}_{S_1}(I)\\ &\quad + \operatorname{sdepth}_{S_2}(S_2/J)\}.\end{aligned}$$

Under the assumption that $\operatorname{sdepth}_{S_2}(J) > \operatorname{sdepth}_{S_2}(S_2/J)$, it turns out that

$$\begin{aligned}\operatorname{sdepth}_S((I'J+I)S) &\geq \operatorname{sdepth}_{S_2}(S_2/J)\\ &\quad + \min\{\operatorname{sdepth}_{S_1}(I')+1, \operatorname{sdepth}_{S_1}(I)\}.\end{aligned}$$

(2) As K-spaces, we have the following decompositions

$$\begin{aligned} S/(I'J+I)S &\cong \frac{S}{(J+I)S} \oplus \frac{(J+I)S}{(I'J+I)S} \\ &= \frac{S}{(J+I)S} \oplus \frac{(J+I)S}{I'S \cap (J+I)S} \\ &\cong \frac{S}{(J+I)S} \oplus \frac{(I'+J+I)S}{I'S} \\ &= \frac{S}{(J+I)S} \oplus \frac{(I'+J)S}{I'S} \\ &\cong \frac{S}{(J+I)S} \oplus \frac{JS}{I'S \cap JS}. \end{aligned}$$

Then, it follows from Lemma 2.3 (3) and (5) that

$$\begin{aligned} &\operatorname{sdepth}_S(S/(I'J+I)S) \\ &\geq \min\{\operatorname{sdepth}_S(S/(J+I)S), \\ &\qquad \operatorname{sdepth}_S(JS/(I'S \cap JS))\} \\ &\geq \min\{\operatorname{sdepth}_{S_1}(S_1/I) + \operatorname{sdepth}_{S_2}(S_2/J), \\ &\qquad \operatorname{sdepth}_{S_2}(J) + \operatorname{sdepth}_{S_1}(S_1/I')\}. \end{aligned}$$

If $\operatorname{sdepth}_{S_1}(S_1/I) > \operatorname{sdepth}_{S_1}(S_1/I')$, then

$$\begin{aligned} \operatorname{sdepth}_S(S/(I'J+I)S) &\geq \operatorname{sdepth}_{S_1}(S_1/I') \\ &\quad + \min\{\operatorname{sdepth}_{S_2}(J), \operatorname{sdepth}_{S_2}(S_2/J)+1\}. \end{aligned}$$ □

Proposition 3.4.

(1) $\operatorname{sdepth}_S((I'J + IJ')S) \geq \min\{\operatorname{sdepth}_{S_1}(I) + \operatorname{sdepth}_{S_2}(J'), \operatorname{sdepth}_{S_1}(I'/I) + \operatorname{sdepth}_{S_2}(J)\}$;

(2) *Suppose that* $\operatorname{sdepth}_{S_1}(S_1/I) > \operatorname{sdepth}_{S_1}(S_1/I')$. *Then*

$$\begin{aligned} &\operatorname{sdepth}_S(S/(I'J+IJ')S) \\ &\geq \min\{\operatorname{sdepth}_{S_1}(S_1/I') + \min\{\operatorname{sdepth}_{S_2}(J), \\ &\qquad \operatorname{sdepth}_{S_2}(S_2/J)+1\}, \operatorname{sdepth}_{S_1}(I) \\ &\qquad + \operatorname{sdepth}_{S_2}(S_2/J')\}. \end{aligned}$$

Proof.

(1) As K-spaces, we have the following decompositions

$$\begin{aligned}(I'J+IJ')S &\cong (IJ')S \oplus \frac{(I'J+IJ')S}{(IJ')S}\\ &\cong (IJ')S \oplus \frac{(I'J)S}{IS\cap J'S\cap I'S\cap JS}\\ &= (IJ')S \oplus \frac{(JI')S}{(JI)S}\\ &\cong (IJ')S \oplus (J\otimes_K \frac{I'}{I})S.\end{aligned}$$

Since $\operatorname{sdepth}_S((J\otimes_K(I'/I))S) \geq \operatorname{sdepth}_{S_1}(I'/I)+\operatorname{sdepth}_{S_2}(J)$ by Lemma 2.2, it follows that

$$\begin{aligned}&\operatorname{sdepth}_S((I'J+IJ')S)\\ &\quad\geq \min\left\{\operatorname{sdepth}_S((IJ')S), \operatorname{sdepth}_S\left(\left(J\otimes_K \frac{I'}{I}\right)S\right)\right\}\\ &\quad\geq \min\{\operatorname{sdepth}_{S_1}(I)+\operatorname{sdepth}_{S_2}(J'),\\ &\qquad\qquad \operatorname{sdepth}_{S_1}(I'/I)+\operatorname{sdepth}_{S_2}(J)\}.\end{aligned}$$

(2) As K-spaces, the following decompositions hold

$$\begin{aligned}S/(I'J+IJ')S &\cong \frac{S}{(I'J+I)S}\oplus\frac{(I'J+I)S}{(I'J+IJ')S}\\ &= \frac{S}{(I'J+I)S}\oplus\frac{(I'J+I)S}{(I'J+I)S\cap J'S}\\ &\cong \frac{S}{(I'J+I)S}\oplus\frac{(I'J+I)S+J'S}{J'S}\\ &= \frac{S}{(I'J+I)S}\oplus\frac{(I+J')S}{J'S}\\ &\cong \frac{S}{(I'J+I)S}\oplus\frac{IS}{IS\cap J'S}.\end{aligned}$$

By the assumption that $\operatorname{sdepth}_{S_1}(S_1/I) > \operatorname{sdepth}_{S_1}(S_1/I')$, from Proposition 3.3 (2), we have

$$\begin{aligned}&\operatorname{sdepth}_S(S/(I'J+IJ')S)\\ &\quad\geq \min\{\operatorname{sdepth}_S(S/(I'J+I)S),\end{aligned}$$

$$\begin{aligned}&\mathrm{sdepth}_S(IS/(IS\cap J'S))\}\\ \geq \min\{&\mathrm{sdepth}_{S_1}(S_1/I')\\ &+\min\{\mathrm{sdepth}_{S_2}(J),\mathrm{sdepth}_{S_2}(S_2/J)+1\},\\ &\mathrm{sdepth}_{S_1}(I)+\mathrm{sdepth}_{S_2}(S_2/J')\}.\end{aligned}$$ □

4. Stanley conjecture on ideals of mixed products. Let $S = K[x_1,\dots,x_N]$ be a polynomial ring over a field K and L a square-free monomial ideal of S. We say that L is an ideal of mixed products if there exists an integer n such that $1 \leq n < N$ and

$$L = (I_qJ_r + I_sJ_t)S, \quad 0 \leq q < s \leq n,\ 0 \leq t < r \leq N-n,$$

where I_q (respectively, I_s) is the square-free monomial ideal of $K[x_1,\dots,x_n]$ generated by all the square-free monomials of degree q (respectively, s), and J_r (respectively, J_t) is the square-free monomial ideal of $K[x_{n+1},\dots,x_N]$ generated by all the square-free monomials of degree r (respectively, t). We use the convention that $I_0 = K[x_1,\dots,x_n]$ and $J_0 = K[x_{n+1},\dots,x_N]$.

Now we can prove that the Stanley conjecture holds for ideals of mixed products. It should be noticed that the following case $L = (I_qJ_t)S = I_qS \cap J_tS$ was done in **[7, 10]**, especially **[6,** Corollary 1.12 and Theorem 2.1]. For the completion, we give a direct proof.

Theorem 4.1. *Let $L \subset S$ be a monomial ideal of mixed products. Then*

(a) $\mathrm{sdepth}_S(L) \geq \mathrm{depth}_S(L)$;
(b) $\mathrm{sdepth}_S(S/L) \geq \mathrm{depth}_S(S/L)$.

Proof. Use the notation as above, and set $S_1 = K[x_1,\dots,x_n]$ and $S_2 = K[x_{n+1},\dots,x_N]$. We will prove (a) and (b) according to L's following forms:

(1) I_qS, J_tS;
(2) $(I_qJ_t)S$;
(3) $(I_q + J_t)S$;
(4) $(I_qJ_r + I_s)S$;
(5) $(I_qJ_r + I_sJ_t)S$,

where $1 \leq q < s \leq n$, $1 \leq t < r \leq N-n$.

(1) When $L = I_qS$ or J_tS, it is clear from Lemma 2.4 and Lemma 2.1 (1).

(2) By Proposition 3.1 and Lemma 2.1 (2), we have

$$\begin{aligned}
&\operatorname{sdepth}_S((I_qJ_t)S) \geq \operatorname{sdepth}_{S_1}(I_q) + \operatorname{sdepth}_{S_2}(J_t)\\
&\geq q + t = \operatorname{depth}_S((I_qJ_t)S); \operatorname{sdepth}_S(S/(I_qJ_t)S)\\
&\geq \operatorname{sdepth}_{S_2}(S_2/J_t) + \min\{\operatorname{sdepth}_{S_1}(I_q), \operatorname{sdepth}_{S_1}(S_1/I_q) + 1\}\\
&\geq t - 1 + \min\{q, q - 1 + 1\}\\
&= q + t - 1 = \operatorname{depth}_S(S/(I_qJ_t)S).
\end{aligned}$$

(3) By Proposition 3.2, Lemma 2.3 (3) and Lemma 2.1 (3), we have

$$\begin{aligned}
&\operatorname{sdepth}_S((I_q + J_t)S)\\
&\geq \operatorname{sdepth}_{S_2}(J_t) + \min\{\operatorname{sdepth}_{S_1}(I_q), \operatorname{sdepth}_{S_1}(S_1/I_q)\}\\
&\geq t + \{q, q - 1\} = q + t - 1\\
&= \operatorname{depth}_S((I_q + J_t)S); \operatorname{sdepth}_S(S/(I_q + J_t)S)\\
&\geq \operatorname{sdepth}_{S_1}(S_1/I_q) + \operatorname{sdepth}_{S_2}(S_2/J_t)\\
&= q - 1 + t - 1 = q + t - 2\\
&= \operatorname{depth}_S(S/(I_q + J_t)S).
\end{aligned}$$

(4) Since $\operatorname{sdepth}_{S_2}(J_r) \geq r > r - 1 = \operatorname{sdepth}_{S_2}(S_2/J_r)$, it follows from Proposition 3.3 (1) and Lemma 2.1 (4) that

$$\begin{aligned}
&\operatorname{sdepth}_S((I_qJ_r + I_s)S)\\
&\geq \operatorname{sdepth}_{S_2}(S_2/J_r) + \min\{\operatorname{sdepth}_{S_1}(I_q) + 1, \operatorname{sdepth}_{S_1}(I_s)\}\\
&\geq r - 1 + \min\{q + 1, s\} = q + r\\
&= \operatorname{depth}_S((I_qJ_r + I_s)S).
\end{aligned}$$

Since $\operatorname{sdepth}_{S_1}(S_1/I_s) = s - 1 > q - 1 = \operatorname{sdepth}_{S_1}(S_1/I_q)$, it follows from Proposition 3.3 (2) and Lemma 2.1 (4) that

$$\begin{aligned}
&\operatorname{sdepth}_S(S/(I_qJ_r + I_s)S)\\
&\geq \operatorname{sdepth}_{S_1}(S_1/I_q)\\
&\qquad + \min\{\operatorname{sdepth}_{S_2}(J_r), \operatorname{sdepth}_{S_2}(S_2/J_r) + 1\}\\
&\geq q - 1 + \min\{r, r - 1 + 1\} = q + r - 1\\
&= \operatorname{depth}_S(S/(I_qJ_r + I_s)S).
\end{aligned}$$

(5) Note that $\mathrm{sdepth}_{S_1}(I_q/I_s) \geq q$ by [**4,** Proposition 3.1]. Then, from Proposition 3.4 (1) and Lemma 2.1 (5), we obtain

$$\begin{aligned}
&\mathrm{sdepth}_S((I_qJ_r + I_sJ_t)S) \\
&\geq \min\{\mathrm{sdepth}_{S_1}(I_s) + \mathrm{sdepth}_{S_2}(J_t), \\
&\qquad\qquad \mathrm{sdepth}_{S_1}(I_q/I_s) + \mathrm{sdepth}_{S_2}(J_r)\} \\
&\geq \min\{s+t, q+r\} \\
&= \mathrm{depth}_S((I_qJ_r + I_sJ_t)S).
\end{aligned}$$

Since $\mathrm{sdepth}_{S_1}(S_1/I_s) = s-1 > q-1 = \mathrm{sdepth}_{S_1}(S_1/I_q)$, it follows from Proposition 3.4 (2) and Lemma 2.1 (5) that

$$\begin{aligned}
&\mathrm{sdepth}_S(S/(I_qJ_r + I_sJ_t)S) \\
&\geq \min\{\mathrm{sdepth}_{S_1}(S_1/I_q) \\
&\qquad + \min\{\mathrm{sdepth}_{S_2}(J_r), \mathrm{sdepth}_{S_2}(S_2/J_r)+1\}, \\
&\qquad\qquad \mathrm{sdepth}_{S_1}(I_s) + \mathrm{sdepth}_{S_2}(S_2/J_t)\} \\
&\geq \min\{q-1 + \min\{r, r-1+1\}, s+t-1\} \\
&= \min\{q+r, s+t\} - 1 \\
&= \mathrm{depth}_S(S/(I_qJ_r + I_sJ_t)S).
\end{aligned}$$ $\square$

It is known that the above theorem holds for any monomial ideal L with $N \leq 5$ or if L is an intersection of four monomial prime ideals (cf., [**8, 9, 10, 14**]). The following example demonstrates that the set of ideals of mixed products involves more ideals on which the Stanley conjecture holds.

Example 4.2. Let $S = K[x_1, \ldots, x_6]$ be a polynomial ring over a field K and

$$\begin{aligned}
L = {} & (I_2J_1 + I_1J_2)S = (x_1x_2x_4, x_1x_3x_4, x_2x_3x_4, x_1x_2x_5, x_1x_3x_5, \\
& x_2x_3x_5, x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_1x_4x_5, x_2x_4x_5, x_3x_4x_5, \\
& x_1x_4x_6, x_2x_4x_6, x_3x_4x_6, x_1x_5x_6, x_2x_5x_6, x_3x_5x_6)
\end{aligned}$$

an ideal of mixed products, where I_2 (respectively, I_1) is the square-free monomial ideal of $K[x_1, x_2, x_3]$ generated by all the square-free monomials of degree 2 (respectively, 1) and J_2 (respectively, J_1) is the square-free monomial ideal of $K[x_4, x_5, x_6]$ generated by all the square-free monomials of degree 2 (respectively, 1). Note that

$$
\begin{aligned}
L \quad = \quad & (x_1, x_2, x_3) \cap (x_4, x_5, x_6) \cap (x_1, x_3, x_5, x_6) \cap (x_2, x_3, x_5, x_6) \\
& \cap (x_1, x_3, x_4, x_5) \cap (x_2, x_3, x_4, x_5) \cap (x_1, x_2, x_4, x_5) \\
& \cap (x_1, x_2, x_4, x_6) \cap (x_1, x_2, x_5, x_6) \\
& \cap (x_1, x_3, x_4, x_6) \cap (x_2, x_3, x_4, x_6)
\end{aligned}
$$

is an intersection of prime ideals of S such that each prime ideal is contained in the sum of the remaining prime ideals, not satisfying [**10,** Theorem 2.4].

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