

KOSZUL PROPERTY OF DIAGONAL SUBALGEBRAS

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ABSTRACT. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and I a homogeneous ideal in S generated by a regular sequence f_1, f_2, \dots, f_k of homogeneous forms of degree d . We study a generalization of a result of Conca et al. [9] concerning Koszul property of the diagonal subalgebras associated to I . Each such subalgebra has the form $K[(I^e)_{ed+c}]$, where $c, e \in \mathbf{N}$. For $k = 3$, we extend [9, Corollary 6.10] by proving that $K[(I^e)_{ed+c}]$ is Koszul as soon as $c \geq d/2$ and $e > 0$. We also extend [9, Corollary 6.10] in another direction by replacing the polynomial ring with a Koszul ring.

1. Introduction. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and I a homogeneous ideal in S . For large c , the algebra $K[(I^e)_c]$ is isomorphic to the coordinate ring of some embedding of the blow up of \mathbf{P}_K^{n-1} along the ideal sheaf \tilde{I} in a projective space [9].

Let $\text{Rees}(I) = \bigoplus_{j \geq 0} I^j t^j$ be the *Rees algebra* of I . Since the polynomial ring $S[t]$ is a bigraded algebra with $S[t]_{(i,j)} = S_i t^j$, we may consider $\text{Rees}(I)$ as a bigraded subalgebra of $S[t]$ with $\text{Rees}(I)_{(i,j)} = (I^j)_i t^j$. Let c and e be positive integers. Let $\Delta = \{(cs, es) : s \in \mathbf{Z}\}$. We call Δ the (c, e) -*diagonal* of \mathbf{Z}^2 [9]. Let $R = \bigoplus_{(i,j) \in \mathbf{Z}^2} R_{(i,j)}$ be a bigraded algebra, where $R_{(i,j)}$ denotes the (i, j) -th bigraded component of R . The (c, e) -*diagonal subalgebra* of R_δ is defined as the \mathbf{Z} -graded algebra $R_\Delta = \bigoplus_{s \in \mathbf{Z}} R_{(cs, es)}$. Similarly, for every bigraded R -module M , one defines the (c, e) -*diagonal submodule* of M as $M_\Delta = \bigoplus_{s \in \mathbf{Z}} M_{(cs, es)}$. Notice that M_Δ is a module over R_Δ .

When I is a homogeneous ideal in S generated by f_1, f_2, \dots, f_k , of homogeneous forms of degree d , then $\text{Rees}(I)$ is a standard bigraded

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algebra by setting $\deg x_i = (1, 0)$ and $\deg f_j t = (0, 1)$. We observe that $K[(I^e)_{ed+c}]$ is the (c, e) -diagonal subalgebra of Rees (I) . We may also view $K[(I^e)_{ed+c}]$ as a K -subalgebra of S generated by the forms of degree $ed + c$ in the ideal I^e .

Given a field K , a positively graded K -algebra $A = \bigoplus_{i \in \mathbf{N}} A_i$ with $A_0 = K$ is Koszul if the field K , viewed as an A -module via the identification $K = A/A_+$, has a linear free resolution. Koszul algebras were introduced by Priddy [15] in 1970. During the last four decades Koszul algebra has been studied in various contexts. A good survey on Koszul algebra was given by Fröberg in [12] during the 90's and recently by Conca, De Negri and Rossi in [10].

Diagonal subalgebras have been intensely studied by several authors (e.g., see [1], [9], [16]) because they naturally appear in Rees algebras and symmetric algebras. In [9], Conca, et al. discuss some algebraic properties of diagonal subalgebras, such as Cohen-Macaulayness and Koszulness. In [14], Kurano et al. showed that Cohen-Macaulayness property holds in [9] even if the polynomial ring is replaced by a Cohen-Macaulay ring of dimension $d \geq 2$. In this article, we study the Koszul property of certain diagonal subalgebras of bigraded algebras, with focus and applications to diagonals of Rees algebras. We generalize some of the important results of [9] regarding the Koszulness of certain diagonal subalgebras of bigraded algebras.

For any homogeneous ideal I , there exist integers c_0, e_0 such that the K -algebra $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq c_0$ and $e \geq e_0$, see [9, Corollary 6.9]. If I is a complete intersection ideal generated by f_1, f_2, \dots, f_k , of homogeneous forms of degree d , then the K -algebra $K[(I^e)_{ed+c}]$ is quadratic if $c \geq d/2$ and $e > 0$; furthermore $K[(I^e)_{ed+c}]$ is Koszul if $c \geq d(k-1)/k$ and $e > 0$, see [9, Corollary 6.10].

The main results of this paper are the following:

- (i) Let I be an ideal of the polynomial ring $K[x_1, \dots, x_n]$ generated by a regular sequence f_1, f_2, f_3 , of homogeneous forms of degree d . Then $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq d/2$, and $e > 0$.
- (ii) Let A be a standard graded Koszul ring. Let I be an ideal of A generated by a regular sequence f_1, f_2, \dots, f_k , of homogeneous forms of degree d . Then $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq d(k-1)/k$ and $e > 0$.

Let R be a standard bigraded K -algebra. In Section 2, we study homological properties of shifted modules $R(-a, -b)_\Delta$, which play an important role in the transfer of homological information from R to R_Δ . It is important to bound the homological invariants of the shifted diagonal module $R(-a, -b)_\Delta$ as an R_Δ -module. For a bigraded polynomial ring R , it is proved in [9] that $R(-a, -b)_\Delta$ has a linear R_Δ resolution. Proposition 2.10 is an extension of [9, Theorem 6.2] for certain bigraded complete intersection ideals and crucial in proving Theorem 3.1.

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring. In [9, page 900], the authors mentioned that, for a complete intersection ideal I in S generated by f_1, f_2, \dots, f_k , of homogeneous forms of degree d , the algebra $K[(I^e)_{ed+c}]$ is expected to be Koszul as soon as $c \geq d/2$. For $k = 1, 2$, it is obvious. The first nontrivial case is $k = 3$. In Section 3, we answer their expectation affirmatively for $k = 3$, see Theorem 3.1. The motivation for such generalization came from the work of Caviglia [7] and Caviglia and Conca [8]. Note that, for $k = 3$, the result of [8] is just the case: $d = 2$ and $c = 1$; furthermore, the main result of [7] correspond to the case: $d = 2, c = 1, f_1 = x_1^2, f_2 = x_2^2, f_3 = x_3^2$ and $n = 3$.

In Section 4, we generalize [9, Theorem 6.2] and some of its relevant corollaries. The main result of this section is Theorem 4.5, which is a generalization of [9, Corollary 6.10] regarding the Koszulness of certain diagonals of the Rees algebra of an ideal in the polynomial ring. We show that the Koszulness property holds even if the polynomial ring is replaced by a Koszul ring. The reason for such a generalization comes from the fact that, from a certain point of view, Koszul algebras behave homologically as polynomial rings.

2. Generalities and preliminary results. Let A be a standard graded K -algebra, i.e., $A = \bigoplus A_i = S/I$, where S is a polynomial ring and I a homogeneous ideal of S . For a finitely generated graded A -module $M = \bigoplus M_i$, set

$$t_i^A(M) = \sup\{j : \text{Tor}_i^A(M, K)_j \neq 0\},$$

with $t_i^A(M) = -\infty$ if $\text{Tor}_i^A(M, K) = 0$.

Definition 2.1. The *Castelnuovo-Mumford regularity* $\text{reg}_A(M)$ of an A -module M is defined to be

$$\text{reg}_A M = \sup\{t_i^A(M) - i : i \geq 0\}.$$

When $A = S$ is a polynomial ring, one has

$$\text{reg}_S M = \max\{t_i^S(M) - i : i \geq 0\}.$$

For a polynomial ring $S = K[x_1, \dots, x_n]$, one can also compute $\text{reg}_S M$ via the local cohomology modules $H_m^i(M)$ for $i = 0, 1, \dots, n$. One has

$$\text{reg}_S M = \max\{j + i : H_m^i(M)_j \neq 0\}.$$

Definition 2.2 (Koszul algebra). A standard graded K -algebra A is said to be a *Koszul algebra* if the residue field K has a linear A -resolution. Equivalently, A is Koszul when $\text{reg}_A(K) = 0$.

Example 2.3. Let $A = K[x]/(x^2)$. Then K has a linear A -resolution

$$\dots \longrightarrow A(-2) \xrightarrow{\bar{x}} A(-1) \xrightarrow{\bar{x}} A \longrightarrow K \longrightarrow 0.$$

The property of being a Koszul algebra is preserved under various constructions, in particular under taking tensor products, Segre products and Veronese subrings, see Backelin and Fröberg [4].

Let A be a Koszul algebra, and let S be the polynomial ring mapping onto A . Then the regularity of any finitely generated graded module M over A is always finite; in fact, $\text{reg}_A M \leq \text{reg}_S M$, see Avramov and Eisenbud [3, Theorem 1]. If $M = \bigoplus_{i=a}^b M_i$ with $M_b \neq 0$, then

$$(1) \quad \text{reg}_A M \leq \text{reg}_S M = b.$$

If $A = \bigoplus_{i \geq 0} A_i$ is a graded algebra, then the c -th *Veronese subalgebra* is $A^{(c)} = \bigoplus_{i \geq 0} A_{ic}$. An element in A_{ic} is considered to have degree i .

Definition 2.4. Consider a standard graded K -algebra A . Given $k, m \in \mathbf{N}$ and $0 \leq k < m$, we set

$$V_A(m, k) = \bigoplus_{i \in \mathbf{N}} A_{im+k}.$$

We observe that $A^{(m)} = V_A(m, 0)$ is the usual m -th Veronese subring of A , and that the $V_A(m, k)$ are $A^{(m)}$ -modules known as the *Veronese modules* of A . For a finitely generated graded A -module M , similarly we define

$$M^{(m)} = \bigoplus_{i \in \mathbf{Z}} M_{im}.$$

We consider $A^{(m)}$ as a standard graded K -algebra with homogeneous component of degree i equal to A_{im} and $M^{(m)}$ as a graded $A^{(m)}$ -module with homogeneous components M_{im} of degree i .

Definition 2.5. Let A and B be a positively graded K -algebra. Denote by $A \underline{\otimes} B$ the *Segre product*

$$A \underline{\otimes} B = \bigoplus_{i \in \mathbf{N}} A_i \otimes_K B_i,$$

of A and B . Given graded modules M and N over A and B , one may form the Segre product

$$M \underline{\otimes} N = \bigoplus_{i \in \mathbf{Z}} M_i \otimes_K N_i,$$

of M and N . Clearly, $M \underline{\otimes} N$ is a graded $A \underline{\otimes} B$ -module.

A beautiful introduction to construction of multigraded objects including Segre products is given by Goto and Watanabe in [13, Chapter 4]. Segre products have been studied in the sense of Koszulness by several authors, e.g., Backelin et al. [4], Eisenbud et al. [11], Conca et al. [9], Fröberg [12] and Blum [5]. We will use their results at several occasions in this paper.

Let A and B be Koszul K -algebras. Let M be a finitely generated graded A -module and N a finitely generated graded B -module. Assume M and N have linear resolutions over A and B , respectively. Also assume $M \underline{\otimes} N \neq 0$. Then, by [9, Lemma 6.5], $M \underline{\otimes} N$ has a linear $A \underline{\otimes} B$ -resolution and

$$(2) \quad \text{reg}_{A \underline{\otimes} B} M \underline{\otimes} N = \max\{\text{reg}_A M, \text{reg}_B N\}.$$

We will use this relation on regularity for Segre products in the proof of Theorems 4.1 and 4.5.

Remark 2.6. It is to be noticed that [9, Lemma 6.5] also needs the hypothesis $M \otimes N \neq 0$. For instance, Assume $M = K$ and $N = K(-1)$. Then $M \otimes N = 0$, and this leads to inconsistency in (2).

The following lemma is very useful. We will use this lemma in the proof of Proposition 2.10, Theorem 3.1 and Theorem 4.5.

Lemma 2.7 (Technical lemma). *Let*

$$\mathbf{M} : \cdots \longrightarrow M_{k+1} \longrightarrow M_k \longrightarrow M_{k-1} \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow 0,$$

be a complex of graded A -modules with maps of degree 0. Set $H_i = H_i(\mathbf{M})$. Then, for every $i \geq 0$, one has

$$(3) \quad t_i^A(H_0) \leq \max\{\alpha, \beta\},$$

where $\alpha = \sup\{t_{i-j}^A(M_j) : j = 0, \dots, i\}$ and $\beta = \sup\{t_{i-j-1}^A(H_j) : j = 1, \dots, i-1\}$. Moreover, one has

$$(4) \quad \text{reg}_A(H_0) \leq \max\{\alpha', \beta'\},$$

where $\alpha' = \sup\{\text{reg}_A(M_j) - j : j \geq 0\}$ and $\beta' = \sup\{\text{reg}_A(H_j) - (j+1) : j \geq 1\}$.

Proof. Let Z_i , B_i and H_i denote the i th cycles, boundaries and homology modules respectively. We have short exact sequences

$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0,$$

and

$$0 \longrightarrow Z_{i+1} \longrightarrow M_{i+1} \longrightarrow B_i \longrightarrow 0,$$

with $Z_0 = M_0$. Therefore, by [6, Lemma 2.2(a)], one has

$$\begin{aligned} t_i^A(H_0) &\leq \max\{t_i^A(M_0), t_{i-1}^A(B_0)\}, \\ t_{i-1}^A(B_0) &\leq \max\{t_{i-1}^A(M_1), t_{i-2}^A(Z_1)\}, \\ t_{i-2}^A(Z_1) &\leq \max\{t_{i-2}^A(B_1), t_{i-2}^A(H_1)\}, \\ t_{i-2}^A(B_1) &\leq \max\{t_{i-2}^A(M_2), t_{i-3}^A(Z_2)\}, \end{aligned}$$

and so on. Summarizing the details, one has:

$$t_i^A(H_0) \leq \max\{t_i^A(M_0), t_{i-1}^A(M_1), \dots, t_{i-j}^A(M_j), t_{i-2}^A(H_1), \dots, t_{i-j-1}^A(H_j)\}.$$

Take $\alpha = \sup\{t_{i-j}^A(M_j) : j = 0, \dots, i\}$ and $\beta = \sup\{t_{i-j-1}^A(H_j) : j = 1, \dots, i - 1\}$. Then we obtain the desired result (3) for $t_i^A(H_0)$. The second inequality (4) follows from (3). \square

Let $R = \bigoplus_{(i,j) \in \mathbf{Z}^2} R_{(i,j)}$ be a bigraded standard K -algebra. Here standard means that $R_{(0,0)} = K$ and R is generated as a K -algebra by the K -vector spaces $R_{(1,0)}$ and $R_{(0,1)}$ of finite dimension.

Definition 2.8 (Diagonal subalgebra). Let R be a bigraded standard K -algebra. Let c and e be positive integers. Let Δ be the (c, e) -diagonal of \mathbf{Z}^2 . The (c, e) -diagonal subalgebra R_Δ of R is defined as

$$R_\Delta = \bigoplus_{s \in \mathbf{Z}} R_{(cs, es)}.$$

We observe that R_Δ is the K -subalgebra of R generated by $R_{(c,e)}$, and hence it is a standard graded K -algebra. Similarly, one defines the (c, e) -diagonal submodule of any bigraded R -module $M = \bigoplus_{(i,j) \in \mathbf{Z}^2} M_{(i,j)}$ as $M_\Delta = \bigoplus_{s \in \mathbf{Z}} M_{(cs, es)}$. Notice that M_Δ is a module over R_Δ . The map $M \mapsto M_\Delta$, being a selection of homogeneous components, defines an exact functor from the category of bigraded R -modules and maps of degree 0 to the category of graded R_Δ -modules with maps of degree 0.

Notation 2.9. We have $\Delta = \{(cs, es) : s \in \mathbf{Z}\}$, but the bounds on c and e change from time to time. Note that c, s, Δ will always be used in this way, with c and s changing as described. For a real number α , we use $\lceil \alpha \rceil$ for the smallest integer m such that $m \geq \alpha$.

For $(a, b) \in \mathbf{Z}^2$, let $R(-a, -b)$ be a shifted copy of R . By definition,

$$R(-a, -b)_\Delta = \bigoplus_{s \in \mathbf{Z}} R_{(-a+cs, -b+es)}.$$

Since R is positively graded, we may consider only those $s \in \mathbf{Z}$ for which $-a+cs \geq 0$ and $-b+es \geq 0$. Assume $\max\{\lceil a/c \rceil, \lceil b/e \rceil\} = \lceil a/c \rceil$. Then

$$R(-a, -b)_\Delta = \bigoplus_{s \geq \lceil a/c \rceil} R_{(-a+cs, -b+es)}.$$

Therefore, $R(-a, -b)_\Delta$ is an R_Δ -submodule of R generated by

$$R_{(-a+c\lceil a/c \rceil, -b+e\lceil a/c \rceil)}.$$

The other case is similar; summarizing the details, one has:

$$R(-a, -b)_\Delta = \begin{cases} R(-a + c\lceil \frac{a}{c} \rceil, -b + e\lceil \frac{a}{c} \rceil)_\Delta(-\lceil \frac{a}{c} \rceil) & \text{if } \lceil \frac{a}{c} \rceil \geq \lceil \frac{b}{e} \rceil; \\ R(-a + c\lceil \frac{b}{e} \rceil, -b + e\lceil \frac{b}{e} \rceil)_\Delta(-\lceil \frac{b}{e} \rceil) & \text{if } \lceil \frac{a}{c} \rceil \leq \lceil \frac{b}{e} \rceil. \end{cases}$$

The homological properties of the shifted diagonal module $R(-a, -b)_\Delta$ play an important role in the transfer of homological information from R to R_Δ . In the following proposition, we try to bound the homological invariant (regularity) of the shifted diagonal module $R(-a, -b)_\Delta$ as an R_Δ -module. The following proposition is crucial in proving Theorem 3.1.

Proposition 2.10. *Let $S = K[x_1, \dots, x_m, t_1, \dots, t_n]$ be a polynomial ring bigraded by $\deg x_i = (1, 0)$ for $i = 1, \dots, m$ and $\deg t_i = (0, 1)$ for $i = 1, \dots, n$. Let I be an ideal of S generated by a regular sequence with elements all of bidegree $(d, 1)$ and $R = S/I$. Let $d/2 \leq c < 2d/3$ and $e > 0$. Then:*

- (a) R_Δ is Koszul.
- (b) $\text{reg}_{R_\Delta} R(-a, -b)_\Delta \leq \max\{\lceil a/c \rceil, \lceil b/e \rceil\}$.

Proof. Let h be the codimension of I . The proof is by induction on h . If $h = 0$, then R_Δ is the Segre product of $K[x_1, \dots, x_m]^{(c)}$ and $K[t_1, \dots, t_n]^{(e)}$. Thus, R_Δ is Koszul by [4]. For (b), see [9, proof of Theorem 6.2].

Assume $h > 0$. We may write $R = T/(f)$ where f is a T -regular element of bidegree $(d, 1)$ and where T is defined as the quotient of S by an S -regular sequence of length $h - 1$ of elements of bidegree $(d, 1)$. We have a short exact sequence of T -modules:

$$(5) \quad 0 \longrightarrow T(-d, -1) \longrightarrow T \longrightarrow R \longrightarrow 0$$

and, applying $-_{\Delta}$, we have an exact sequence of T_{Δ} -modules:

$$0 \longrightarrow T(-d, -1)_{\Delta} \longrightarrow T_{\Delta} \longrightarrow R_{\Delta} \longrightarrow 0.$$

By induction, we know T_{Δ} is Koszul and that $\text{reg}_{T_{\Delta}} T(-d, -1)_{\Delta} \leq \lceil d/c \rceil$. As $d/c \leq 2$, one has

$$\text{reg}_{T_{\Delta}} R_{\Delta} \leq 1.$$

By [8, Lemma 2.1 (3)], we may conclude that R_{Δ} is Koszul, as T_{Δ} is Koszul by induction. Now, to prove (b), we consider the following two cases.

Case 1. Assume $\lceil a/c \rceil < \lceil b/e \rceil$. Shifting (5) by $(-a, -b)$ and then applying $-_{\Delta}$, we get a short exact sequence of T_{Δ} -modules:

$$0 \longrightarrow T(-a - d, -b - 1)_{\Delta} \longrightarrow T(-a, -b)_{\Delta} \longrightarrow R(-a, -b)_{\Delta} \longrightarrow 0.$$

So we have:

$$\begin{aligned} \text{reg}_{T_{\Delta}} R(-a, -b)_{\Delta} &\leq \max\{\text{reg}_{T_{\Delta}} T(-a, -b)_{\Delta}, \text{reg}_{T_{\Delta}} T(-a - d, -b - 1)_{\Delta} - 1\}. \end{aligned}$$

By induction, one has $\text{reg}_{T_{\Delta}} T(-a, -b)_{\Delta} \leq \lceil b/e \rceil$, and

$$\text{reg}_{T_{\Delta}} T(-a - d, -b - 1)_{\Delta} \leq \max\left\{\left\lceil \frac{a+d}{c} \right\rceil, \left\lceil \frac{b+1}{e} \right\rceil\right\}.$$

Since

$$\left\lceil \frac{a+d}{c} \right\rceil \leq \left\lceil \frac{a}{c} \right\rceil + \left\lceil \frac{d}{c} \right\rceil \leq \left(\left\lceil \frac{b}{e} \right\rceil - 1\right) + 2 = \left\lceil \frac{b}{e} \right\rceil + 1$$

and

$$\left\lceil \frac{b+1}{e} \right\rceil \leq \left\lceil \frac{b}{e} \right\rceil + 1,$$

we conclude that $\text{reg}_{T_{\Delta}} T(-a - d, -b - 1)_{\Delta} \leq \lceil b/e \rceil + 1$. Thus, we have

$$\text{reg}_{T_{\Delta}} R(-a, -b)_{\Delta} \leq \left\lceil \frac{b}{e} \right\rceil.$$

Since we have already shown that $\text{reg}_{T_{\Delta}} R_{\Delta} \leq 1$, we may conclude by [8, Lemma 2.1 (1)] that

$$\text{reg}_{R_{\Delta}} R(-a, -b)_{\Delta} \leq \left\lceil \frac{b}{e} \right\rceil.$$

Case 2. Assume $\lceil a/c \rceil \geq \lceil b/e \rceil$. Setting $P = (t_1, \dots, t_n) \subset S$, we have

$$R(-a, -b)_\Delta = R\left(-a + c\left\lceil \frac{a}{c} \right\rceil, -b + e\left\lceil \frac{a}{c} \right\rceil\right)_\Delta \left(-\left\lceil \frac{a}{c} \right\rceil\right).$$

So we have to prove that $\text{reg}_{R_\Delta} R(\alpha, \beta)_\Delta \leq 0$ where $\alpha = -a + c\lceil a/c \rceil$ and $\beta = -b + e\lceil a/c \rceil$. Consider the minimal free (bigraded) resolution of S/P^β as an S -module:

$$\mathbf{F} : 0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0,$$

with $F_0 = S$ and $F_i = S^\sharp(0, -\beta - i + 1)$ for $i > 0$ where \sharp denotes some integer depending on n, β and i that is irrelevant in our discussion. The homology of $\mathbf{F} \otimes R$ is $\text{Tor}_\bullet^S(S/P^\beta, R)$. We may as well compute $\text{Tor}_\bullet^S(S/P^\beta, R)$ as the homology of $S/P^\beta \otimes \mathbf{G}$ where \mathbf{G} is a free resolution of R as an S -module. By assumption, we may take \mathbf{G} to be a Koszul complex on a sequence of elements of bidegree $(d, 1)$. It follows that:

$$H_i(\mathbf{F} \otimes R) = \begin{cases} \text{a subquotient of } (S/P^\beta)^\sharp(-di, -i) & \text{if } 0 \leq i \leq h; \\ 0 & \text{if } i > h. \end{cases}$$

Shifting with (α, β) and applying $-\Delta$, we have a complex $(\mathbf{F} \otimes R(\alpha, \beta))_\Delta$. We claim this complex has no homology. Shifting and applying $-\Delta$ are compatible operations with taking homology. Therefore, to prove $(\mathbf{F} \otimes R(\alpha, \beta))_\Delta$ has no homology at all, we only need to check that

$$(6) \quad [(S/P^\beta)(-di + \alpha, -i + \beta)]_\Delta = 0 \quad \text{for all } i.$$

To prove (6), take the j th degree component

$$[[(S/P^\beta)(-di + \alpha, -i + \beta)]_\Delta]_j = (S/P^\beta)_{(cj - di + \alpha, ej - i + \beta)}.$$

We will show that

$$(7) \quad (S/P^\beta)_{(cj - di + \alpha, ej - i + \beta)} = 0.$$

The case $i = 0$ will be dealt with separately. Assume $i > 0$. Clearly (7) holds if $ej - i + \beta \geq \beta$, that is, if $ej \geq i$. To complete the argument for (7), it is enough to show that $cj - di + \alpha < 0$ for $ej < i$. Let $a = qc + r$; $0 \leq r < c$. Then

$$\left\lceil \frac{a}{c} \right\rceil = \begin{cases} q + 1 & \text{if } r \neq 0; \\ q & \text{if } r = 0. \end{cases}$$

Therefore,

$$(8) \quad cj - di + \alpha = \begin{cases} cj - di + c - r & \text{if } r \neq 0; \\ cj - di & \text{if } r = 0. \end{cases}$$

Assume $r = 0$ and $ej < i$. It is easy to see that

$$cj - di < i\left(\frac{c}{e} - d\right) < i\left(\frac{2d}{3e} - d\right) < 0.$$

Assume $r \neq 0$ and $ej < i$. By assumption, we have $j \leq i/e - 1/e$ and $cj - di + c - r < c(j + 1) - id$. One has

$$(9) \quad c(j + 1) - di \leq c\left(\frac{i}{e} - \frac{1}{e} + 1\right) - di = i\left(\frac{c}{e} - d\right) - \frac{c}{e} + c.$$

As $d/2 \leq c < 2d/3$ and $e > 0$, we may write

$$(10) \quad \frac{c}{e} - d < \frac{2d}{3e} - d = \frac{d(2 - 3e)}{3e} \quad \text{and} \quad -\frac{c}{e} \leq -\frac{d}{2e}.$$

Thus, by (9) and (10), we have

$$\begin{aligned} c(j + 1) - di &< i\left(\frac{d(2 - 3e)}{3e}\right) - \frac{d}{2e} + \frac{2d}{3} \\ &= \frac{d}{6e}[i(4 - 6e) + (4e - 3)]. \end{aligned}$$

It is easy to see that $i(4 - 6e) + (4e - 3) < 0$ for all $i > 0$. Assume $i = 0$. Denote by \mathbf{C} the complex $(\mathbf{F} \otimes R(\alpha, \beta))_\Delta$. If $H_0(\mathbf{C}) \neq 0$, then either $ej + \beta < \beta$ or $cj + \alpha \geq 0$ in (7). Thus, $H_0(\mathbf{C}) \neq 0$ if $-\alpha/c \leq j < 0$, which is impossible since j has to be an integer and, by previous discussion in (8), one has $-1 < -\alpha/c \leq 0$. Since $H_i(\mathbf{C}) = 0$ for all $i \geq 0$, we have the following exact complex \mathbf{C} :

$$\begin{aligned} 0 \longrightarrow R(\alpha, -i + 1)_\Delta \longrightarrow \cdots \longrightarrow R(\alpha, -1)_\Delta \\ \longrightarrow R(\alpha, 0)_\Delta \longrightarrow R(\alpha, \beta)_\Delta \longrightarrow 0. \end{aligned}$$

From the exact complex \mathbf{C} , we build another complex:

$$\mathbf{T} : 0 \longrightarrow R(\alpha, -i + 1)_\Delta \longrightarrow \cdots \longrightarrow R(\alpha, -1)_\Delta \longrightarrow R(\alpha, 0)_\Delta \longrightarrow 0.$$

Then the homology of the new complex \mathbf{T} is given by

$$H_i(\mathbf{T}) = \begin{cases} R(\alpha, \beta)_\Delta & \text{if } i = 0; \\ 0 & \text{if } i > 0. \end{cases}$$

Thus, applying Lemma 2.7 to the complex \mathbf{T} , one has

$$\text{reg}_{R_\Delta} R(\alpha, \beta)_\Delta \leq \max\{\text{reg}_{R_\Delta} R(\alpha, -i)_\Delta - i : i \geq 0\}.$$

Note that, by Case 1, we have $\text{reg}_{R_\Delta} R(\alpha, -i)_\Delta \leq \lceil i/e \rceil$, since $[-\alpha/c] \leq \lceil i/e \rceil$. Thus, we conclude that $\text{reg}_{R_\Delta} R(\alpha, \beta)_\Delta \leq 0$. Hence, Claim (b) follows. □

Remark 2.11. Note that Proposition 2.10 is an extension of [9, Theorem 6.2] for certain bigraded complete intersection ideal. Note also that the statement of Proposition 2.10 is similar to (and more general than) [8, Proposition 2.2].

3. Improvement of bounds. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring. Let I be a homogeneous ideal in S generated by a regular sequence f_1, f_2, \dots, f_k , of homogeneous forms of degree d . Let c and e be positive integers. Consider the K -subalgebra of S generated by the homogeneous forms of degree $ed + c$ in the ideal I^e , that is, $K[(I^e)_{ed+c}]$. We have seen that $K[(I^e)_{ed+c}]$ is the (c, e) -diagonal subalgebra of Rees (I) .

By [9, Corollary 6.10], $K[(I^e)_{ed+c}]$ is shown to be quadratic if $c \geq d/2$ and Koszul if $c \geq d(k-1)/k$. In [9, page 900], the authors mentioned that they expect $K[(I^e)_{ed+c}]$ to be Koszul also for $d/2 \leq c < d(k-1)/k$. For $k = 1, 2$, it is obvious that $K[(I^e)_{ed+c}]$ is Koszul. The very first nontrivial instance of this problem occurs for $d = 2$ and $k = 3$, in which case the only possible value is $c = 1$. For $c = 1$, the answer is positive and solved by Caviglia and Conca [8].

In this Section, for $k = 3$ and for any d , we prove that $K[(I^e)_{ed+c}]$ is Koszul as soon as $c \geq d/2$ and $e > 0$. The main theorem of this section is as follows.

Theorem 3.1. *Let I be an ideal of the polynomial ring $K[x_1, \dots, x_n]$ generated by a regular sequence f_1, f_2, f_3 , of homogeneous forms of degree d . Then $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq d/2$ and $e > 0$.*

We consider the Rees algebra, $\text{Rees}(I) \subset S[t]$ of I with its standard bigraded structure induced by $\deg(x_i) = (1, 0)$ and $\deg(f_j t) = (0, 1)$. It can be realized as a quotient of the polynomial ring $S' = K[x_1, \dots, x_n, t_1, t_2, t_3]$ bigraded with $\deg(x_i) = (1, 0)$ and $\deg(t_j) = (0, 1)$, by the ideal J generated by the 2-minors of

$$M = \begin{pmatrix} f_1 & f_2 & f_3 \\ t_1 & t_2 & t_3 \end{pmatrix}.$$

Let h_1, h_2 and h_3 be the 2-minors of M with the appropriate sign, say h_i equal to $(-1)^{i+1}$ times the minor of M obtained by deleting the i th column. Hence,

$$J = I_2(M) = (h_1, h_2, h_3).$$

The sign convention is chosen so that the rows of the matrix M are syzygies of h_1, h_2 and h_3 . We will use the following lemma to prove Theorem 3.1.

Lemma 3.2. [8, Lemma 3.1] (Technical lemma).

- (i) h_1, h_2 form a regular S' -sequence.
- (ii) $(h_1, h_2) : h_3 = (f_3, t_3)$.
- (iii) $(h_1, h_2) : t_3 = J$.
- (iv) $(t_3, h_1, h_2) : f_3 = (t_1, t_2, t_3)$.

Remark 3.3. Note that Lemma 3.2 is proved for $d = 2$ in [8, Lemma 3.1]. We observe that the proof of [8, Lemma 3.1] is independent of the degrees of polynomial. Hence, Lemma 3.2 also holds for all d .

We are now ready for the proof of Theorem 3.1:

Proof. Recall that $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq 2d/3$ and $e > 0$ [9, Corollary 6.10]. We will show that $K[(I^e)_{ed+c}]$ is Koszul also for $d/2 \leq c < 2d/3$ and $e > 0$.

We set $B = S'/(h_1, h_2)$. By Lemma 3.2 (i), we may apply to B the result of Proposition 2.10. Hence, B_Δ is Koszul. One also has $B/h_3 B = \text{Rees}(I)$. It is enough to show for the (c, e) -diagonal subalgebra of $\text{Rees}(I)$, one has

$$\text{reg}_{B_\Delta}(\text{Rees}(I)_\Delta) \leq 1.$$

Since $h_3t_3 = 0$ in B , we have a complex

$$(11) \quad \mathbf{F} : \dots \xrightarrow{t_3} B(-2d, -3) \xrightarrow{h_3} B(-d, -2) \xrightarrow{t_3} B(-d, -1) \xrightarrow{h_3} B \longrightarrow 0,$$

where $F_0 = B$, $F_{2i} = B(-id, -2i)$ and $F_{2i+1} = B(-(i + 1)d, -2i - 1)$. The homology of \mathbf{F} can be described by using Lemma 3.2

$$H_k(\mathbf{F}) = \begin{cases} \text{Rees}(I) & \text{if } k = 0, \\ 0 & \text{if } k = 2i \text{ and } i > 0, \\ [S'/(t_1, t_2, t_3)]_{(-(i+1)d-d, -2i-1)} & \text{if } k = 2i+1 \text{ and } i \geq 0. \end{cases}$$

The assertion for $k = 0$ holds by construction. For k even and positive, it holds because of Lemma 3.2. For k odd and positive by Lemma 3.2 (ii), we have

$$H_{2i+1}(\mathbf{F}) = \frac{(t_3, f_3)}{(t_3, h_1, h_2)}_{(-(i + 1)d, -2i - 1)}.$$

Hence, $H_{2i+1}(\mathbf{F})$ is cyclic generated by the residue class of $f_3 \bmod (t_3, h_1, h_2)$ that has degree $(-i + 1)d - d, -2i - 1)$. Using Lemma 3.2 (iv) and keeping track of the degree, we get the desired result. Applying the $-_\Delta$ functor to (11), we obtain a complex \mathbf{F}_Δ :

$$(12) \quad \dots \longrightarrow B(-2d, -3)_\Delta \longrightarrow B(-d, -2)_\Delta \longrightarrow B(-d, -1)_\Delta \longrightarrow B_\Delta \longrightarrow 0,$$

where

$$(F_k)_\Delta = \begin{cases} B_\Delta & \text{if } k = 0, \\ B(-id, -2i)_\Delta & \text{if } k = 2i, \\ B(-(i + 1)d, -2i - 1)_\Delta & \text{if } k = 2i + 1. \end{cases}$$

Note that $H_{2i}(\mathbf{F}_\Delta) = 0$ and $H_0(\mathbf{F}_\Delta) = \text{Rees}(I)_\Delta$. We observe that $H_{2i+1}(\mathbf{F}_\Delta)$ is not necessarily zero for all $e \geq 1$. Assume $e \geq 2$. Then we claim that $H_{2i+1}(\mathbf{F}_\Delta) = 0$. Take the j -th degree component

$$(H_{2i+1}(\mathbf{F}_\Delta))_j = [S'/(t_1, t_2, t_3)]_{(-(i+1)d-d+jc, -2i-1+je)}.$$

We will show that

$$(13) \quad [S'/(t_1, t_2, t_3)]_{(-(i+1)d-d+jc, -2i-1+je)} = 0.$$

Clearly, (13) holds if $-2i - 1 + ej \geq 1$, that is, if $ej \geq 2(i + 1)$. So, it is enough to show that $-(i + 1)d - d + jc < 0$ for $ej < 2(i + 1)$, that is, $j < (i + 2)d/c$ for $j < 2(i + 1)/e$. This is an easy consequence of the following inequalities:

$$\frac{2(i + 1)}{e} < \frac{3(i + 2)}{2} < \frac{(i + 2)d}{c}.$$

Assume $e = 1$. We know that $H_{2i+1}(\mathbf{F}_\Delta) = H_{2i+1}(\mathbf{F}_\Delta)$. Take the j -th degree component of $H_{2i+1}(\mathbf{F}_\Delta)$. Then $(H_{2i+1}(\mathbf{F}_\Delta))_j = 0$ if $-(2i + 1) + j \geq 1$, that is, if $j \geq 2i + 2$. So, the largest degree of a non zero component of $H_{2i+1}(\mathbf{F}_\Delta)$ is at most $2i + 1$. Therefore, by (1), one has $\text{reg}_{B_\Delta} H_k(\mathbf{F}_\Delta) \leq \text{reg}_{S'} H_k(\mathbf{F}_\Delta) \leq k$ for all $k \geq 1$. Applying Lemma 2.7 to (12), we obtain

$$(14) \quad \text{reg}_{B_\Delta}(\text{Rees}(I)_\Delta) \leq \sup\{\alpha', \beta'\},$$

where

$$\alpha' = \sup\{\text{reg}_{B_\Delta}(F_k)_\Delta - k : k \geq 0\}$$

and

$$\beta' = \sup\{\text{reg}_{B_\Delta} H_k(\mathbf{F}_\Delta) - (k + 1) : k \geq 1\}.$$

Since B is defined by a regular sequence of elements of bidegree $(d, 1)$, we may apply Proposition 2.10 to (12):

$$\text{reg}_{B_\Delta}(F_k)_\Delta \leq \begin{cases} \max\{\lceil \frac{id}{c} \rceil, \lceil \frac{2i}{e} \rceil\} & \text{if } k = 2i, \\ \max\{\lceil \frac{(i+1)d}{c} \rceil, \lceil \frac{2i+1}{e} \rceil\} & \text{if } k = 2i + 1. \end{cases}$$

Since $3/2 < d/c \leq 2$, we conclude that $\alpha' \leq 1$. Since $\text{reg}_{B_\Delta} H_k(\mathbf{F}_\Delta) \leq k$ for all $k \geq 1$, we conclude that $\beta' \leq -1$. Therefore, by (14), one has

$$\text{reg}_{B_\Delta}(\text{Rees}(I)_\Delta) \leq 1.$$

Thus, we conclude that $\text{Rees}(I)_\Delta$ is Koszul. □

Remark 3.4. We observe that, in the proof of [8, Theorem 3.2], $H_k(\mathbf{F})_\Delta = 0$ for all k , whereas in our case this is true for all $e \geq 2$, and not for $e = 1$. This affects the proof of Theorem 3.1 much more than that of [8, Theorem 3.2]. To achieve our goal, we first have to deduce Lemma 2.7. We use the fact that, if the homology module is non zero and its regularity is bounded by the homology module at zero position, then by (1), Lemma 2.7 and Proposition 2.10, we conclude the proof.

4. More general base rings. In this section, the two main results that we generalize are [9, Theorem 6.2] and [9, Corollary 6.10]. We show that the Koszulness property holds even if the assumption of the polynomial ring is replaced by a Koszul ring.

Conca et al. in [9, page 900] posed two interesting questions, one of which was positively answered by Aramova, Crona and De Negri [1] who showed that for an arbitrary bigraded standard algebra R , the defining ideal of R_Δ has quadratic Grobner basis for $c, e \gg 0$, and another one by Blum [5], who showed that all the diagonal algebras of bigraded standard Koszul algebra R are Koszul. We will use these results in the proofs of Theorem 4.1 and Theorem 4.5.

Let A and B be two standard graded Koszul algebras. Let $A = K[A_1]$, where $A_1 = \langle X_1, \dots, X_m \rangle$ is a K vector space generated by linear forms with $\deg(X_i) = 1$. Similarly, let $B = K[B_1]$, where $B_1 = \langle Y_1, \dots, Y_n \rangle$ is a K vector space generated by linear forms with $\deg(Y_j) = 1$. We set $T = A \otimes_K B$. Then T is bigraded standard by setting $\deg(X_i) = (1, 0)$ and $\deg(Y_j) = (0, 1)$. Let R be a bigraded quotient of T , that is, $R = T/I$ for some bihomogeneous ideal I of T .

Let c and e be positive integers. We will study the Koszul property of (c, e) -diagonal subalgebra R_Δ of bigraded algebra R . Consider the bigraded free resolution of R over T :

$$(15) \quad \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow T \longrightarrow R \longrightarrow 0,$$

where

$$F_i = \bigoplus_{(a,b) \in \mathbf{N}^2} T(-a, -b)^{\beta_{i,(a,b)}}.$$

Set

$$(16) \quad t_{i,1} = \max\{a : \text{there exists } b \text{ such that } \beta_{i,(a,b)} \neq 0\},$$

and

$$t_{i,2} = \max\{b : \text{there exists } a \text{ such that } \beta_{i,(a,b)} \neq 0\}.$$

With this notation, the following is a generalization of [9, Theorem 6.2].

Theorem 4.1. *Let A and B be standard graded Koszul algebras. We set $T = A \otimes_K B$ and R a bi-graded quotient of T . Then:*

- (i) T is Koszul and $\text{reg}_T R$ is finite.
- (ii) If $c \geq \sup\{t_{i,1}/(i+1) : i \geq 1\} \in \mathbf{R}$ and $e \geq \sup\{t_{i,2}/(i+1) : i \geq 1\} \in \mathbf{R}$, then R_Δ is Koszul.
- (iii) In particular, if $c \geq (\text{reg}_T R - 1)/2$ and $e \geq (\text{reg}_T R - 1)/2$, then R_Δ is Koszul.

Proof. For (i), T is Koszul by [4] and $\text{reg}_T R$ is finite by [3, Theorem 1]. For (ii), applying $-\Delta$ functor to the bigraded free resolution (15) of R over T , one obtains an exact complex:

$$(17) \quad \cdots \longrightarrow (F_i)_\Delta \longrightarrow \cdots \longrightarrow (F_1)_\Delta \longrightarrow T_\Delta \longrightarrow R_\Delta \longrightarrow 0,$$

of T_Δ -modules, where

$$(F_i)_\Delta = \bigoplus_{(a,b) \in \mathbf{N}^2} T(-a, -b)_\Delta^{\beta_{i,(a,b)}}.$$

Note that $T_\Delta = A^{(c)} \otimes B^{(e)}$, where $A^{(c)}$ denotes the c -th Veronese subring of A , and $B^{(e)}$ denotes the e -th Veronese subring of B . Note that T_Δ is Koszul [5, Theorem 2.1]. To show that R_Δ is Koszul, it is enough to show that

$$\text{reg}_{T_\Delta} R_\Delta \leq 1.$$

Applying [9, Lemma 6.3 (ii)] to (17), we get

$$\text{reg}_{T_\Delta} R_\Delta \leq \sup\{\text{reg}_{T_\Delta} (F_i)_\Delta - i : i \geq 1\}.$$

Thus, it is enough to show that

$$(18) \quad \text{reg}_{T_\Delta} (F_i)_\Delta - i \leq 1 \text{ for all } i \geq 1.$$

Since

$$(F_i)_\Delta = \bigoplus_{(a,b) \in \mathbf{N}^2} T(-a, -b)_\Delta^{\beta_{i,(a,b)}},$$

one has

$$(19) \quad \text{reg}_{T_\Delta} (F_i)_\Delta = \max\{\text{reg}_{T_\Delta} T(-a, -b)_\Delta : \beta_{i,(a,b)} \neq 0\}.$$

Now we need to evaluate $\text{reg}_{T_\Delta} T(-a, -b)_\Delta$. We denote by $V_A(c, \alpha)$, the Veronese modules of A , that is, $V_A(c, \alpha) = \bigoplus_{s \in \mathbf{N}} A_{sc+\alpha}$ for $\alpha = 0, \dots, c - 1$. Similarly, denote $V_B(e, \beta)$, the Veronese modules

of B . Hence, for the shifted module $T(-a, -b)_\Delta$ we can write

$$\begin{aligned} T(-a, -b)_\Delta &= \bigoplus_s [A_{sc-a} \otimes B_{se-b}] \\ &= V_A(c, \alpha) \left(- \left[\frac{a}{c} \right] \right) \otimes V_B(e, \beta) \left(- \left[\frac{b}{e} \right] \right), \end{aligned}$$

where $\alpha = -a \bmod (c)$, $0 \leq \alpha \leq c - 1$ and $\beta = -b \bmod (e)$, $0 \leq \beta \leq e - 1$.

The Veronese modules $V_A(c, \alpha)$ and $V_B(e, \beta)$ have a linear resolutions as an $A^{(c)}$ -module and $B^{(e)}$ -module, respectively, see [6, Lemma 5.1]. Hence, by (2), one has

$$(20) \quad \text{reg}_{T_\Delta} T(-a, -b)_\Delta = \max \left\{ \left\lceil \frac{a}{c} \right\rceil, \left\lceil \frac{b}{e} \right\rceil \right\}.$$

Thus, by (16), (18), (19) and (20), we conclude that R_Δ is Koszul, provided that

$$(21) \quad \max \left\{ \left\lceil \frac{t_{i,1}}{c} \right\rceil, \left\lceil \frac{t_{i,2}}{e} \right\rceil \right\} \leq i + 1 \text{ for all } i \geq 1.$$

From (21), we conclude that if $c \geq \sup\{t_{i,1}/(i+1) : i \geq 1\}$ and $e \geq \sup\{t_{i,2}/(i+1) : i \geq 1\}$, then R_Δ is Koszul. By definition, one has $t_{i,1} \leq t_i \leq \text{reg}_T R - i$. Thus, we have:

$$(22) \quad \frac{t_{i,1}}{i+1} \leq \frac{t_i}{i+1} \leq \frac{\text{reg}_T R - i}{i+1}.$$

We know that $\text{reg}_T R$ is finite. Notice that $\frac{\text{reg}_T R - i}{i+1}$ is a decreasing function of i , as i varies over the natural numbers. Taking the sup in (22), we get

$$\sup \left\{ \frac{t_{i,1}}{i+1} : i \geq 1 \right\} \leq \sup \left\{ \frac{t_i}{i+1} : i \geq 1 \right\} \leq \sup \left\{ \frac{\text{reg}_T R - i}{i+1} : i \geq 1 \right\}.$$

Similarly, the other case

$$\sup \left\{ \frac{t_{i,2}}{i+1} : i \geq 1 \right\} \leq \sup \left\{ \frac{t_i}{i+1} : i \geq 1 \right\} \leq \sup \left\{ \frac{\text{reg}_T R - i}{i+1} : i \geq 1 \right\}.$$

Note that

$$\sup \left\{ \frac{\text{reg}_T R - i}{i+1} : i \geq 1 \right\} \leq \frac{\text{reg}_T R - 1}{2}.$$

Thus, we observe that the numbers $\sup\{t_{i,1}/(i + 1) : i \geq 1\}$ and $\sup\{t_{i,2}/(i + 1) : i \geq 1\}$ are in fact some finite real numbers, bounded by $(\text{reg}_T R - 1)/2$. Thus, claims (ii) and (iii) follow. \square

Remark 4.2. Note that in the Theorem 4.1, if we take $A = K[x_1, \dots, x_m]$ and $B = K[y_1, \dots, y_n]$, then we will re-obtain [9, Theorem 6.2].

The following is the generalization of [9, Corollary 6.9].

Corollary 4.3. *Let I be the homogeneous ideal of a standard graded Koszul ring A . Let d denote the highest degree of a minimal generator of I . Then there exist integers c_0, e_0 such that the K -algebra $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq c_0$ and $e \geq e_0$.*

Proof. Let $B = K[t_1, \dots, t_k]$ in Theorem 4.1. Then $T = A[t_1, \dots, t_k]$ is a polynomial extension of A . Note that T is a bigraded standard algebra by setting $\text{deg } X_i = (1, 0)$ and $\text{deg } t_j = (0, 1)$. By replacing I with the ideal generated by I_d , we may assume that I is generated by forms of degree d . Then $\text{Rees}(I) \subset A[t]$ is a bigraded standard algebra by setting $\text{deg } X_i = (1, 0)$ and $\text{deg } ft = (0, 1)$ for all $f \in I_d$. Moreover, $\text{Rees}(I)$ can also be realized as the bigraded quotient of T .

By Theorem 4.1 (i), we have T is Koszul and $\text{reg}_T \text{Rees}(I)$ is finite. Note that the numbers c and e exist from Theorem 4.1 (ii) such that $\text{Rees}(I)_\Delta$ is Koszul. In particular, if $c, e \geq (\text{reg}_T \text{Rees}(I) - 1)/2$, then $\text{Rees}(I)_\Delta$ is Koszul by Theorem 4.1 (iii). Thus, the claim follows, since $\text{Rees}(I)_\Delta = K[(I^e)_{ed+c}]$. \square

Remark 4.4. In Corollary 4.3, the integers c_0 and e_0 can be computed explicitly whenever one knows the shifts in the bigraded free resolution of $\text{Rees}(I)$ over the Koszul ring T . For instance, when I is a complete intersection ideal generated by homogeneous forms of degree d , one has Theorem 4.5.

The following is a generalization of [9, Corollary 6.10].

Theorem 4.5. *Let A be a standard graded Koszul ring. Let I be an ideal of A generated by a regular sequence f_1, f_2, \dots, f_k , of homogeneous*

forms of degree d . Then $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq d(k-1)/k$ and $e > 0$.

Proof. Let $A = K[A_1]$, where $A_1 = \langle X_1, \dots, X_m \rangle$ is a K vector space generated by linear forms with $\deg(X_i) = 1$. Let $A' = A[t_1, \dots, t_k]$ be a polynomial extension of A . Then A' is a bigraded standard algebra by setting $\deg X_i = (1, 0)$ and $\deg t_j = (0, 1)$. Let I be an ideal of A generated by a regular sequence f_1, f_2, \dots, f_k , of homogeneous forms of degree d . Then $\text{Rees}(I) \subset A[t]$ is a bigraded standard algebra by setting $\deg X_i = (1, 0)$ and $\deg f_j t = (0, 1)$. Let

$$\phi : A[t_1, t_2, \dots, t_k] \longmapsto A[It]$$

be the surjective map by sending t_j to $f_j t$. Since I is a complete intersection ideal, one has

$$\ker(\phi) = I_2 \begin{pmatrix} f_1 & f_2 & \cdots & f_k \\ t_1 & t_2 & \cdots & t_k \end{pmatrix}.$$

The resolution of $A[It]$ over A' is given by the Eagon-Northcott complex (23). We know that $\ker(\phi)$ is a determinantal ideal and $\text{grade}(\ker(\phi)) = k - 1$; hence, the Eagon-Northcott complex (23) is the minimal free resolution of $A[It]$ over A' :

$$(23) \quad 0 \longrightarrow F_{k-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A[It] \longrightarrow 0,$$

where

$$F_i = \bigoplus_{j=1}^i A'(-jd, -i - 1 + j)^{\sharp_i}.$$

Here \sharp_i denotes some integer which is irrelevant in our discussion. Applying the $-\Delta$ functor to (23), one obtains an exact complex:

$$(24) \quad 0 \longrightarrow (F_{k-1})_\Delta \longrightarrow \cdots \longrightarrow (F_1)_\Delta \longrightarrow (F_0)_\Delta \longrightarrow A[It]_\Delta \longrightarrow 0$$

of A'_Δ -modules, where

$$(F_i)_\Delta = \bigoplus_{j=1}^i A'(-jd, -i - 1 + j)^{\sharp_i}_\Delta.$$

From the exact complex (24), we build another complex:

$$\mathbf{F} : 0 \longrightarrow (F_{k-1})_\Delta \longrightarrow \cdots \longrightarrow (F_1)_\Delta \longrightarrow (F_0)_\Delta \longrightarrow 0$$

of A'_Δ -modules. Then the homology of the new complex \mathbf{F} is given by

$$H_i(\mathbf{F}) = \begin{cases} A[It]_\Delta & \text{if } i = 0; \\ 0 & \text{if } i > 0. \end{cases}$$

Thus, applying Lemma 2.7 to the complex \mathbf{F} , one has

$$\text{reg}_{A'_\Delta}(A[It]_\Delta) \leq \sup\{\text{reg}_{A'_\Delta}(F_i)_\Delta - i : i = 1, 2, \dots, k - 1\}.$$

Note that A'_Δ is Koszul [5, Theorem 2.1]. To show that $A[It]_\Delta$ is Koszul, it is enough to show that

$$(25) \quad \text{reg}_{A'_\Delta}(F_i)_\Delta - i \leq 1 \text{ for all } i = 1, 2, \dots, k - 1.$$

One has

$$\text{reg}_{A'_\Delta}(F_i)_\Delta = \max\{\text{reg}_{A'_\Delta} A'(-jd, -i - 1 + j)_\Delta : j = 1, 2, \dots, i\}.$$

By a similar argument as used in Theorem 4.1 to obtain equation (20), we get

$$\text{reg}_{A'_\Delta}(F_i)_\Delta = \max \left\{ \left\lceil \frac{jd}{c} \right\rceil, \left\lceil \frac{i + 1 - j}{e} \right\rceil : j = 1, 2, \dots, i \right\}.$$

Thus, we have

$$(26) \quad \text{reg}_{A'_\Delta}(F_i)_\Delta = \max \left\{ \left\lceil \frac{id}{c} \right\rceil, \left\lceil \frac{i}{e} \right\rceil \right\}.$$

Therefore, by (25) and (26), we conclude that $K[(I^e)_{ed+c}]$ is Koszul if $c \geq d(k - 1)/k$ and $e > 0$. □

We conclude the section with one final remark, and with an open question.

Remark 4.6. The claim by Conca, et al. in [9, page 900] together with Theorems 3.1 and 4.5 in this paper suggest that the following question may have a positive answer.

Question 4.7. *Let I be an ideal of a Koszul ring A generated by a regular sequence f_1, f_2, \dots, f_k , of homogeneous forms of degree d . Is it true that $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq d/2$ and $e > 0$?*

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