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ON SEPARABLE $\widetilde{p}^{\alpha}$-BOUNDED PRIMARY ABELIAN GROUPS

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ABSTRACT. The notion of $\widetilde{p}^{\alpha}$-boundedness is used to study several topics in the theory of separable abelian p-groups, including the torsion product of torsion-complete groups, the generalized core class property, and in the context of the constructible universe, groups whose socles are essentially finitely indecomposable as valuated vector spaces.

1. Introduction. By the term *group* we will mean an abelian p-group, where p is some fixed prime. Our terminology will, in the main, be standard, and consistent with that found in **[5]** or **[8]**.

In this paper we apply some invariants from **[19]** to the class of separable (i.e., p^{ω}-bounded) groups. In fact, these invariants go back at least to **[1, 18]**, where they were used to answer *Nunke's problem*, which asks when the torsion product of two groups is a direct sum of cyclics (hereafter shortened to Σ-*cyclic*). Although we review these ideas below, the reader will need to refer to **[18, 19]** for most of the details.

If $\mathcal{A}$ is the class of all groups, then a non-empty proper subclass $\mathcal{B} \subset \mathcal{A}$ was said to be *additively bounded* if it is closed under subgroups and direct sums. If $\mathcal{U} = \mathcal{A} - \mathcal{B}$, then $E = (\mathcal{B},\mathcal{U})$ was termed a *B/U-pair*; the elements of $\mathcal{B}$ were called *E-bounded* and those of $\mathcal{U}$ were called *E-unbounded*. If $E' = (\mathcal{B}',\mathcal{U}')$ is a second B/U-pair, we write $E \subseteq E'$ when $\mathcal{B}' \subseteq \mathcal{B}$. The most important examples are where α is in $\mathcal{O}$ (i.e., the ordinals with the symbol ∞ adjoined), and $\mathcal{B}$ consists of the p^{α}-bounded groups; we denote this by $p^{\alpha} = (\mathcal{B}^{\alpha},\mathcal{U}^{\alpha})$.

If $E = (\mathcal{B},\mathcal{U})$ is any additively bounded B/U-pair, then for every group G there is defined (using transfinite induction, subgroups and filtrations) an invariant L_G^E, which will be a subclass of $\mathcal{R}_f$, the

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collection of finite sets consisting of regular cardinals. For simplicity, $L_G^{p^\alpha}$ is denoted by L_G^α. If $0_\mathcal{R} = \emptyset$ and $\widetilde{\mathcal{B}}$ is the collection of all groups G such that $L_G^E = 0_\mathcal{R}$, then $\widetilde{E} = (\widetilde{\mathcal{B}}, \mathcal{A} - \widetilde{\mathcal{B}})$ will be the smallest additively bounded B/U-pair containing E that is *perfect* in the sense that $\widetilde{\mathcal{B}}$ is also closed under filtrations. We write $\widetilde{p}^\alpha$ for $\widetilde{p^\alpha}$; it is a fundamental result from [**18**] that a group is $\widetilde{p}^\omega$-bounded if and only if it is Σ-cyclic.

If $\alpha \in \mathcal{O}$, then there is another important perfect B/U-pair which is denoted by $p_*^\alpha = (\mathcal{W}^\alpha, \mathcal{A} - \mathcal{W}_\alpha)$, where $\mathcal{W}_\alpha$ consists of those groups that can be embedded in a p^α-bounded simply presented group. The groups in $\mathcal{W}^\alpha$ were studied in [**15**] under the name p_*^α*-projectives*, where they were shown to be the projectives with respect to a natural collection of short-exact sequences. They have also been studied under the name *weak* p^α*-projectives* (e.g., [**17**]). The class $\mathcal{W}^\infty$ was first described by Nunke in [**20**]; note that $G \in \mathcal{W}^\infty$ if and only if it is p_*^∞-projective if and only if it a subgroup of some reduced simply presented group. If $n < \omega$, a group is $p_*^{\omega+n}$-projective if and only if it is $p^{\omega+n}$-projective, and G is $p_*^{\omega+\omega}$-projective if and only if it has a subgroup A such that both A and G/A are Σ-cyclic.

If $E = (\mathcal{B}, \mathcal{U})$ is any additively bounded B/U-pair, then the *length* λ of E is the supremum of the lengths of the elements of $\mathcal{B}$. In [**19**] it was shown that λ is the unique element of $\mathcal{O}$ such that $p^\lambda \subseteq E \subseteq p_*^\lambda$. In addition, if E is perfect, then $\widetilde{p}^\lambda \subseteq E \subseteq p_*^\lambda$. In particular, this means that every p_*^λ-projective group is $\widetilde{p}^\lambda$-bounded. It was also shown that $\widetilde{p}^\lambda = p_*^\lambda$ if and only if $\lambda \leq \omega$.

In this paper the above ideas are applied in three directions. Section 2 is a discussion of the torsion product of torsion-complete groups. Our main result (Theorem 2.1) implies that such a product will either be so well-behaved as to be Σ-cyclic, or it will be so misbehaved as to fail to be p_*^∞-projective (i.e., it is impossible to embed it in a reduced simply presented group). This observation allows us to provide an interesting example of a group that is *almost* Σ*-cyclic* (as defined by Hill in [**10**]), but not p_*^∞-projective (Corollary 2.4).

In Section 3 we study the following long-standing question: Does every reduced group satisfy the "generalized core class property," i.e., if G is a reduced group and $n < \omega$, is it true that either G is $p^{\omega+n}$-projective, or that it has a proper $p^{\omega+n+1}$-projective subgroup? This question has been studied by a number of authors (see, for example, [**2, 3, 11, 16, 17**]). Though we do not completely settle the question, we do prove the following slightly weaker version: If G is a reduced group and $n < \omega$, then G is either $\widetilde{p}^{\omega+n}$-bounded, or it has a proper $p^{\omega+n+1}$-projective subgroup (Corollary 3.1).

Section 4 is devoted to addressing some issues left over from [**19**]. A useful property of $p^{\omega+1}$-projective $= p_*^{\omega+1}$-projective groups is that they are *C-decomposable* (i.e., they have Σ-cyclic summands with the same final rank as the group itself, see [**7**]). On the other hand, in [**4**], it was shown (using somewhat different terminology) that, in the context of the constructible universe (V=L), there is a separable $\widetilde{p}^{\omega+1}$-bounded group that is *essentially finitely indecomposable* (i.e., it does not have a summand that is an unbounded Σ-cyclic group). We simplify and improve this construction to show that such a separable $\widetilde{p}^{\omega+1}$-bounded group G can be constructed such that its *socle* $G[p]$ is essentially finitely decomposable as a *valuated vector space* (Theorem 4.1; see [**6**] for definitions of these terms). On the other hand, it is an easy consequence of results from [**14**] that no unbounded separable p_*^{∞}-projective group has a socle that is essentially finitely indecomposable (Corollary 4.2). This means that in $V = L$ there are separable $\widetilde{p}^{\omega+1}$-bounded groups that are not p_*^{∞}-projective (Corollary 4.3). As a result, we are able to settle some open questions and conjectures from [**19**] (Corollaries 4.4 and 4.5).

2. The torsion product of torsion complete groups. If $\lambda, \lambda' \in \mathcal{O}$ with $\lambda \leq \lambda'$, then it easily follows that $L_G^{\lambda'} \subseteq L_G^{\lambda}$ for all groups G (or see [**19,** Proposition 2.2]). The following sharpens this observation a bit.

Proposition 2.1. *If G is a group and $\gamma = |G|^+$, then $L_G^{\infty} = L_G^{\gamma}$.*

Proof. The definitions of L_G^{γ} and L_G^{∞} only refer to groups X of cardinality at most $|G|$. Since $p^{\gamma}X = p^{\infty}X$ for all such groups, the computations of L_G^{γ} and L_G^{∞} start with the same initial data, so they yield the same final results. $\square$

By the *$\mathcal{L}$-length* of a group G, we will mean the smallest ordinal λ such that $L_G^{\infty} = L_G^{\lambda}$. (This is not to be confused with the length of an additive B/U-pair defined in [**19**].) By Proposition 2.1, every group

G has an $\mathcal{L}$-length, which never exceeds $|G|^+$. Since $L_G^0 = 1_{\mathcal{R}}$ (where $1_{\mathcal{R}} = \mathcal{R}_f$) for all groups $G \neq \{0\}$, G has $\mathcal{L}$-length 0 if and only if $G = \{0\}$ or it is not reduced. If G is reduced of $\mathcal{L}$-length $0 < \lambda < \infty$, then $L_G^\lambda = L_G^\infty \neq 1_{\mathcal{R}}$. This implies that G must be p^λ-bounded; so if λ' is the (customary) length of G, then $\lambda' \leq \lambda$. In addition, when G has $\mathcal{L}$-length λ and $\lambda \leq \alpha \leq \infty$, then $L_G^\lambda = L_G^\alpha = L_G^\infty$.

Observe that, if λ is an ordinal and H is a reduced simply presented group of length λ, then for $\alpha \in \mathcal{C}$ we have $L_G^\alpha = 1_{\mathcal{R}}$ when $\alpha < \lambda$ and $L_G^\alpha = 0_{\mathcal{R}}$ when $\lambda \leq \alpha$ (see, for example, [**19,** Corollary 3.8]). It follows that such an H always has $\mathcal{L}$-length λ. We wish to strengthen this observation. We will denote the torsion product of the groups A and B by the convenient, albeit non-standard, notation $A \bigtriangledown B$.

Proposition 2.2. *For every ordinal λ there is a separable group of $\mathcal{L}$-length λ.*

Proof. Let H be any reduced simply presented group of length λ, B a Σ-cyclic group of rank and final rank $\gamma > |H|$ such that $\gamma^{\aleph_0} > \gamma$ (e.g., γ has countable cofinality). We claim that $G = \overline{B} \bigtriangledown H$ is as advertised.

First, observe that G is p^λ-projective. So it must be p_*^λ-projective and hence $\widetilde{p}^\lambda$-bounded. Therefore, $0_{\mathcal{R}} \subseteq L_G^\infty \subseteq L_G^\lambda = 0_{\mathcal{R}}$.

On the other hand, let $\kappa = \gamma^+ \in \mathcal{R}$ and $A \subseteq \overline{B}$ be a pure subgroup containing B of cardinality κ. If $\{X_i\}_{i<\kappa}$ is a pure filtration of A such that $B \subseteq X_0$, then each A/X_i is divisible. It follows that $\{X_i \bigtriangledown H\}_{i<\kappa}$ is a pure filtration of $A \bigtriangledown H \subseteq G$ and there are pure-exact sequences

$$0 \longrightarrow X_i \bigtriangledown H \longrightarrow A \longrightarrow (A/X_i) \bigtriangledown H \longrightarrow 0.$$

Note that $(A/X_i) \bigtriangledown H$ will be isomorphic to a direct sum of copies of H. So if $\alpha < \lambda$, then $\{\kappa\} \in L_G^\alpha$. Therefore, G is not $\widetilde{p}^\alpha$-bounded, which means that it has $\mathcal{L}$-length exactly λ. $\square$

The next statement is an analogue of the useful observation that, if $n < \omega$, α is an ordinal, G is p^α-bounded and P is a subgroup of $G[p^n]$, then G/P is $p^{\alpha+n}$-bounded.

Proposition 2.3. *If $n < \omega$, G is a group of $\mathcal{L}$-length λ and P is a subgroup of $G[p^n]$, then G/P has $\mathcal{L}$-length at most $\lambda + n$ and $L_G^\infty = L_{G/P}^\infty$.*

Proof. If we set $G' = G/P$ and $P' = G[p^n]/P \subseteq G'[p^n]$, then $G'/P' \cong p^n G$. Using ([**19,** Corollary 3.1]), for all $\mu \geq \lambda$, we have

$$\begin{aligned} L_G^\lambda = L_G^{\mu+2n} &\subseteq L_{G'}^{\mu+n} \cup L_P^n \\ = L_{G'}^{\mu+n} &\subseteq L_{p^n G}^{\mu} \cup L_{P'}^n \\ = L_{p^n G}^{\mu} &\subseteq L_G^\mu = L_G^\lambda. \end{aligned}$$

This implies that $L_G^\infty = L_G^\lambda = L_{G'}^{\lambda+n} = L_{G'}^\infty$, as stated. □

Corollary 2.1. *If G is a group and P is a bounded subgroup of G, then G is $\widetilde{p}^{\infty}$-bounded if and only if G/P is $\widetilde{p}^{\infty}$-bounded*

Clearly, a group G has $\mathcal{L}$-length at most ω if and only if $L_G^\omega = L_G^\infty$. If, in addition, G is reduced, then its $\mathcal{L}$-length agrees with its normal length. Here is our primary example:

Proposition 2.4. *A torsion-complete group has $\mathcal{L}$-length at most ω.*

Proof. This is actually shown by the computation in ([**18,** Proposition 3.1]). □

Corollary 2.2. *If $n < \omega$, then a group that is $p^{\omega+n}$-injective in the category of abelian p-groups will have $\mathcal{L}$-length at most $\omega + n$.*

Proof. If G is such a group, then we may clearly assume that it is reduced. It follows (say, from [**13,** Lemma 3]) that $p^{\omega+n}G = \{0\}$ and $G/p^\omega G$ is torsion-complete. Find a group A such that $p^n A = G$. So $p^\omega A = p^\omega G$ and $G' = A/p^\omega A$ are torsion-complete. Note $P = A[p^n]/p^\omega A \subseteq G'$ is p^n-bounded and $G'/P \cong A/A[p^n] \cong p^n A = G$. The result then follows from Propositions 2.3 and 2.4. □

In the solution to Nunke's problem, which used the invariants L_G^ω, ([**18,** Theorem 2.1]) was an important step. Because of ([**19,** Proposition 3.5]), exactly the same proof will establish the following for p^λ when λ is countable or ∞.

Lemma 2.1. *Let $\lambda \in \mathcal{O}$ be either countable or ∞. If G and H are groups, then $L^\lambda_G \cdot L^\lambda_H \subseteq L^\lambda_{G \,\triangledown\, H}$.*

Torsion products of torsion-complete groups have provided interesting examples of various phenomena. For example, suppose B is a countable unbounded Σ-cyclic group with torsion completion $G = \overline{B}$. In ([**12,** Proposition 5]), it was shown that $G \triangledown G$ is Σ-cyclic if and only if the continuum hypothesis holds (i.e., $2^{\aleph_0} = \aleph_1$). In addition, in ([**18,** Corollary 3.3]) it was shown that $L^\omega_{G \,\triangledown\, G} = L^\omega_G \cdot L^\omega_G$ if and only if $2^{\aleph_0}$ is smaller than the first weakly Mahlo cardinal. The following shows that the torsion product of torsion-complete groups is either Σ-cyclic, or not p^∞_*-projective; in other words, such products are either very simple or rather wild.

Theorem 2.1. *Suppose $G_1, \ldots, G_k$ are groups with $\mathcal{L}$-length at most ω, and let $G = G_1 \triangledown \cdots \triangledown G_k$. Then the following are equivalent*:

(a) *G is Σ-cyclic*;
(b) *G is p^∞_*-projective*;
(c) *G is $\widetilde{p}^{\,\infty}$-bounded.*

Proof. Certainly (a) implies (b). Next, since any p^∞_*-projective group is $\widetilde{p}^{\,\infty}$-bounded, (b) implies (c). We will be done if we can show that (c) implies (a).

If (c) holds, then by Lemma 2.1 we have an inclusion

$$L^\omega_{G_1} \cdots L^\omega_{G_k} = L^\infty_{G_1} \cdots L^\infty_{G_k} \subseteq L^\infty_{G_1 \,\triangledown\, \cdots \,\triangledown\, G_k} = 0_{\mathcal{R}}.$$

Therefore, $L^\omega_{G_1} \cdots L^\omega_{G_k} = 0_{\mathcal{R}}$. However, by ([**18,** Corollary 3.7]), this implies that G is Σ-cyclic, and (a) follows. □

For torsion-complete groups, we restate the above in a couple of equivalent ways.

Corollary 2.3. *Suppose $\overline{B}_1, \ldots, \overline{B}_k$ are torsion-complete groups, and let $G = \overline{B}_1 \triangledown \cdots \triangledown \overline{B}_k$.*

(a) *If G can be embedded in a reduced simply presented group, then it is Σ-cyclic.*
(b) *If $L_G^\infty = 0_{\mathcal{R}}$, then it is Σ-cyclic.*

The group G is *almost Σ-cyclic* if it has a collection of subgroups $\mathcal{C}$ such that

(a) $\{0\} \in \mathcal{C}$;
(b) If $C \in \mathcal{C}$, then C is closed in G in the p-adic topology, i.e., G/C is p^ω-bounded;
(c) $\mathcal{C}$ is inductive, i.e., closed under unions of chains;
(d) If $X \subseteq G$ is countable, then there is a countable $C \in \mathcal{C}$ such that $X \subseteq C$.

These groups were defined in **[10]** using the (clearly equivalent) terminology *almost coproducts of finite cyclic groups.*

In **[1]** the proof of the next result was embedded in a rather involved discussion of what were termed *K_G-invariants.* We present the following more straightforward and essentially self-contained argument.

Proposition 2.5. ([**1,** Corollary 35]). *If G and H are separable groups, then $G \bigtriangledown H$ is almost Σ-cyclic.*

Proof. We may clearly assume that G and H are unbounded (otherwise, $G \bigtriangledown H$ is Σ-cyclic). We begin with a simple observation.

Claim A. If A, A' are unbounded subgroups of G and B, B' are unbounded subgroups of H, then $A \bigtriangledown B \subseteq A' \bigtriangledown B'$ if and only if $A \subseteq A'$ and $B \subseteq B'$.

Sufficiency being trivial, let $a \in A$. Choose $b \in B$ of the same order as a. So $\langle a \rangle \bigtriangledown \langle b \rangle \subseteq A \bigtriangledown B \subseteq A' \bigtriangledown B'$. And, by ([**21,** Corollary 9]), $a \in A'$, which implies that $A \subseteq A'$. Showing $B \subseteq B'$ is analogous.

We now turn to proving the result. Let $\mathcal{C}$ consist of $\{0\}$ together with all subgroups of $G \bigtriangledown H$ of the form $C = A \bigtriangledown B$, where A and B are unbounded subgroups of G and H, respectively.

We show $\mathcal{C}$ satisfies (a)–(d) for $G \bigtriangledown H$. Note first that (a) and (d) are quite easy. Regarding (b), if $A \bigtriangledown B \in \mathcal{C}$, then there is an embedding

$$(G \triangledown H)/(A \triangledown B) \longrightarrow [(G/A) \triangledown H] \oplus [G \triangledown (H/B)].$$

Since the group on the right is clearly p^ω-bounded (in fact, it will be a subgroup of a direct sum of a collection of copies of H and G), (b) immediately follows.

Finally, if $\{A_i \triangledown B_i\}_{i\in I}$ is a chain in $\mathcal{C}$, then by Claim A, $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ will be chains in G and H, respectively. So

$$\bigcup_{i\in I}(A_i \triangledown B_i) = \left(\bigcup_{i\in I} A_i\right) \triangledown \left(\bigcup_{i\in I} B_i\right) \in \mathcal{C},$$

which establishes (c) and completes the argument. □

Corollary 2.4. *There is a group G that is almost Σ-cyclic, but not p_*^∞-projective.*

Proof. Let $\overline{B}$ be a torsion-complete group of final rank at least $\aleph_2$ and $G = \overline{B} \triangledown \overline{B}$. It follows from Proposition 2.5 that G is almost Σ-cyclic. On the other hand, by ([**18,** Proposition 3.1]), $\{\aleph_1\}$ and $\{\aleph_2\}$ are in $L^\omega_{\overline{B}}$, so it follows that $\{\aleph_1, \aleph_2\} \in L^\omega_G$. So G is not Σ-cyclic; and by Corollary 2.3 (a), it is not p_*^∞-projective. □

Recall that a group is Σ-cyclic if and only if it is p^ω-projective. On the other hand, the group we have just constructed is *almost* Σ-cyclic, but not only does it fail to be p^λ-projective for any ordinal λ, it cannot even be embedded in such a group.

3. The generalized core class property. Recall that, if $n < \omega$ and G is a group that is $p^{\omega+n}$-projective, then any subgroup of G is also $p^{\omega+n}$-projective. On the other hand, consider the statement:

$\mathrm{CCP}_n(G)$ – *Either G is $p^{\omega+n}$-projective or it has a subgroup which is $p^{\omega+n+1}$-projective but not $p^{\omega+n}$-projective.*

In other words, if G is not $p^{\omega+n}$-projective, then it has a $p^{\omega+n+1}$-projective subgroup that is *proper*, in the sense that it is not $p^{\omega+n}$-projective. It is known that $\mathrm{CCP}_0(G)$ holds for every reduced group

G, that is, every reduced group that is not Σ-cyclic has a proper $p^{\omega+1}$-projective subgroup. This is the so-called "core class theorem" of [**2**]. A group G is said to have the *generalized core class property* (or GCCP) if $\mathrm{CCP}_n(G)$ holds for all $n < \omega$.

Does every reduced group have the GCCP, i.e., does $\mathrm{CCP}_n(G)$ hold for all reduced groups G and all $n < \omega$? The class of reduced groups with the GCCP is quite extensive and contains, for example, the C-decomposable groups, the groups whose final ranks have countable cofinality, and the weak $p^{\omega\cdot 2}$-projective $= p_*^{\omega\cdot 2}$-projective groups. In fact, it is arguably the case that what is most significant about this question is not necessarily its intrinsic importance, but rather the number of interesting techniques and results that have come from studying it.

Since this paper is primarily concerned with separable groups, we include the next essentially well-known observation.

Proposition 3.1. *If all separable groups satisfy the* GCCP*, then all reduced groups satisfy the* GCCP*.*

Proof. Suppose G is any reduced group. If G is not $p^{\omega+k}$-bounded for some $k < \omega$, it follows easily from the theory of simply presented groups that, for each $n < \omega$, G has a countable subgroup H of length $\omega + n + 1$. Clearly, such a group will be a proper $p^{\omega+n+1}$-projective so that G satisfies the GCCP.

So assume $k < \omega$ and G is $p^{\omega+k}$-bounded. By hypothesis, $G/p^{\omega}G$ satisfies the GCCP. Suppose $n < \omega$ and G is not $p^{\omega+n}$-projective. If $G/p^{\omega}G$ is $p^{\omega+k+n}$-projective, then G is $p^{\omega+2k+n}$-projective, and all $p^{\omega+2k+n}$-projective groups satisfy the GCCP by ([**11,** Corollary 28]). And, if $G/p^{\omega}G$ is not $p^{\omega+k+n}$-projective, then it must have a proper $p^{\omega+k+n+1}$-projective subgroup $H' \subseteq G/p^{\omega}G$. If $H \subseteq G$ is defined by the equation $H/p^{\omega}G = H'$, then it easily follows that H is $p^{\omega+2k+n+1}$-projective, but not $p^{\omega+n}$-projective. So, again, by ([**11,** Corollary 28]), H will have a subgroup that is a proper $p^{\omega+n+1}$-projective. Therefore, G has the GCCP, completing the argument. □

By a classical result of Hill ([**9**]), if G is the ascending union of a sequence of pure subgroups $\{G_i\}_{i<\omega}$ and each G_i is Σ-cyclic, then so is G. This is the $\alpha = \omega$ case of the next result, whose proof is a simplified version of ([**1,** Theorem 14]). We will use without extensive explanation the notation of [**19**].

Theorem 3.1. *If $\alpha < \omega \cdot 2$ and G is a group which is the ascending union of a sequence of pure subgroups $\{G_j\}_{j<\omega}$, then $L^\alpha_G = \cup_{j<\omega} L^\alpha_{G_j}$.*

Proof. If α is finite, this reduces to the obvious statement that G is p^α-bounded if and only if each G_j is p^α-bounded; so assume $\alpha = \omega + n$ is infinite. Since the containment $\supseteq$ is routine, we consider the inclusion $\subseteq$. We prove by induction on $\kappa \stackrel{\text{def}}{=} \mu(T)$ that, if $T \in L^\alpha_G$, then there is a $j < \omega$ such that $T \in L^\alpha_{G_j}$.

First, if $\kappa = \aleph_0$, then $T = \emptyset$, which means that $p^{\omega+n}G \neq \{0\}$. So $p^\omega G$ has an element x of order p^{n+1}. Clearly, $x \in G_j$ for some $j < \omega$. The purity of G_j in G then implies that $x \in p^\omega G_j$. This means that $p^{\omega+n}G_j \neq \{0\}$, so that $T \in L^\alpha_{G_j}$, as required.

Suppose, therefore, that $\kappa > \aleph_0$, and we have verified the result for all $S \in L^\alpha_G$ with $\mu(S) < \kappa$ and $T \in L^\alpha_G$. Consider the reason that $T \in L^\alpha_G$.

Suppose first that (L$^\alpha$-1) is true. If $i < \kappa$, then since $\mu(T_i) < \mu(T)$, we can conclude by induction on κ that

$$\Upsilon^\alpha_T(G) = \bigcup_{j<\omega} \Upsilon^\alpha_T(G_j) \subseteq \kappa.$$

Since $\Upsilon^\alpha_T(G)$ is stationary in κ, $\Upsilon^\alpha_T(G_j)$ is also stationary for some $j < \omega$, and we have $T \in L^\alpha_{G_j}$.

Suppose next that (L$^\alpha$-2) is true; let A be as in the statement of that condition. After possibly expanding A a bit (without changing its cardinality), we can also assume that:

(a) For all $j < \omega$, $A \cap G_j$ is pure in G_j, and hence in G and in A.

Next, since κ has uncountable cofinality, if we ignore a few G_js at the beginning, we may assume that

(b) $|A \cap G_j| = \kappa$ for all $j < \omega$;

Let $\{A_i\}_{i<\kappa}$ be a filtration of A. Note that $[A_i + (A \cap G_j)]/(A \cap G_j)$ is a smoothly ascending chain of groups of cardinality less than κ whose union is $A/(A \cap G_j)$. The fact that κ is an uncountable regular cardinal easily implies that, by restricting to a closed and unbounded subset, we may assume that, for all $i < \kappa$ and $j < \omega$, that

(c) $[A_i+(A\cap G_j)]/(A\cap G_j)$ is pure in $A/(A\cap G_j)$, and so $A_i+(A\cap G_j)$ will be pure in A.

This implies that $[A_i + (A \cap G_j)]/A_i$ is a pure subgroup of A/A_i, and their union over $j < \omega$ will be A/A_i.

Let $\mathcal{S} = \Lambda^{\alpha}_{T}(A) \subseteq \kappa$. By induction, for all $i \in \mathcal{S}$, we can conclude that there is a $j_i < \omega$ such that T_i is in the $\mathcal{R}_f$-invariant corresponding to

$$[A_i + (A \cap G_{j_i})]/A_i \cong (A \cap G_{j_i})/(A_i \cap G_{j_i}).$$

It follows that there is a fixed $j' < \omega$ such that $\mathcal{S}' \overset{\text{def}}{=} \{i \in \mathcal{S} : j_i = j'\}$ is stationary in κ.

Since $\{A_i \cap G_{j'}\}_{i<\kappa}$ is a filtration of $A\cap G_{j'}$, it follows that $\Lambda^{\alpha}_{T}(A\cap G_{j'})$ is stationary in κ (since it contains $\mathcal{S}'$). Therefore, applying (L^{α}-2) to $A\cap G_{j'}$, we have $T \in L^{\alpha}_{A\cap G_{j'}} \subseteq L^{\alpha}_{G_{j'}}$, as required. □

The last result fails for larger ordinals. Here is an easy counterexample.

Proposition 3.2. *There is a separable group G that is the ascending union of a sequence of pure subgroups $\{G_j\}_{j<\omega}$, such that $\cup_{j<\omega} L^{\omega\cdot 2}_{G_j} = 0_{\mathcal{R}}$, but $L^{\omega\cdot 2}_{G} \neq 0_{\mathcal{R}}$.*

Proof. Let H be any countable reduced group of length $p^{\omega\cdot 2+1}$; in particular, $L^{\omega\cdot 2}_{H} = 1_{\mathcal{R}}$. Next, let X be the generators in some simple presentation of H.

If $j < \omega$, let Y_j consist of all $y \in X$ such that, for all $k < \omega$, either $|p^k y|_H < \omega+j$ or $|p^k y|_H \geq \omega\cdot 2$. If $H_j \overset{\text{def}}{=} \langle Y_j\rangle$, then it is readily checked that H_j is pure in H and that H is the ascending union of the H_js. In addition, H_j is a $p^{\omega+j+1}$-bounded simply presented group.

Let T be a separable group such that $L^{\omega\cdot 2}_{T} \neq 0_{\mathcal{R}}$ (e.g., a torsion-complete group). If we let $G = H \bigtriangledown T$ and, for all $j < \omega$, we let $G_j = H_j \bigtriangledown T$, then these groups are separable, G is the union of its pure subgroups G_j, and, by Lemma 2.1,

$$0_{\mathcal{R}} \neq L^{\omega\cdot 2}_{T} = 1_{\mathcal{R}} \cdot L^{\omega\cdot 2}_{T} = L^{\omega\cdot 2}_{H} \cdot L^{\omega\cdot 2}_{T} \subseteq L^{\omega\cdot 2}_{G}.$$

On the other hand, each G_j is $p^{\omega+j+1}$-projective, so that $L^{\omega\cdot 2}_{G_j} \subseteq L^{\omega+j+1}_{G_j} = 0_{\mathcal{R}}$; therefore, $\cup_{j<\omega} L^{\omega\cdot 2}_{G_j} = 0_{\mathcal{R}}$. □

Notice that, in Proposition 3.2, since G is separable, the subgroups G_j will actually be *isotype* in G.

Question. *Suppose E is an additive B/U-pair such that whenever G is a group that is the ascending union of a sequence of pure subgroups $\{G_j\}_{j<\omega}$, then $L^E_G = \cup_{j<\omega} L^E_{G_j}$. Can we conclude that $E = p^\alpha$ for some $\alpha < \omega \cdot 2$?*

We are not able to determine whether all reduced groups satisfy the GCCP, but we do show that in $\mathrm{CCP}_n(G)$, if "not $p^{\omega+n}$-projective" is replaced by the more restrictive condition "$\widetilde{p}^{\omega+n}$-unbounded," then the resulting statement *does* hold for all reduced groups G and all $n < \omega$.

Theorem 3.2. *Suppose $n < \omega$ and G is a reduced group. If G is $\widetilde{p}^{\omega+n}$-unbounded, then it has a $p^{\omega+n+1}$-projective subgroup that is $\widetilde{p}^{\omega+n}$-unbounded.*

Proof. Suppose G has final rank κ. After possibly discarding a bounded summand, we may assume that G also has rank κ.

Let B be a lower basic subgroup of G, so that $G/B \cong \bigoplus_{i\in I} Z_i$, where $|I| = \kappa$ and each Z_i is a copy of the infinite cocyclic group $\mathbb{Z}_{p^\infty}$. Express I as the ascending union of subsets I_j for $j < \omega$ with the property that $|I - I_j| = \kappa$ for each $j < \omega$; and define $G_j \subseteq G$ by the equation $G_j/B = \bigoplus_{i\in I_j} Z_i$. It follows that each G_j is pure in G, G is the union of the G_j and each G/G_j is isomorphic to the direct sum of κ copies of $\mathbb{Z}_{p^\infty}$.

By Theorem 3.1, there is a $j_0 < \omega$ such that $L^{\omega+n}_{G_{j_0}} \neq 0_{\mathcal{R}}$. If we let H be a countable group with $p^\omega H \cong \mathbb{Z}_{p^{n+1}}$ and we define $A = G_{j_0} \bigtriangledown H$, then clearly A is $p^{\omega+n+1}$-projective. By Lemma 2.1,

$$0_{\mathcal{R}} \neq L^{\omega+n}_{G_{j_0}} = L^{\omega+n}_{G_{j_0}} \cdot 1_{\mathcal{R}} = L^{\omega+n}_{G_{j_0}} \cdot L^{\omega+n}_{H} \subseteq L^{\omega+n}_{A},$$

so that A is $\widetilde{p}^{\omega+n}$-unbounded.

If we let $P = G_{j_0} \bigtriangledown\ p^\omega H \subseteq A$, then it is easy to check that there is an isomorphism $f : P = G_{j_0} \bigtriangledown\ p^\omega H \cong G_{j_0}[p^{n+1}]$ that preserves all finite heights (computed in A and G_{j_0}, respectively). And, since

$$A/P = (G_{j_0} \bigtriangledown H)/(G_{j_0} \bigtriangledown p^{\omega}H) \subseteq G_{j_0} \bigtriangledown (H/p^{\omega}H),$$

this quotient is Σ-cyclic. This easily implies that f extends to a homomorphism $g : G_{j_0} \bigtriangledown H \to G_{j_0}$.

Next, observe that there is a subgroup $M \subseteq G/G_{j_0}$ such that M is isomorphic to A/P. Define the subgroup $N \subseteq G$ by the equation $N/G_{j_0} = M$. Since G_{j_0} is pure in G, it is also pure in N. And, since M is Σ-cyclic, there is a decomposition $N = G_{j_0} \oplus C$, where $C \cong M \cong A/P$. We let h be the composite $A \to A/P \to C$.

Define $\phi : A \to G$ by $\phi(x) = g(x) + h(x) \in G_{j_0} \oplus C \subseteq G$. Since P is the kernel of h and g is injective on P, it follows that ϕ is an embedding, as desired. □

Recall that, if $n = 0$, then a group is $\widetilde{p}^{\omega}$-bounded if and only if it is Σ-cyclic. So the $n = 0$ case of Theorem 3.2 is simply the core class theorem of [**2**]. Next, since any $p^{\omega+n}$-projective group is $\widetilde{p}^{\omega+n}$-bounded, Theorem 3.2 has the following immediate consequence.

Corollary 3.1. *Suppose $n < \omega$ and G is a reduced group. If G is $\widetilde{p}^{\omega+n}$-unbounded, then it has a subgroup that is a proper $p^{\omega+n+1}$-projective.*

Corollary 3.2. *If G is a reduced group that is $\widetilde{p}^{\omega+n}$-unbounded for all $n < \omega$, then G has the* GCCP.

The following shows that, if there is a reduced group without the GCCP, then it lies in a narrow range. In fact, it must be an example of the proper inclusion discussed in ([**19,** Theorem 3.14]) at the ordinal $\alpha = \omega \cdot 2$.

Corollary 3.3. *If G is a reduced group that does not have the* GCCP, *then G is $\widetilde{p}^{\omega\cdot 2}$-bounded, but not $p_*^{\omega\cdot 2}$-projective.*

Proof. Consider the contrapositive. If G is $\widetilde{p}^{\omega\cdot 2}$-unbounded, then it is $\widetilde{p}^{\omega+n}$-unbounded for all $n < \omega$; and by Corollary 3.2, G has the GCCP. On the other hand, if G is $p_*^{\omega\cdot 2}$-projective, then by ([**17,** Corollary 3.4 (b)]), G again has the GCCP. □

Recall that a group G is *fully starred* if, for every subgroup A of G, if B is a basic subgroup of A, then $|A| = |B|$. One of the earliest observations regarding the GCCP was the fact that any reduced group that is *not* fully starred satisfies the GCCP (see [**3,** Corollary 7]). Corollaries 3.2 and 3.3 can be viewed as more sophisticated versions of this statement. To see this, if the reduced group G is not fully starred, B is a basic subgroup of $A \subseteq G$ with $|B| < |A|$, then let $\kappa = |B|^+ \in \mathcal{R}$. After possibly replacing A by one of its subgroups, we may clearly assume that $|A| = \kappa$. Now, if $T = \{\kappa\} \in \mathcal{R}_f$, it easily follows that $\Lambda^\infty_T(A)$ is stationary (in fact, it must contain all but an initial segment of κ). It follows that $T \in L^\infty_G \subseteq L^{\omega\cdot 2}_G$; so by Corollary 3.3, G has the GCCP.

In a similar vein, we have the next statement.

Corollary 3.4. *Any reduced group G with $L^\omega_G = L^{\omega\cdot 2}_G$ has the* GCCP.

Proof. Suppose G is such a group. If G is $\widetilde{p}^\omega$-bounded, then it is Σ-cyclic, and so it trivially has the GCCP. On the other hand, if it is $\widetilde{p}^\omega$-unbounded, then $0_\mathcal{R} \neq L^\omega_G = L^{\omega\cdot 2}_G$, and the result follows from Corollary 3.3. □

4. $\widetilde{p}^{\omega+1}$-bounded groups in V=L. Recall that a B/U-pair $E = (\mathcal{B},\mathcal{U})$ is *perfect* if $\mathcal{B}$ is closed under subgroups, direct sums and filtrations. Summarizing the results of [**19**], if $\lambda \leq \omega$, then there is a unique perfect class of length λ, namely the p^λ-bounded Σ-cyclics. On the other hand, if $\lambda > \omega$, then this no longer holds. However, we will always have $\widetilde{p}^\lambda \subseteq E \subseteq p^\lambda_*$, so that $\widetilde{p}^\lambda$ and p^λ_* are the unique smallest and largest perfect B/U-pairs of length λ. The purpose of this section is to show that, even for $\lambda = \omega + 1$, in the context of the constructible universe (V=L), there are some $\widetilde{p}^{\omega+1}$-bounded groups that are not well behaved. In the process, we answer a couple of open problems and conjectures stated in [**19**].

We will use the language of *valuated vector space* (see [**6**]). For example, if G is a group and V is a subgroup of $G[p]$, then we have a valuation $|x|_V = |x|_G$, and if α is an ordinal, we define $V(\alpha) = \{x \in V : |x|_G \geq \alpha\}$. The category of valuated vector spaces has direct sums, its morphisms are non value-reducing homomorphisms and its isomorphisms are the bijective homomorphisms that preserve values; we will refer to these as *isometries*. A valuated vector space is *free* if it is isometric to a valuated direct sum of cyclic valuated vector spaces; in particular, any countable valuated vector space will actually be free.

The valuated vector space V will be said to be *essentially finitely indecomposable*, or efi, if it is not isometric to a valuated direct sum $F \oplus V'$, where F is free and unbounded (i.e., for all $m < \omega$, $F(m) \neq \{0\}$). So if G is a group and $G[p]$ is efi as a valuated vector space, then it readily follows that G is efi as a group (i.e., it does not have an unbounded Σ-cyclic summand).

The following proof is a fairly straightforward application of the diamond principle ($\diamondsuit$) that is valid in the constructible universe ($V = L$). It is simplified version of a construction for groups (as opposed to valuated vector spaces) contained in [**4**].

Theorem 4.1. ($V = L$) *Assuming the axiom of constructibility, there is a separable $\widetilde{p}^{\omega+1}$-bounded group G such that $G[p]$ is essentially finitely indecomposable* (*as a valuated vector space*).

Proof. Let $\{X_\alpha\}_{\alpha<\omega_1}$ be a smoothly ascending chain of countably infinite sets such that, for all $\alpha < \omega_1$, $X_{\alpha+1} - X_\alpha$ is countably infinite. Our objective is to construct G so that $G[p] = \cup_{\alpha<\omega_1} X_\alpha \overset{\text{def}}{=} X$; clearly, $|X| = \aleph_1$. We do this inductively by defining a smoothly ascending chain $\{G_\alpha\}_{\alpha<\omega_1}$ of countable groups such that:

(a) $G_\alpha[p] = X_\alpha$;

(b) G_α is separable, and hence Σ-cyclic;

(c) $G_{\alpha+1}/G_\alpha$ is unbounded (i.e., not p^j-bounded for any $j < \omega$) and $p^{\omega+1}$-bounded;

(d) if $\beta < \alpha$, then G_β is pure in G_α;

(e) if $\beta < \alpha$ and β is isolated, then G_β is a summand of G_α, so that G_α/G_β is an unbounded Σ-cyclic group.

Regarding the last condition, since G_β is pure in the countable Σ-cyclic group G_α, it will be a summand if and only if G_α/G_β is separable; if and only if G_β is closed (in the p-adic topology) in G_α; if and only if X_β is closed in X_α.

Suppose first that α is a limit and we have defined G_β for all $\beta < \alpha$; so we must clearly set $G_\alpha = \cup_{\beta<\alpha} G_\beta$. To verify that our conditions continue to hold up to α, certainly (a) is trivial. For all $\beta < \alpha$, it is easy to check that G_β remains pure in G_α, so that (d) holds, and we already know that each $G_{\beta+1}/G_\beta$ is unbounded and $p^{\omega+1}$-bounded; i.e., (c) is valid. Since each G_β is separable (and pure) in G_α, it easily follows that G_α is also separable; and since it is countable, (b) must hold, as well. As for (e), let $\{\beta_k\}_{k<\omega}$ be a strictly ascending sequence of isolated ordinals, starting at $\beta_0 = \beta$, with limit α. It follows that each G_{β_k} is a summand of $G_{\beta_{k+1}}$, so that G_β is a summand of G_α.

We now need to show how to define $G_{\alpha+1}$ once we have constructed G_α. In fact, we will need to be a bit more careful, adding yet one more condition because of our use of $\diamondsuit$. Let $\{Y_\alpha, Z_\alpha\}_{\alpha<\omega_1}$ be a double $\diamondsuit$-system for X (i.e., for any pair of subsets $Y, Z \subseteq X$, if $C \subseteq \omega_1$ is closed and unbounded, then there is an $\alpha \in C$ such that $Y_\alpha = X_\alpha \cap Y$ and $Z_\alpha = X_\alpha \cap Z$–a double $\diamondsuit$-system for X can easily be constructed from a $\diamondsuit$-system for $X \times X$). We have two cases:

Case I. α is a limit ordinal, $Y_\alpha \oplus Z_\alpha$ is a valuated decomposition of $G_\alpha[p] = X_\alpha$, Y_α is unbounded and $Y_\alpha \subseteq X_\gamma$ for some $\gamma < \alpha$.

Let $\{\beta_k\}_{k<\omega}$ be a strictly increasing sequence of isolated ordinals with limit α such that $\gamma \le \beta_0$. Since, for all $k < \omega$, we have $X_{\beta_k} = Y_\alpha \oplus (X_{\beta_k} \cap Z_\alpha)$, and $X_{\beta_k} = G_{\beta_k}[p]$ is a valuated summand of $X_{\beta_{k+1}} = G_{\beta_{k+1}}[p]$, we can conclude that there are valuated decompositions $X_{\beta_{k+1}} = X_{\beta_k} \oplus Z'_k$, where each $Z'_k \subseteq X_{\beta_{k+1}} \cap Z_\alpha$ is unbounded. It follows that there are valuated decompositions

$$G_\alpha[p] = X_\alpha = Y_\alpha \oplus Z_\alpha = Y_\alpha \oplus (X_{\beta_0} \cap Z_\alpha) \oplus \left(\bigoplus_{j<\omega} Z'_j \right),$$

where $X_{\beta_k} = X_{\beta_0} \oplus (\bigoplus_{j<k} Z'_j)$.

In the torsion-completion $\overline{G}_\alpha$, let $y_\alpha \in \overline{Y}_\alpha - Y_\alpha$. Next, for each $j < \omega$, choose a non-zero $z_j \in Z'_j$ such that $|z_j|_{G_\alpha}$ is a strictly increasing sequence. Now, let

$$x_\alpha = y_\alpha + \sum_{j<\omega} z_j \in \overline{G}_\alpha[p],$$

and define $W_\alpha = X_\alpha + \langle x_\alpha \rangle \subseteq \overline{G}_\alpha[p]$.

The projection $X_\alpha \to Y_\alpha$ extends to a valuated homomorphism $W_\alpha \to Y_\alpha + \langle y_\alpha \rangle \subseteq \overline{G}_\alpha[p]$ whose kernel is Z_α; so Z_α is closed in W_α. Similarly, if $k < \omega$, then the projection $X \to \bigoplus_{k \le j < \omega} Z'_j$ extends to a valuated homomorphism

$$W_\alpha \longrightarrow \bigoplus_{k \le j < \omega} Z'_j + \left\langle \sum_{k \le j < \omega} z_j \right\rangle \subseteq \overline{G}_\alpha[p],$$

with kernel X_{β_k}; so again, X_{β_k} will be closed in W_α. In addition, if $\beta < \alpha$ is isolated, then there is a $k < \omega$ such that $\beta < \beta_k$. By induction, X_β will be closed in X_{β_k}, so that, in fact, X_β will be closed in W_α whenever $\beta < \alpha$ is isolated.

Let H_α be a pure subgroup of $\overline{G}_\alpha$ containing G_α so that $H_\alpha[p] = W_\alpha = X_\alpha + \langle x_\alpha \rangle$. Since H_α is countable and separable, it also is Σ-cyclic.

Let $G_{\alpha+1} = H_\alpha \oplus pG_\alpha$; in particular, we think of H_α as a subgroup of $G_{\alpha+1}$. Define a map $\phi : G_\alpha \to G_{\alpha+1}$ by $\phi(x) = (x, px)$. For all $x \in G_\alpha[p] = X_\alpha$, we have $|x|_{G_\alpha} = |x|_{H_\alpha} = |\phi(x)|_{G_{\alpha+1}}$; so ϕ is an embedding and $\phi(G_\alpha)$ is pure in $G_{\alpha+1}$. We identify G_α with $\phi(G_\alpha)$; so we are no longer thinking of G_α as a subgroup of H_α, but rather as a subgroup of $G_{\alpha+1} = H_\alpha \oplus pG_\alpha$. However, the socle $G_\alpha[p]$ remains the same, namely, X_α, which is a dense subsocle of $W_\alpha = H_\alpha[p]$ of corank 1.

If we identify $G_{\alpha+1}[p]$ with $X_{\alpha+1}$, then we need to check (b)–(e) continue to hold for $\alpha + 1$. First, since H_α and pG_α are countable and Σ-cyclic, so is $G_{\alpha+1}$, i.e., (b) is satisfied.

Turning to (c), it is easy to check that

$$G_{\alpha+1}/G_\alpha = (H_\alpha + G_\alpha)/G_\alpha \cong H_\alpha/(H_\alpha \cap G_\alpha) = H_\alpha/X_\alpha,$$

is unbounded and $p^{\omega+1}$-bounded, as required.

To verify (d), we already know that G_α is pure in $G_{\alpha+1}$. And, if $\beta < \alpha$, then G_β is pure in G_α, so that it is also pure in $G_{\alpha+1}$, as stated.

Considering (e), if $\beta < \alpha + 1$ is isolated, then in fact, we must have $\beta < \alpha$. We already know that $X_\beta = G_\beta[p]$ is closed in $W_\alpha = H_\alpha[p]$. Since $H_\alpha[p]$ is a valuated summand of $G_{\alpha+1}[p]$, it follows that X_β is also closed in $G_{\alpha+1}[p]$, as required.

Case II. The first case does not apply. If this occurs, we just set $G_{\alpha+1} = G_\alpha \oplus K_\alpha$, where K_α is an unbounded countable Σ-cyclic group. If we identify $G_{\alpha+1}[p]$ with $X_{\alpha+1}$, it is easy to check that (b)–(e) continue to hold.

We now define $G = \cup_{\alpha<\omega_1} G_\alpha$; we need to verify that $X = G[p]$ is efi. So, assume that there is a valuated decomposition $G[p] = Y \oplus Z$, where Y is countable and unbounded. Choose $\gamma < \omega_1$ such that $Y \subseteq X_\gamma$. Since the limit ordinals greater than γ form a closed unbounded subset of ω_1, it follows that there is a limit ordinal $\alpha > \gamma$ such that $Y = Y \cap X_\alpha = Y_\alpha$ and $Z \cap X_\alpha = Z_\alpha$. This shows that there is a valuated decomposition $X_\alpha = Y_\alpha \oplus Z_\alpha$, so that we are in the above Case I.

Using the notation there, we will have a valuated decomposition $W_\alpha = Y \oplus (Z \cap W_\alpha)$, where $Z_\alpha \subseteq Z \cap W_\alpha$. We observed before that Z_α is closed in W_α, so it is also closed in $Z \cap W_\alpha$. Since Y is obviously closed in Y, we can conclude that $X_\alpha = Y \oplus Z_\alpha$ is closed in $W_\alpha = Y \oplus (Z \cap W_\alpha)$. However, as this is obviously not the case ($x_\alpha \in W_\alpha$ is in $\overline{X}_\alpha - X_\alpha$), we can conclude that $G[p]$ cannot have a countable unbounded valuated summand such as Y. So $G[p]$ is efi.

We will therefore be done if we can show that G is $\widetilde{p}^{\omega+1}$-bounded. Assume otherwise, and let $T \in L_G^{\omega+1}$ be some element minimal under inclusion. Since $p^{\omega+1}G = \{0\}$, we can conclude that $T \neq \emptyset$; and since $|G| = \aleph_1$, by [**19,** Lemma 2.1 (g)] we can conclude $T = \{\aleph_1\}$. For every $i < \aleph_1$, $T_i = \emptyset$, which implies that $\Upsilon_T^{\omega+1}(G) = \emptyset$. Therefore, we can conclude that

$$\Lambda_T^{\omega+1}(G) = \{i < \omega_1 : T_i \in L_{G/G_i}^{\omega+1}\} = \{i < \omega_1 : p^{\omega+1}(G/G_i) \neq 0\}$$

is stationary in $\aleph_1$. However, if $\gamma > i$, then by (e) there is a decomposition

$$G_\gamma/G_i \cong (G_\gamma/G_{i+1}) \oplus (G_{i+1}/G_i).$$

The first term in the above is actually Σ-cyclic; and by (c), the second is $p^{\omega+1}$-bounded. Considering this condition for all $\gamma > i$ implies that G/G_i is always $p^{\omega+1}$-bounded. Therefore, $\Lambda_T^{\omega+1}(G)$ is actually empty; and this contradiction completes the argument. $\square$

Recall from [**14**] that a group G is *cothin* if, whenever $A = \bigoplus_{i\in I} A_i$ is a direct sum, then a homomorphism $G \to A$ is small if and only if for each $i \in I$, the composition $G \to A \to A_i$ is small. It was shown that, if G is cothin, then it is thick ([**14,** Proposition 13]), and if G is a separable group such that $G[p]$ is efi, then it is cothin ([**14,** Proposition 18]). This immediately implies the following.

Corollary 4.1. ($V = L$) *Assuming the axiom of constructibility, there is an unbounded, $\widetilde{p}^{\omega+1}$-bounded separable group that is cothin, and hence thick.*

We next apply these ideas to p_*^{∞}-projectives.

Corollary 4.2. *If G is an unbounded separable group that is cothin, then G is not p_*^{∞}-projective.*

Proof. If G is p_*^{∞}-projective, then there is an embedding $G \subseteq H$ where H is a reduced simply presented group. It follows from ([**14,** Corollary 11]) that this inclusion must be a small homomorphism, which cannot be if G is unbounded. □

Putting together Corollaries 4.1 and 4.2, we have the following statement.

Corollary 4.3. ($V = L$) *Assuming the axiom of constructibility, there is a separable $\widetilde{p}^{\omega+1}$-bounded group that is not p_*^{∞}-projective.*

To repeat, in [**4**], a group was constructed similar to the one constructed in Theorem 4.1. In this previous paper, it was noted that the resulting group was not $p^{\omega+1}$-projective. In contrast, Corollary 4.3 states that our group not only fails to be $p^{\omega+1}$-projective, it actually fails to be p^{λ}-projective for *any* ordinal λ; and, further, one cannot even *embed* it in such a p^{λ}-projective.

This discussion resolves a couple of the Questions/Conjectures of [**19**]. For example, if α is an ordinal, let $\mathcal{C}^{\alpha}$ be the class of groups that can be embedded in a direct sum $\bigoplus_{i \in I} H_i$, where each H_i is p^{α}-bounded and has a countable subgroup C_i such that H_i/C_i is p_*^{α}-projective. Conjecture 18 states that a group G is $\widetilde{p}^{\alpha}$-bounded if and only if it is in $\mathcal{C}^\alpha$. By ([**19,** Proposition 3.11]), any group in $\mathcal{C}^\alpha$ is $\widetilde{p}^\alpha$-bounded. On the other hand, for $\alpha = \omega + 1$, the reverse containment does not hold in $V = L$.

Corollary 4.4. ($V = L$) *Assuming the axiom of constructibility, there is a separable $\widetilde{p}^{\omega+1}$-bounded group that is not in $\mathcal{C}^{\omega+1}$.*

Proof. The class of p_*^∞-projective groups is additively bounded, closed under extension (i.e., if X is a subgroup of Y and both X and Y/X are p_*^∞-projective, then so is Y), and it contains the reduced countable groups, as well as the $p^{\omega+1}$-projective groups. This means that every group in $\mathcal{C}^{\omega+1}$ is p_*^∞-projective. So, the group from Theorem 4.1 satisfies our requirements. □

In [**19**], Question 19 asks if a group is $\widetilde{p}^\infty$-bounded if and only if it is p_*^∞-projective. Since every p_*^∞-projective group is $\widetilde{p}^\infty$-bounded, this question is restated as Conjecture 20, which is answered in the constructible universe by the following.

Corollary 4.5. ($V = L$) *Assuming the axiom of constructibility, there is a $\widetilde{p}^\infty$-bounded group that is not p_*^∞-projective.*

Proof. Since any group that is $\widetilde{p}^{\omega+1}$-bounded is clearly $\widetilde{p}^\infty$-bounded, the group from Theorem 4.1 again satisfies our requirements. □

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