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MUHLY LOCAL DOMAINS AND ZARISKI'S THEORY OF COMPLETE IDEALS

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ABSTRACT. Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain, that is, a two-dimensional integrally closed Noetherian local domain with algebraically closed residue field and with the associated graded ring $\mathrm{gr}_{\mathfrak{M}} R$ an integrally closed domain. In this paper we show that a number of fundamental results of Zariski's theory of complete ideals in two-dimensional regular local rings are not necessarily valid in R. However, if the associated graded ring $\mathrm{gr}_{\mathfrak{M}} R$ satisfies an additional assumption as in work of Muhly and Sakuma, then we are able to show that "any product of contracted ideals is contracted" holds in R if and only if R has minimal multiplicity.

1. Introduction. The theory of complete (i.e., integrally closed) ideals in two-dimensional regular local rings has been founded and developed by Zariski. In [**26**, Appendix 5] Zariski has proved a number of fundamental results, including the following:

(1) The product of contracted ideals is contracted.

(2) The product of complete ideals is complete.

(3) The transform of a simple complete ideal is a simple complete ideal.

(4) Every complete ideal factors uniquely, up to order, as a product of simple complete ideals.

(5) There exists a one-to-one correspondence between the set of simple complete ideals and the set of prime divisors.

Here "ideal" will always mean "$\mathfrak{M}$-primary ideal," unless specified otherwise. Further explanation of terminology can be found in Section 2.

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For more information about Zariski's theory and its further developments, the reader is referred to Swanson and Huneke [**24**, Chapter 14], Huneke [**9**], Lipman [**12, 13**] and Noh [**17**].

In the early 1960's Muhly (jointly with Sakuma) has attempted to extend some of Zariski's results to nonregular two-dimensional local domains (see [**14–16**]).

More precisely, Muhly considered two-dimensional normal local domains $(R, \mathfrak{M})$ with infinite residue field and with the associated graded ring $\mathrm{gr}_{\mathfrak{M}} R$ an integrally closed domain. Actually, Muhly imposed one further condition on $\mathrm{gr}_{\mathfrak{M}} R$ in order to ensure, for example, that the transform of any complete ideal is again complete. This additional condition, denoted "condition (MS)" in this paper, is explained in Section 4.

This paper will be concerned with two-dimensional normal local domains $(R, \mathfrak{M})$ with algebraically closed residue field and with the associated graded ring $\mathrm{gr}_{\mathfrak{M}} R$ an integrally closed domain. Because of Muhly's pioneering work mentioned above, such a local domain $(R, \mathfrak{M})$ will be called a *two-dimensional Muhly local domain* throughout this paper.

The aim of this paper is twofold. Firstly, to show that properties (1)–(5) do not necessarily hold in a two-dimensional Muhly local domain $(R, \mathfrak{M})$. (It will turn out that (3)–(5) are not necessarily valid, even if R has minimal multiplicity.)

Secondly, to prove that in a two-dimensional Muhly local domain $(R, \mathfrak{M})$, satisfying the supplementary condition (MS), the property that any product of contracted ideals is contracted holds if and only if $(R, \mathfrak{M})$ has minimal multiplicity.

2. Preliminaries. Throughout this paper, a Noetherian local ring R with maximal ideal $\mathfrak{M}$ will be denoted by $(R, \mathfrak{M})$. Let I be an ideal of R. An element x of R is called integral over I if $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ for some elements $a_i \in I^i$ with $i = 1, 2, \ldots, n$. The set $\overline{I}$ of all elements of R that are integral over I is an ideal of R. This ideal $\overline{I}$ is called the integral closure of I and I is called *integrally closed* or *complete* if $\overline{I} = I$. An ideal I is said to be *simple* if it cannot be factored as a product of proper ideals of R. An ideal J contained in I is called a *reduction* of I if $JI^n = I^{n+1}$ for some $n \in \mathbf{N}$. A reduction J of I is called a *minimal reduction* of I if no ideal properly contained in J is a reduction of I. It can be shown that an ideal $J \subseteq I$ is a reduction of I if and only if $I \subseteq \overline{J}$ (see [**24**, page 6, Corollary 1.2.5]). From now on, let us suppose that the Noetherian local ring $(R, \mathfrak{M})$ is an integral domain.

Let I be an $\mathfrak{M}$-primary ideal of R. Then Rees has associated with I an integer valued function d_I on $\mathfrak{M}\backslash\{0\}$ defined by

$$d_I(x) = e\bigg(\frac{I + xR}{xR}\bigg),$$

where $e((I + xR)/xR)$ denotes the multiplicity of $(I + xR)/(xR)$. This function d_I is called the *degree function* defined by I.

Following Rees and Sharp in [**21**, page 454], we define a *prime divisor* v of R as a discrete valuation v of the quotient field K of R with value group $\mathbf{Z}$, whose valuation ring $(V, \mathfrak{M}_V)$ dominates $(R, \mathfrak{M})$ (i.e., $V \supset R$ and $\mathfrak{M}_V \cap R = \mathfrak{M}$) and such that the transcendence degree $\operatorname{trdeg}_{R/\mathfrak{M}} V/\mathfrak{M}_V = \dim R - 1$.

It is shown in [**20**] that, with every prime divisor v of R there is associated a non-negative integer $d(I, v)$, with $d(I, v) = 0$ for all except finitely many prime divisors, such that

$$d_I(x) = \sum_v d(I, v) v(x),$$

for all $0 \neq x \in \mathfrak{M}$ and the sum is over all the prime divisors of R.

In [**21**] Rees and Sharp have proved that the integers $d(I, v)$ occurring in the sum above, are uniquely determined and they are called the *degree function coefficients* of I.

If R is analytically unramified, then $d(I, v) \neq 0$ for each prime divisor v of R that is a Rees valuation of I, while $d(I, v') = 0$ for all other prime divisors v' of R (see [**20**, Theorem 2.3]). If, in addition, R is normal and quasi-unmixed, then all the Rees valuations of I are prime divisors of R (see [**23**]), and thus $d(I, v) \neq 0$ if and only if v is a Rees valuation of I.

The reader is referred to [**23**, page 437] or [**24**, Chapter 10] for the definition of the Rees valuation rings and the Rees valuations of an ideal I of a Noetherian local domain. The set of Rees valuations of an ideal I will be denoted by $T(I)$ in the rest of this paper.

Next let us recall some background material concerning two-dimensional Muhly local domains. Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain (for the definition, see the Introduction). The fact that the associated graded ring $\mathrm{gr}_{\mathfrak{M}} R$ is an integral domain implies that the $\mathfrak{M}$-adic order function ord_R is a valuation. It will mostly be denoted by $v_{\mathfrak{M}}$ in this paper. Further, all powers of $\mathfrak{M}$ are integrally closed, in other words, $\mathfrak{M}$ is a normal ideal. Since the integral domain $\mathrm{gr}_{\mathfrak{M}} R$ is integrally closed, it follows that the *blowup* $\mathrm{Bl}_{\mathfrak{M}} R$ *of* R *at* $\mathfrak{M}$ is a desingularization of R. A *desingularization* X of R is a regular scheme X together with a proper birational morphism $f : X \to \mathrm{Spec}\,(R)$. Here by $\mathrm{Bl}_{\mathfrak{M}} R$ we mean the following collection of local rings lying between R and its quotient field K:

$$\mathrm{Bl}_{\mathfrak{M}} R = \left\{ R\left[\frac{\mathfrak{M}}{x}\right]_P \;\Big|\; 0 \neq x \in \mathfrak{M}, P \in \mathrm{Spec}\left(R\left[\frac{\mathfrak{M}}{x}\right]\right) \right\}.$$

Any local ring of $\mathrm{Bl}_{\mathfrak{M}} R$ of the form

$$R\left[\frac{\mathfrak{M}}{x}\right]_N$$

with N a maximal ideal of $R[\mathfrak{M}/x]$ such that $N \cap R = \mathfrak{M}$, is called a *first* (or an *immediate*) *quadratic transform* of $(R, \mathfrak{M})$. Since $\mathrm{Bl}_{\mathfrak{M}} R$ is a desingularization of $(R, \mathfrak{M})$, all immediate quadratic transforms of $(R, \mathfrak{M})$ are two-dimensional regular local rings.

Let $(R', \mathfrak{M}')$ be such an immediate quadratic transform of R, i.e.,

$$R' = R\left[\frac{\mathfrak{M}}{x}\right]_N$$

for some $x \in \mathfrak{M} \setminus \mathfrak{M}^2$ and some maximal ideal N of $R[\mathfrak{M}/x]$ lying over $\mathfrak{M}$ (i.e., $N \cap R = \mathfrak{M}$). We then say that $(R', \mathfrak{M}')$ is lying on the chart $R[\mathfrak{M}/x]$. The $\mathfrak{M}'$-adic order function $\mathrm{ord}_{R'}$ is a prime divisor of R and it is called an *immediate prime divisor* of R. If I is an $\mathfrak{M}$-primary ideal of R with $\mathrm{ord}_R(I) = r$, then we have in $R[\mathfrak{M}/x]$ that

$$IR\left[\frac{\mathfrak{M}}{x}\right] = x^r J$$

with J an ideal of $R[\mathfrak{M}/x]$. This ideal J is called the *transform of I in $R[\mathfrak{M}/x]$*. Note that J cannot be contained in any height one prime ideal of $R[\mathfrak{M}/x]$. So, if $J \neq R[\mathfrak{M}/x]$, then J can only be contained in maximal ideals of $R[\mathfrak{M}/x]$ lying over $\mathfrak{M}$. Suppose N is such a maximal ideal. Then in R' we have

$$IR\left[\frac{\mathfrak{M}}{x}\right]_N = x^r J_N.$$

The ideal $I := J_N$ is an $\mathfrak{M}'$-primary ideal of R' and it is called the *transform of I in R'*. Since $I' \neq R'$, the immediate quadratic transform $(R', \mathfrak{M}')$ of $(R, \mathfrak{M})$ is called an *immediate base point of I*. It has been shown in [**5**, Proposition 1.1] that the immediate base points of I are among the centers of the Rees valuations of I on $\mathrm{Bl}_{\mathfrak{M}}R$. Since the residue field $R/\mathfrak{M}$ is infinite, there always exists an element $x \in \mathfrak{M} \setminus \mathfrak{M}^2$ such that $R[\mathfrak{M}/x]$ is contained in every Rees valuation ring of I. This implies that *all* immediate base points of I are lying on the chart $R[\mathfrak{M}/x]$.

If $x \in \mathfrak{M} \setminus \mathfrak{M}^2$ and I is an $\mathfrak{M}$-primary ideal of R such that

$$IR\left[\frac{\mathfrak{M}}{x}\right] \cap R = I,$$

then I is said to be *contracted from* $R[\mathfrak{M}/x]$. Similarly, if $R' = R[\mathfrak{M}/x]_N$ is an immediate quadratic transform of R and if $IR' \cap R = I$, then we will say that I *is contracted from* R'. An $\mathfrak{M}$-primary ideal I of $(R, \mathfrak{M})$ is said to be *contracted from the blowup* $\mathrm{Bl}_{\mathfrak{M}}R$ if

$$I = \bigcap_{S \in \mathrm{Bl}_{\mathfrak{M}}R} IS \cap R.$$

Observe that this is equivalent to saying that I is contracted from $R[\mathfrak{M}/x]$, where $x \in \mathfrak{M} \setminus \mathfrak{M}^2$ is any element, such that all immediate base points of I are lying on $R[\mathfrak{M}/x]$.

If I is a *complete* $\mathfrak{M}$-primary ideal of $(R, \mathfrak{M})$, then I is contracted from $\mathrm{Bl}_{\mathfrak{M}}R$. This follows from the fact that $\overline{I} = \bigcap_{V \in T(I)} IV \cap R$ (see [**24**, page 188, Discussion 10.1.3]).

In this paper, a two-dimensional Muhly local domain $(R, \mathfrak{M})$ is said to satisfy property (1) from the Introduction (i.e., any product of contracted ideals is contracted) if, for any $\mathfrak{M}$-primary ideals I and J contracted from $\mathrm{Bl}_{\mathfrak{M}}R$, their product IJ is also contracted from $\mathrm{Bl}_{\mathfrak{M}}R$. (Equivalently, if I and J are contracted from $R[\mathfrak{M}/x]$, where $x \in \mathfrak{M} \setminus \mathfrak{M}^2$ is such that all immediate base points of I and J are lying on $R[\mathfrak{M}/x]$, then IJ is also contracted from $R[\mathfrak{M}/x]$).

3. Minimal multiplicity. Let $(R, \mathfrak{M})$ be a two dimensional Cohen-Macaulay local domain (with infinite residue field). By Abhyankar's inequality (see [**1**, page 1073]), we have that

$$\operatorname{emb}\dim R \leq e(R) + 1.$$

If equality holds, then $(R, \mathfrak{M})$ is said to have *minimal multiplicity.*

The first result in this section, which is an immediate consequence of the work of Lipman, Ooishi, Huneke and Sally, will give some characterizations of two-dimensional Muhly local domains having minimal multiplicity. To state this result, we need some further preparation.

Let $(R, \mathfrak{M})$ be a two-dimensional analytically unramified local domain (with infinite residue field). If I is an $\mathfrak{M}$-primary ideal of R, then length $(R/\overline{I^n})$ is finite for all $n \geq 1$. There exists a polynomial

$$\overline{P}_I(n) = \overline{e}_0(I)\binom{n+1}{2} - \overline{e}_1(I)\binom{n}{1} + \overline{e}_2(I),$$

with $\overline{e}_0(I)$, $\overline{e}_1(I)$, $\overline{e}_2(I)$ integers, such that

$$\operatorname{length}\left(\frac{R}{\overline{I^n}}\right) = \overline{P}_I(n)$$

for all n sufficiently large.

This polynomial $\overline{P}_I(n)$ is called the *normal Hilbert polynomial of I*. Its constant term $\overline{e}_2(I)$ is said to be the *normal genus of I* and is denoted by $\overline{g}(I)$ (see [**18**, page 54]).

If $(R, \mathfrak{M})$ is a two-dimensional analytically normal local domain, then there exists a desingularization X of R and the length of $H^1(X, \mathcal{O}_X)$, as an R-module, is independent of desingularizations (see [**11**, page 158] and [**18**, page 65]). The length of $H^1(X, \mathcal{O}_X)$ is called the *geometric genus of R*, and it is denoted by $p_g(R)$ (cf. [**18**, page 65]).

If $p_g(R) = 0$, then $(R, \mathfrak{M})$ is said to be a *rational singularity* (see [**19**, page 1373]).

Theorem 3.1. *Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain. Then the following assertions are equivalent*:

(i) *$(R, \mathfrak{M})$ has minimal multiplicity.*

(ii) *$(x, y)\mathfrak{M} = \mathfrak{M}^2$ for a (any) minimal reduction (x, y) of $\mathfrak{M}$.*

(iii) $\overline{g}(\mathfrak{M}) = 0$.

(iv) *$(R, \mathfrak{M})$ is a rational singularity.*

Proof. (i) ⇔ (ii). This follows from Sally [**22**, Theorem 1].

(ii) ⇔ (iii). To see this, observe that it follows from Theorem 4.5 (ii) and Remark 4.2 [**8**, pages 307, 308], that

$$\overline{g}(\mathfrak{M}) = 0$$

is equivalent to

$$(x, y)\mathfrak{M}^n = \mathfrak{M}^{n+1}$$

for all $n \geq 1$ and (x, y) a (any) minimal reduction of $\mathfrak{M}$.

(iii) ⇔ (iv). Since $(R, \mathfrak{M})$ is a two-dimensional Muhly local domain, we know that the blowup X of R at $\mathfrak{M}$ is a desingularization of Spec (R). Thus, the geometric genus $p_g(R)$ of R is given by

$$p_g(R) = \text{length}\,(H^1(X, \mathcal{O}_X))$$

(cf. [**18**, page 65]). On the other hand, Theorem 3.1 [**18**, page 54] implies that the normal genus $\overline{g}(\mathfrak{M})$ of $\mathfrak{M}$ is also equal to length $(H^1(X, \mathcal{O}_X))$. So, here we have

$$\overline{g}(\mathfrak{M}) = p_g(R),$$

and this shows, that (iii) is equivalent to (iv). □

Corollary 3.2. *Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain having minimal multiplicity. Then $(R, \mathfrak{M})$ satisfies the property that any product of contracted ideals is contracted.*

Proof. This follows immediately from Theorem 3.1 above and Lipman's Theorem 7.2 [**10**, page 209]. □

Corollary 3.3. *Let* $(R, \mathfrak{M})$ *be a two-dimensional Muhly local domain. Then the following assertions are equivalent*:

(i) $(R, \mathfrak{M})$ *has minimal multiplicity.*

(ii) $(R, \mathfrak{M})$ *satisfies the property that any product of complete ideals is complete.*

Proof. (i) $\Rightarrow$ (ii). This follows from Theorem 3.1 above and Theorem (7.1) [**10**, page 209].

(ii) $\Rightarrow$(i). Since $(R, \mathfrak{M})$ is a two-dimensional analytically normal local domain with algebraically closed residue field, satisfying the property that any product of complete ideals is complete, it follows from Cutkosky [**2**, Theorem 1] that $(R, \mathfrak{M})$ is a rational singularity. Hence, by Theorem 3.1 ((iv) $\Leftrightarrow$ (i)), we have that $(R, \mathfrak{M})$ has minimal multiplicity. □

It follows from the preceding results that, if a two-dimensional Muhly local domain has minimal multiplicity, then it satisfies (1) and (2) (see Introduction) from Zariski's theory of complete ideals in two-dimensional regular local rings. The remaining properties (3)–(5), however, will not necessarily hold in such a Muhly local domain, as the example below will show.

Example 3.4. Let

$$R := \frac{k[X, Y, Z]_{(X,Y,Z)}}{(XY - Z^2)_{(X,Y,Z)}}$$

where k is an algebraically closed field, and X, Y and Z are indeterminates. Let x, y, z denote the images of X, Y, Z in R. Then

$$R = k[x, y, z]_{(x,y,z)} \quad \text{with } xy = z^2,$$

and

$$\mathfrak{M} = (x, y, z)$$

is the unique maximal ideal of R. By the Jacobian criterion it can be seen that $k[X,Y,Z]/(XY-Z^2)$ is an integrally closed domain.

It follows that $(R,\mathfrak{M})$ is a two-dimensional normal local domain (with algebraically closed residue field), and its associated graded ring $\mathrm{gr}_{\mathfrak{M}}(R) \cong k[X,Y,Z]/(XY-Z^2)$ is an integrally closed domain. It is clear that $\operatorname{embdim} R = e(R) + \dim R - 1$, so $(R,\mathfrak{M})$ has minimal multiplicity. Now we consider the immediate quadratic transform $(R',\mathfrak{M}')$ of $(R,\mathfrak{M})$ given by

$$R' = R\left[\frac{\mathfrak{M}}{x}\right]_M$$

with $M = (x,(y/x),(z/x))$. Then we have that $\mathfrak{M}' = (x,(z/x))R'$. The inverse transform I of $\mathfrak{M}'$ in R (which is by definition $x\mathfrak{M}' \cap R$) is the first neighborhood complete ideal of R corresponding to the immediate prime divisor $w := \mathrm{ord}_{R'}$ (see [**4**, Theorem 3.2]. The following facts concerning I have been shown in [**4**, Lemma 3.1 and Corollary 3.4]:

- $I = (x^2,y,z)R$.
- I is a simple complete $\mathfrak{M}$-primary ideal and $I \subset \mathfrak{M}$ are adjacent (i.e., length $(\mathfrak{M}/I) = 1$.)
- $(R',\mathfrak{M}')$ is the unique immediate base point of I.
- $\mathfrak{M}'$ is the transform of I in R' (thus, $I^{R'} = \mathfrak{M}'$).
- The set of Rees valuations $T(I) = \{v_{\mathfrak{M}}, w\}$.

Since the simple complete $\mathfrak{M}$-primary ideal I has two Rees valuations, Zariski's one-to-one correspondence (i.e., property (5)) does not hold in R. Further, since $(R,\mathfrak{M})$ is a two-dimensional rational singularity, R satisfies condition (N), i.e., for every prime divisor v of R there is an $\mathfrak{M}$-primary ideal J of R such that $T(J) = \{v\}$ (see [**7**, Corollary 3.11]). Moreover, the set of complete $\mathfrak{M}$-primary ideals having v as unique Rees valuation consists of the powers of an ideal A_v that is uniquely determined by v (see again [**7**, Corollary 3.11]). In [**4**, page 1150], it has been shown that

$$I^2 : \mathfrak{M}$$

is a simple, complete $\mathfrak{M}$-primary ideal such that

$$I^2 = \mathfrak{M}(I^2 : \mathfrak{M}),$$

implying that $T((I^2 : \mathfrak{M})) = \{w\}$ (by [**4**, Theorem 3.6 (ii)]). It follows that

$$I^2 : \mathfrak{M} = A_w,$$

where A_w is the complete $\mathfrak{M}$-primary ideal corresponding to w in the sense of Corollary 3.11 in [**7**] (see [**4**, page 1150]).

Since $(R, \mathfrak{M})$ is a two-dimensional rational singularity (thus any product of complete ideals is complete), I^2 is a complete $\mathfrak{M}$-primary ideal of R and $I^2 = \mathfrak{M}(I^2 : \mathfrak{M})$ shows that this complete ideal has two distinct factorizations in simple complete ideals. Hence, property (4) from Zariski's theory (cf. Introduction) does not hold in R.

Now we want to obtain an explicit description of Göhner's ideal A_w in terms of x, y and z. To do so, we first recall that A_w is the integral closure of the ideal $(x^3, y)R$, because $I^2 : \mathfrak{M} = \overline{(x^3, y)R}$ (see [**4**, page 1150]). Next we observe that $I\mathfrak{M} \subset A_w$, and thus we have the following chain of ideals

$$I\mathfrak{M} \subset A_w \subset (x^3, y, z) \subset I = (x^2, y, z).$$

Here the inclusion $A_w \subset (x^3, y, z)$ follows from the fact that A_w is the integral closure of (x^3, y) and that (x^3, y, z) is integrally closed (see Lemma 2.2 (ii) in [**3**]). This chain has the following properties:

- $(x^3, y, z) \subset I = (x^2, y, z)$ are adjacent, thus $\text{length}\,(I/(x^3, y, z)) = 1$.
- $\text{length}\,(I/I\mathfrak{M}) = 3$.
- $A_w \neq I\mathfrak{M}$ since A_w is simple.
- $A_w \neq (x^3, y, z)$ since $T(A_w) = \{w\}$ while $T((x^3, y, z)) = \{(v_{\mathfrak{M}}, w)\}$ (see [**5**, Proposition 3.13]).

It follows that $I\mathfrak{M} \subset A_w$ are adjacent. From the inclusions

$$I\mathfrak{M} \subset (x^3, y, xz) \subset A_w,$$

we can conclude that

$$A_w = (x^3, y, xz)$$

because $I\mathfrak{M} \subset A_w$ are adjacent and $(x^3, y, xz) \neq I\mathfrak{M}$. We now consider the transform of A_w in $R[\mathfrak{M}/x]$:

$$A_w R\left[\frac{\mathfrak{M}}{x}\right] = x\left(x^2, \frac{y}{x}, z\right).$$

Using the relation $y/x = (z/x)^2$, it is readily seen that

$$\left(x^2, \frac{y}{x}, z\right) = M^2$$

where $M = (x, (y/x), (z/x))$.

Hence, the transform $(x^2, (y/x), z)$ of the simple complete ideal A_w in $R[\mathfrak{M}/x]$ is not simple. This shows that property (3) from Zariski's theory (see Introduction) does not hold in R.

Finally, there is still another deviation from the two-dimensional regular case. In the regular case, any simple complete $\mathfrak{M}$-primary ideal $\mathfrak{q}$ has a unique Rees valuation v (cf. Zariski's one-to-one correspondence). So, $\mathfrak{q}$ has only one nonzero degree function coefficient, namely $d(\mathfrak{q}, v)$. In [**6**, Proposition 3.5, page 128], it has been proved that

$$d(\mathfrak{q}, v) = 1.$$

In this example, the simple complete $\mathfrak{M}$-primary ideal A_w has a unique Rees valuation w. But, in contrast to the regular case, its unique degree function coefficient $d(A_w, w) \neq 1$. To see this, we observe that $I^2 = \mathfrak{M}A_w$ implies that

$$d(I^2, w) = d(\mathfrak{M}A_w, w);$$

hence,

$$2d(I, w) = d(\mathfrak{M}, w) + d(A_w, w).$$

Now $d(I, w) = 1$, since $d(I, w) = d(I^{R'}, w)$ (see [**5**,Proposition 3.3]), and $d(I^{R'}, w) = 1$ since $I^{R'}$ is a simple complete $\mathfrak{M}'$-primary ideal in the two-dimensional regular local ring $(R', \mathfrak{M}')$ with $T(I^{R'}) = \{w\}$. Further, $d(\mathfrak{M}, w) = 0$ since $w \notin T(\mathfrak{M})$. Thus,

$$d(A_w, w) = 2,$$

although A_w is a simple complete $\mathfrak{M}$-primary ideal with unique Rees valuation w.

4. Condition (MS). As we have already seen in the previous section, the property "any product of contracted ideals is contracted" holds in any two-dimensional Muhly local domain $(R, \mathfrak{M})$ with minimal multiplicity. This does not necessarily hold if the multiplicity of $(R, \mathfrak{M})$ is not minimal, as the next example will show.

Example 4.1. Let

$$R := \frac{k[X, Y, Z]_{(X,Y,Z)}}{(X^3 - Y^3 - Z^3)_{(X,Y,Z)}}$$

with k an algebraically closed field of characteristic zero, and X, Y, Z are indeterminates. Let x, y, z denote the images of X, Y, Z in R. Then

$$R = k[x, y, z]_{(x,y,z)} \quad \text{with } z^3 = x^3 - y^3,$$

and the maximal ideal $\mathfrak{M} = (x, y, z)R$.

Using the Jacobian criterion for regularity, one can see that $\mathrm{gr}_{\mathfrak{M}} R = k[X, Y, Z]/(X^3 - Y^3 - Z^3)$ is an integrally closed domain by Serre's normality criterion. It follows that $(R, \mathfrak{M})$ is a two-dimensional Muhly local domain, and the multiplicity of $(R, \mathfrak{M})$ is not minimal (or equivalently, $(R, \mathfrak{M})$ is not a rational singularity).

Hence by [**2**, page 157, Theorem 1], we know that there must exist complete $\mathfrak{M}$-primary ideals I in R such that I^2 is *not* complete. So, there is some hope that there might exist a complete $\mathfrak{M}$-primary ideal I in R such that I^2 is *not* contracted. Below, we will give an example of such an ideal I. Actually, we will do a little more. We will consider a certain first neighborhood complete ideal I of R and prove that

$$I^2 R\left[\frac{\mathfrak{M}}{x}\right] \cap R = \overline{I^2},$$

where $R[\mathfrak{M}/x]$ is such that the unique immediate base point of I is lying on $R[\mathfrak{M}/x]$. Then we will show that I^2 is *not* contracted from $R[\mathfrak{M}/x]$ (thereby also showing that I^2 is *not* complete).

Let us consider the following $\mathfrak{M}$-primary ideal of R

$$I := (x^2, x - y, z).$$

Then I has just one immediate base point, denoted $(R', \mathfrak{M}')$, and it is lying on the chart $R[\mathfrak{M}/x]$. To see this, first observe that I

has no immediate base points on $R[\mathfrak{M}/(x-y)]$ and $R[\mathfrak{M}/z]$, since $IR[\mathfrak{M}/(x-y)]$ and $IR[\mathfrak{M}/z]$ are principal ideals. Hence, I can have immediate base points only on $R[\mathfrak{M}/x]$. In $R[\mathfrak{M}/x]$, we have

$$IR\left[\frac{\mathfrak{M}}{x}\right] = xM,$$

where $M = (x, ((x-y)/x), (z/x))$. Since $x \notin \operatorname{rad}(x-y, z)$, the ideal M is a maximal ideal of $R[\mathfrak{M}/x]$ lying over $\mathfrak{M}$ (see [**3**, page 904]). It follows that I has a unique immediate base point, namely,

$$R' := R\left[\frac{\mathfrak{M}}{x}\right]_M$$

and

$$\mathfrak{M}' = \left(x, \frac{z}{x}\right)R'.$$

Since $I \subset \mathfrak{M}$ are adjacent and $IR' \cap R \neq \mathfrak{M}$, we have that

$$I = x\mathfrak{M}' \cap R.$$

This shows that I is complete, and it also shows that I is the inverse transform of $\mathfrak{M}'$ in R. So, I is a first neighborhood complete $\mathfrak{M}$-primary ideal of R (see [**4**, Theorem 3.2]).

Now, I being a first neighborhood complete $\mathfrak{M}$-primary ideal of a two-dimensional Muhly local domain that is not regular, we have that

$$T(I) = \{v_{\mathfrak{M}}, w\},$$

where $v_{\mathfrak{M}} = \operatorname{ord}_R$-valuation and $w = \operatorname{ord}_{R'}$-valuation (see [**4**, Lemma 3.1 and Theorem 3.3]). Using the fact that $T(I^2) = \{v_{\mathfrak{M}}, w\}$, we can prove our claim that

$$I^2R\left[\frac{\mathfrak{M}}{x}\right] \cap R = \overline{I^2}.$$

To do so, we first observe that

$$I^2R\left[\frac{\mathfrak{M}}{x}\right] = x^2M^2,$$

and thus

$$I^2R\left[\frac{\mathfrak{M}}{x}\right]\cap R = I^2R\left[\frac{\mathfrak{M}}{x}\right]_M\cap R = I^2R'\cap R.$$

Since $I^2R' = x^2\mathfrak{M}'^2$ and $(R',\mathfrak{M}')$ is a two-dimensional regular local ring, it follows that I^2R' is complete. This implies that

$$\overline{I^2}\subseteq I^2R'\cap R = I^2R\left[\frac{\mathfrak{M}}{x}\right]\cap R.$$

Since $T(I^2) = \{v_{\mathfrak{M}}, w\}$ with $v_{\mathfrak{M}} = \mathrm{ord}_R$-valuation and $w = \mathrm{ord}_{R'}$-valuation, $R[\mathfrak{M}/x]$ is contained in all Rees valuation rings of I^2. Hence $I^2R[\mathfrak{M}/x]\cap R\subseteq\overline{I^2}$. Thus $I^2R[\mathfrak{M}/x]\cap R = \overline{I^2}$, and this proves our claim.

Finally, we now will prove that I^2 is not contracted from $R[\mathfrak{M}/x]$ (and hence that I^2 is not complete). To do this, it is sufficient to find an element of $I^2\mathfrak{M}/x$ that is not in I^2. To this end we consider the element $\alpha := x(x-y)$ of the ring R. We have that $\alpha\in I^2\mathfrak{M}/x$, since

$$x\alpha = x^2(x-y) = \left(\frac{2}{3}x+\frac{1}{3}y\right)(x-y)^2+\left(\frac{1}{3}z\right)z^2.$$

On the other hand, $\alpha\notin I^2$. Suppose not. Then $\alpha\in I^2$ implies that in the polynomial ring $k[X,Y,Z]$ we have that

$$X(X-Y)\in(X-Y,Z)^2.$$

It follows that $X-Y\in(X-Y,Z)^2$, since $(X-Y,Z)^2$ is $(X-Y,Z)$-primary. Hence, in the ring $(R,\mathfrak{M})$ we have that $x-y\in\mathfrak{M}^2$, contradicting the fact that $x-y$ is an element of a minimal ideal basis of $\mathfrak{M}$. So I^2 is not contracted from $R[\mathfrak{M}/x]$ (and thus I^2 is not complete).

The discussions in the above example, combined with those in Section 3, seem to suggest that there might be some connection between the assertions "R satisfies the property that any product of contracted ideals is contracted" and "R has minimal multiplicity." In the next result we will prove that these two assertions are in fact equivalent, provided $(R, \mathfrak{M})$ satisfies an additional condition occurring in work of Muhly and Sakuma. So let us begin by introducing this condition.

Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain. Muhly and Sakuma [**15**, page 211] make the following assumption concerning the associated graded ring $\mathrm{gr}_{\mathfrak{M}}R$: *If α and β are two homogeneous elements of $\mathrm{gr}_{\mathfrak{M}}R$ of degree r and s respectively such that the ideal $(\alpha, \beta)\mathrm{gr}_{\mathfrak{M}}R$ is irrelevant, then the ideal $(\alpha, \beta)\mathrm{gr}_{\mathfrak{M}}R$ contains all homogeneous elements of $\mathrm{gr}_{\mathfrak{M}}R$ of degree not less than $r + s$.* We will call this assumption *condition* (*MS*) in the rest of this paper. By using this condition, Muhly and Sakuma have proved the following result (see [**15**, page 217, Corollary 1]):

Proposition 4.2. *Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain that satisfies condition* (MS). *For any complete $\mathfrak{M}$-primary ideal I of R, we have that $\mathfrak{M}^n I$ is complete when n is sufficiently large.*

It follows from this result that, in a two-dimensional Muhly local domain satisfying condition (MS), the transform of any complete $\mathfrak{M}$-primary ideal is complete. This is the reason why we impose $(R, \mathfrak{M})$ to satisfy condition (MS) in the next result.

Proposition 4.3. *Let $(R, \mathfrak{M})$ be a two-dimensional Muhly local domain that satisfies condition* (MS). *Then the following assertions are equivalent*:

(i) *R has minimal multiplicity.*

(ii) *R satisfies the property that any product of contracted ideals is contracted.*

Proof. (i) $\Rightarrow$ (ii). This implication holds because of Corollary 3.2.

(ii) $\Rightarrow$ (i). By Theorem 3.1, to prove that R has minimal multiplicity amounts to the same thing as proving that R is a rational singularity. To do this, it is sufficient to show that, for any complete $\mathfrak{M}$-primary ideal I of R, I^2 is also complete (see [**2**, Theorem 1]). This is immediate if I is a power of $\mathfrak{M}$. So we can assume that I is a complete $\mathfrak{M}$-primary ideal that is not a power of $\mathfrak{M}$. Since I is complete, I is contracted from $R[\mathfrak{M}/x]$, where $x \in \mathfrak{M} \setminus \mathfrak{M}^2$ is any element such that $R[\mathfrak{M}/x]$

is contained in all Rees valuation rings of I (see Section 2). Since we assume that (ii) holds, I^2 is also contracted from $R[\mathfrak{M}/x]$, i.e.,

$$I^2 R\left[\frac{\mathfrak{M}}{x}\right] \cap R = I^2.$$

In $R[\mathfrak{M}/x]$ we have that

$$IR\left[\frac{\mathfrak{M}}{x}\right] = x^r J,$$

where $r := \operatorname{ord}_R(I)$ and J is an ideal of $R[\mathfrak{M}/x]$ not contained in any height one prime ideal of $R[\mathfrak{M}/x]$. Hence, J can only be contained in a (finite) number of maximal ideals of $R[\mathfrak{M}/x]$ lying over $\mathfrak{M}$, say

$$M_1, \dots , M_n.$$

Note also that J is a complete ideal, since $(R, \mathfrak{M})$ satisfies condition (MS).

Localizing at M_i yields

$$IR\left[\frac{\mathfrak{M}}{x}\right]_{M_i} = x^r J_{M_i}.$$

So, in the immediate base point $R_i := R[\mathfrak{M}/x]_{M_i}$ of I, the transform I^{R_i} of I is complete. Since $(R_i, \mathfrak{M}_i)$ is a two-dimensional regular local ring, this implies that

$$(I^{R_i})^2 = (J^2)_{M_i}$$

is also complete.

Since

$$J^2 = \left((J^2)_{M_1} \cap R\left[\frac{\mathfrak{M}}{x}\right]\right) \cap \cdots \cap \left((J^2)_{M_n} \cap R\left[\frac{\mathfrak{M}}{x}\right]\right),$$

it follows that J^2 is complete. Thus,

$$I^2 R\left[\frac{\mathfrak{M}}{x}\right] = x^{2r} J^2$$

is a complete ideal. As $I^2R[\mathfrak{M}/x] \cap R = I^2$, this implies that I^2 is complete. □

A natural question concerning the relationship between "condition (MS)" and "minimal multiplicity" is the following.

Do there exist two-dimensional Muhly local domains that satisfy condition (MS) but fail to have minimal multiplicity?

In the next example we will see that the answer is yes.

Example 4.4. Let

$$R := \frac{\mathbf{C}\,[X,Y,Z]_{(X,Y,Z)}}{(XZ^2 - Y(Y-X)(Y-2X))_{(X,Y,Z)}},$$

where X, Y, Z are indeterminates over $\mathbf{C}$, and let $\mathfrak{M}$ denote the maximal ideal of R.

Then the associated graded ring $\mathrm{gr}_{\mathfrak{M}}R$ of the local ring $(R, \mathfrak{M})$ is given by

$$\mathrm{gr}_{\mathfrak{M}}R = \frac{\mathbf{C}\,[X,Y,Z]}{(XZ^2 - Y(Y-X)(Y-2X))}.$$

This is the homogeneous coordinate ring of an elliptic curve V defined in the projective plane $\mathbf{P}_2(\mathbf{C})$ by the equation

$$XZ^2 - Y(Y-X)(Y-2X) = 0.$$

The ring $\mathrm{gr}_{\mathfrak{M}}R$ is a two-dimensional integrally closed integral domain. Hence $(R, \mathfrak{M})$ is a two-dimensional Muhly local domain. Since V is an elliptic curve in $\mathbf{P}_2(\mathbf{C})$, the linear system cut out on V by the lines of $\mathbf{P}_2(\mathbf{C})$ is non-special (see [**25**, page 187]).

In [**15**, pages 211 and 212], Muhly and Sakuma have proved that this implies that condition (MS) is satisfied.

Since $\mathrm{emb\,dim}\, R = 3$, $e(R) = 3$ and $\dim R = 2$, we have

$$\mathrm{emb\,dim}\, R < e(R) + \dim R - 1,$$

i.e., the two-dimensional Muhly local domain $(R, \mathfrak{M})$ does not have minimal multiplicity.

Questions 4.5. (1) In a two-dimensional regular local ring $(R, \mathfrak{M})$ (with algebraically closed residue field), any simple complete $\mathfrak{M}$-primary ideal I has a unique Rees valuation, say w. Hence, $T(I) = \{w\}$ and I is said to be one-fibered. If $(R, \mathfrak{M})$ is a two-dimensional Muhly local domain (*not regular*), then Example 3.4 shows that there may exist simple complete $\mathfrak{M}$-primary ideals I in R such that $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ with $w \in T(I)$ (both possibilities, $T(I) = \{v_{\mathfrak{M}}, w\}$ and $T(I) = \{w\}$, do occur). We have called such ideals I quasi-one-fibered. Is every simple complete $\mathfrak{M}$-primary ideal in a two-dimensional Muhly local domain quasi-one-fibered?

(2) In a two-dimensional regular local ring $(R, \mathfrak{M})$ (with algebraically closed residue field), every simple complete $\mathfrak{M}$-primary ideal I has a unique nonzero degree function coefficient $d(I, w)$, where w denotes the unique Rees valuation of I. It can be shown that $d(I, w) = 1$ (see [**6**, Proposition 3.5]). Moreover, I has a unique immediate base point, say $(R', \mathfrak{M}')$, and the transform $I^{R'}$ of I in R' is again a simple complete ideal.

By contrast, if $(R, \mathfrak{M})$ is a two-dimensional Muhly local domain (*not regular*), then Example 3.4 shows that there may exist at least two types of simple complete $\mathfrak{M}$-primary ideals I in R:

- Type 1. $T(I) = \{v_{\mathfrak{M}}, w\}$ and $d(I, w) = 1$.
- Type 2. $T(I) = \{w\}$ and $d(I, w) \neq 1$.

Recall that, in Example 3.4, we have found examples of both of these types:

- $I = (x^2, y, z)R$ with $T(I) = \{v_{\mathfrak{M}}, w\}$ and $d(I, w) = 1$.
- $A_w = (x^3, y, xz)R$ with $T(A_w) = \{w\}$ and $d(A_w, w) = 2$.

The simple complete ideals I and A_w have the same unique immediate base point $(R', \mathfrak{M}')$ and $I^{R'}\mathfrak{M}'$ while $A_w^{R'} = \mathfrak{M}'^2$.

Thus, A_w is a simple complete $\mathfrak{M}$-primary ideal whose transform $A_w^{R'}$ in its unique immediate base point is *not* simple (observe that its degree function coefficient $d(A_w, w) \neq 1$). This raises the following question. Let I be a simple complete $\mathfrak{M}$-primary ideal in a two dimensional Muhly local domain $(R, \mathfrak{M})$, and suppose that I is *quasi-one-fibered* (i.e., $T(I) \subseteq \{v_{\mathfrak{M}}, w\}$ and $w \in T(I)$). Let $(R', \mathfrak{M}')$ denote the unique immediate base point of I.

If $d(I, w) = 1$, then does this imply that the transform $I^{R'}$ is again simple?

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