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PROJECTIVE STAR OPERATIONS ON POLYNOMIAL RINGS OVER A FIELD

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ABSTRACT. We consider the polynomial ring $S := K[X_0, \dots, X_n]$ over a field K and the rings $R_i := K[(X_0/X_i), \dots, (X_n/X_i)]$ for $0 \leq i \leq n$. We introduce the notion of a projective star operation on S and relate it to the classical star operations on the R_i's. We show that the projective Kronecker function ring $\mathrm{PKr}\,(S, \star)$ of S is the intersection of the Kronecker function rings $\mathrm{Kr}\,(R_i, \star_i)$, $0 \leq i \leq n$, where the $\star_i$'s are pairwise compatible e.a.b. star operations on the R_i's and $\star$ is a projective star operation on S built from the $\star_i$'s.

1. Introduction. Let R be an integral domain with quotient field F. Let $\mathfrak{F}(R)$ denote the set of nonzero fractional ideals of R. We recall that a star operation on R is defined as a mapping $\star : \mathfrak{F}(R) \to \mathfrak{F}(R)$, $I \mapsto I^\star$, such that for all $I, J \in \mathfrak{F}(R)$ and $x \in F \setminus \{0\}$:

($\star_1$) $R^\star = R$ and $(xI)^\star = xI^\star$;

($\star_2$) $I \subseteq I^\star$, and $I \subseteq J \Rightarrow I^\star \subseteq J^\star$;

($\star_3$) $I^{\star\star} := (I^\star)^\star = I^\star$.

A star operation $\star$ is called *endlich arithmetisch brauchbar* (in brief *e.a.b.*) if for any finitely generated $I, J, H \in \mathfrak{F}(R)$, $(IJ)^\star \subseteq (IH)^\star$ implies $J^\star \subseteq H^\star$. Given an e.a.b. star operation $\star$, the ring $\mathrm{Kr}\,(R, \star) := \{f/g : f, g \in R[X] \setminus \{0\}, C(f)^\star \subseteq C(g)^\star\} \cup \{0\}$, where $C(f)$ denotes the content of the polynomial $f(X)$, is called the *Kronecker function of R* with respect to $\star$. It is known that $\mathrm{Kr}\,(R, \star)$ is a Bézout domain (a domain for which every proper nonzero finitely generated ideal is principal) with quotient field $F(X)$ and such that $\mathrm{Kr}\,(R, \star) \cap F = R$ (for an overview on star operations and Kronecker function rings see [**7**, Section 32]).

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Recently, two generalizations of the concept of Kronecker function rings were proposed: one by Halter-Koch [**8**], and the other one by Fontana and Loper [**5, 6**]. Halter-Koch introduced the notion of F-function ring using only two axioms; the ring turns out to share many properties with the classical Kronecker function ring (the two axioms and properties of F-function rings can be found in Section 5 of this article), whilst the Fontana-Loper approach uses semistar operations (see [**14, 15**]).

The more general nature of F-function rings, compared to Kronecker function rings, is due to the fact that they do not necessarily depend on star operations. For instance, let $K \subseteq F$ be a field extension which is not algebraic and denote by $\Sigma(F/K)$ the set of valuation rings of F containing K. The F-function ring $\mathrm{Kr}\,(F/K) := \cap_{V\in\Sigma(F/K)} V^b$, studied in [**11**], where $V^b = V(X)$ is the Gauss (also called trivial) extension of V to $F(X)$, cannot be associated to a star operation and it is not a classical Kronecker function ring. Motivated by such examples, in this paper we rely on Halter-Koch's approach to overcome this restriction by introducing the notions of projective star operation and projective Kronecker function ring, which is an example of an F-function ring.

Let $S := K[X_0, \dots, X_n]$ be a polynomial ring over a field K. We consider the (relevant) coherent sheaves of ideals on $\mathrm{Proj}\,(S)$. The idea is to define a projective star operation as an application from the set of (relevant) coherent sheaves of ideals into itself, satisfying the same properties as classical star operations. But, motivated by the bijection between the set of coherent sheaves of ideals of $\mathrm{Proj}\,(S)$ and homogeneous saturated ideals of S (see [**3**, Exercises III-15 and III-16]), we restrict our attention to the set of homogeneous ideals of S. Generalities and basic properties of homogeneous and saturated ideals of S are provided in Section 2.

In Section 3, we define homogeneous star operations on S as maps from the set of homogeneous ideals of S into itself satisfying the properties of classical star operation $(\star_1)$, $(\star_2)$ and $(\star_3)$ above. We provide examples of classical star operations, such as the b-operation and the v-operation, that are homogeneous. On the other hand, for the $v(I)$ operation defined as $J^{v(I)} := (I : (I : J))$, it is possible to choose a suitable I so that $v(I)$ is not a homogeneous star operation (see Example 3.6).

A *projective star operation* is a homogeneous star operation $\star$ on S such that $sat \circ \star = \star$, where sat is the saturation. We observe that the v-operation and the composition $sat \circ b$ are projective star operations, but the b-operation is not necessarily a projective star operation (see Example 4.1).

We prove that each homogeneous (or projective) star operation $\star$ induces (by *dehomogenization*) a star operation $\star_i$ on $R_i := K[X_0/X_i, \ldots, X_n/X_i]$, for each $i = 0, \ldots, n$. Conversely, if we have a compatibility condition between star operations on different R_i's, then we can build from those (*homogenization* process) a homogeneous star operation which turns out to be projective. We then obtain a bijection between the $(n+1)$-tuples of compatible star operations, each defined on one of the R_i's and projective star operations on S:

$$\left\{ \begin{array}{c} \{\star_0, \ldots, \star_n\} \\ \star_i = \text{star operation on } R_i, \\ \star_i \text{compatible with } \star_j, \ \forall i, j \end{array} \right\} \\ \longleftrightarrow \left\{ \star = \text{projective star operation on } S \right\}.$$

In Section 4, we show that the b-operation (respectively, the v-operation) dehomogenizes at the integral closure of ideals (respectively, divisorial closure of ideals) on R_i for each $i = 0, \ldots, n$, and observe that the saturation sat is a projective star operation that dehomogenizes at the identity star operation.

In Section 5, we define an *e.a.b. projective star operation* as a projective star operation that dehomogenizes at e.a.b. star operations (e.a.b. in the classical sense) and we prove that an e.a.b. projective star operation on S satisfies the usual cancelation property. We can therefore associate to an e.a.b. projective star operation $\star$ a projective Kronecker function ring with respect to $\star$, denoted $\mathrm{PKr}\,(S, \star)$, which turns out to be an F-function ring (F is the quotient field of the rings R_i's) and has a natural interpretation in terms of valuations of F.

2. Preliminaries and notations. First of all, we fix the notation that will be used throughout. Let K be a field. Let $S := K[X_0, \ldots, X_n]$ be the polynomial ring in $n+1$ indeterminates over K. For i ranging from 0 to n, we shall denote by R_i the ring $K[X_0/X_i, \dots, X_n/X_i]$. All the domains R_i are integrally closed and have the same quotient field, namely , $F := K(X_0/X_i, \dots, X_n/X_i)$.

Let $f \in S$. The *dehomogenization* of f in R_i is the element

$$ {}^{a_i}f := f\left(\frac{X_0}{X_i}, \dots, 1, \dots, \frac{X_n}{X_i}\right) $$

of R_i. The application ^{a_i} is a ring homomorphism for each $i = 0, \dots, n$. Conversely, given an element g in R_i, its *homogenization* in S is the homogeneous element

$$ {}^{h}g := X_i^{n_i} g\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) $$

of S, where n_i is the degree of g.

Since S is graded, we can write each element $f \in S$ as $f = f_0 + \cdots + f_d$, with f_i homogeneous of degree i. An ideal I of S is homogeneous if it is generated by homogeneous elements, or equivalently, if for any $f \in I$, $f = f_0 + \cdots + f_d$, then each $f_i \in I$. A homogeneous prime ideal of S is a prime ideal of S which is also homogeneous.

A useful characterization of homogeneous prime ideals follows:

Proposition 2.1 [**2**, Exercise 2.15(c)]. *Let S be a $\mathbf{Z}$-graded ring. A homogeneous ideal P of S is prime if and only if whenever $fg \in P$ for homogeneous polynomials $f, g \in S$ then $f \in P$ or $g \in P$.*

Given a homogeneous ideal I of S, the *dehomogenization* of I (in R_i):

$$ {}^{a_i}I := \{{}^{a_i}f : f \text{ is homogeneous in } I\} $$

is an ideal of R_i.

Remark 2.2. We note that, using the fact that the operation ^{a_i} is a ring homomorphism for each $i = 0, \dots, n$, and I is generated by some homogeneous elements of I, it is clear that ${}^{a_i}I$ is just the same as the set $\{{}^{a_i}f : f \in I\}$. So, from now on, for a homogeneous ideal I of S, we will say $x \in {}^{a_i}I$ if and only if $x = {}^{a_i}f$ for some $f \in I$.

For each $i = 0, \dots, n$, the operation $I \mapsto {}^{a_i}I$ maps the set of all homogeneous ideals of S onto the set of all ideals of R_i and preserves inclusion and the usual ideal-theoretic operations: addition, multiplication, intersection, radical and colon [**17**, Theorem 18, Chapter VII, Section 5].

Given an ideal I of R_i, we denote by ${}^hI$ the homogeneous ideal of S which is generated by the set of homogeneous polynomials:

$$\left\{X_i^{mh} f : m \geq 0, f \in I\right\}.$$

The operation $I \mapsto {}^hI$ that assigns to each ideal of R_i a homogeneous ideal of S is one-to-one and preserves inclusion and the usual ideal-theoretic operations: addition, multiplication, intersection, radical and colon [**17**, Theorem 17, Chapter VII, Section 5].

Remark 2.3. We recall the properties of the composite operations ^{a_ih} and ^{ha_i}, for each $i = 0, \dots, n$, for each ideal I of R_i and each homogeneous ideal J of S (see [**17**, Chapter VII, Section 5, page 182]):

(H1) ${}^{a_i}({}^hI) = I$;

(H2) ${}^h({}^{a_i}J) \supseteq J$;

(H3) $X_i^m({}^h({}^{a_i}J)) \subseteq J$, for some integer $m \geq 1$.

In particular, if I is a homogeneous ideal of S, for all $i, j = 0, \dots, n$, we have ${}^{a_i}I \subseteq {}^{a_iha_j}I$.

Definition 2.4. Let I be an ideal of S, the *saturation* of I is the ideal:

$${}^{sat}I := \{y \in S : \quad \text{for all } i = 0, \dots, n, \text{there exists a} t_i \geq 0,\ yX_i^{t_i} \in I\}.$$

An ideal I of S is *saturated* if ${}^{sat}I = I$.

By [**9**, Exercise 5.10], the saturation of a homogeneous ideal is homogeneous.

Remark 2.5. Let I be an ideal of S. Then

$$\begin{aligned} y \in {}^{sat}I &\iff y \in IS\left[\frac{1}{X_i}\right] \cap S \text{ for all } i = 0, \dots, n. \\ &\iff y \in \left(IS\left[\frac{1}{X_0}\right] \cap S\right) \cap \dots \cap \left(IS\left[\frac{1}{X_n}\right] \cap S\right). \end{aligned}$$

Thus

$${}^{sat}I = \left(IS\left[\frac{1}{X_0}\right]\right) \cap \dots \cap \left(IS\left[\frac{1}{X_n}\right]\right).$$

Furthermore, denoting by $\mathfrak{m}$ the irrelevant ideal $(X_0, \dots, X_n)$, the saturation of an ideal I can be also defined as: $\cup_k (I : \mathfrak{m}^k) =: (I : \mathfrak{m}^\infty)$. It is easily seen that this definition agrees with the one proposed above.

By using this latest definition, it is not hard to prove that, for an ideal I of S, the following are equivalent:

(a) $\mathfrak{m}$ is an embedded component of I;

(b) the height of I is less than or equal to n and I is not saturated.

Proposition 2.6. *Let I be a homogeneous ideal of S. Then ${}^{sat}I = \cap_{i=0}^n {}^{ha_i}I$.*

Proof. Let $f \in {}^{sat}I$. Then for each $i = 0, \dots, n$, there is a nonnegative integer n such that $X_i^n f \in I$. Set $g := X_i^n f$. Then we have $g = {}^h({}^{a_i}g)X_i^m$, where m is the degree of X_i in polynomial g. Clearly, as $f = X_i^{-n} g \in S$, we have $m \geq n$. So $f = {}^h({}^{a_i}g)X_i^{m-n} \in {}^h({}^{a_i}I)$ for each $i = 0, \dots, n$. Therefore ${}^{sat}I \subseteq \cap_{i=0}^n {}^{ha_i}I$.

Since ${}^{sat}I$ and $\cap_{i=0}^n {}^{ha_i}I$ are homogeneous ideals, it is enough to prove that all the homogeneous elements of $\cap_{i=0}^n {}^{ha_i}I$ are also in ${}^{sat}I$. For, let $f \in \cap_{i=0}^n {}^{ha_i}I$ with f a homogeneous polynomial. Then, for each $i = 0, \dots, n$, we can assume, without loss of generality, that $f = X_i^m({}^h g_i)$ with m_i a nonnegative integer and $g_i \in {}^{a_i}I$ (i.e., $g_i = {}^{a_i}\varphi$, $\varphi \in I$). Thus, $f = X_i^{m_i} {}^h({}^{a_i}\varphi) = X_i^{m_i} X_i^{-m_{0i}} \varphi$, where m_{0i} is the highest power of X_i that divides φ. Therefore, it is enough to choose a nonnegative integer s such that $s \geq m_{0i} - m_i$ to have $X_i^s f = X_i^{s+m_i-m_{0i}} \varphi \in I$, as $\varphi \in I$. Hence $f \in {}^{sat}I$, and ${}^{sat}I = \cap_{i=0}^n {}^{ha_i}I$. $\square$

Proposition 2.7. *Given I, J homogeneous ideals of S the following properties hold*:

(a) *for each* $i = 0, \dots, n$, ${}^{a_i}I = {}^{a_i\,sat}I$;

(b) ${}^{sat}I \subseteq {}^{sat}J$ *if and only if* ${}^{a_i}I \subseteq {}^{a_i}J$ *for all* $i = 0, \dots, n$;

(c) ${}^{sat}I$ *is homogeneous*;

(d) *for all* (*homogeneous*) *polynomials* $f \in S$, ${}^{sat}(fI) = f\,{}^{sat}I$;

(e) $I \subseteq {}^{sat}I$ *and if* $I \subseteq J$ *then* ${}^{sat}I \subseteq {}^{sat}J$;

(f) ${}^{sat}(I \cap J) = {}^{sat}I \cap {}^{sat}J$.

In particular, from (d), (e), (f) *together with the fact that* ${}^{sat}S = S$, *it follows that the saturation* ^{sat} *is a star operation on* S.

Proof. (a) The inclusion ${}^{a_i}I \subseteq {}^{a_i\,sat}I$ is trivial since dehomogenization preserves inclusions. For the converse, since ^{a_i} commutes with intersections:

$$ {}^{a_i\,sat}I = {}^{a_i}\left(\bigcap_{j=0}^{n} {}^{ha_j}I\right) = \bigcap_{j=0}^{n} {}^{a_i ha_j}I \subseteq {}^{a_i}I. $$

(b) Suppose that ${}^{sat}I \subseteq {}^{sat}J$. We have $I \subseteq {}^{sat}I \subseteq {}^{sat}J$. Thus, for each $i = 0, \dots, n$: ${}^{a_i}I \subseteq {}^{a_i\,sat}J = {}^{a_i}J$, by (a). Conversely, suppose that ${}^{a_i}I \subseteq {}^{a_i}J$ for all $i = 0, \dots, n$. Since the operation ^{h} preserves inclusion, we can conclude by using Proposition 2.6 that ${}^{sat}I \subseteq {}^{sat}J$.

(c) It is straightforward by Proposition 2.6 and the fact that an intersection of homogeneous ideals is homogeneous.

(d) Let f be a polynomial in S.

$$ \begin{aligned} {}^{sat}(fI) &= \left((fI)\,S\left[\frac{1}{X_0}\right]\right) \cap \cdots \cap \left((fI)\,S\left[\frac{1}{X_n}\right]\right) \\ &= f\left(\left(IS\left[\frac{1}{X_0}\right]\right) \cap \cdots \cap \left(IS\left[\frac{1}{X_n}\right]\right)\right) \\ &= f\,{}^{sat}I. \end{aligned} $$

(e) This is clear by the definition of saturation or by Proposition 2.6.

(f) It is clear by combining Proposition 2.6 and the fact that the operations ^{a_i} and ^{h} preserve intersections. □

3. Star operations on homogeneous and non-homogeneous ideals.

Definition 3.1. A fractional ideal J of S is *homogeneous*, respectively *saturated*, if a homogeneous polynomial $f \in S$ exists such that fJ is a homogeneous, respectively saturated, ideal of S.

Definition 3.2. If J is a homogeneous fractional ideal of S, then the *dehomogenization* of J is ${}^{a_i}J := 1/({}^{a_i}f)^{a_i}(fJ)$, where f is a homogeneous element of S such that fJ is a homogeneous ideal of S.

For each $i = 0, \dots, n$, if J is a fractional ideal of R_i, i.e., J is an R_i-module in F and there is an $f \in R_i$ such that fJ is an ideal of R_i, then the *homogenization* of J is ${}^{h}J := 1/({}^{h}f)^{h}(fJ)$.

Remark 3.3. For each $i = 0, \dots, n$, operation ^{a_i} is well defined for homogeneous fractional ideals of S. For instance, let J be a homogeneous fractional ideal of S. Suppose that there are homogeneous polynomials f and g such that fJ and gJ are homogeneous ideals of S. Then:

$$\begin{aligned}\frac{1}{{}^{a_i}f}{}^{a_i}(fJ) = \frac{1}{{}^{a_i}g}{}^{a_i}(gJ) &\Longleftrightarrow {}^{a_i}g^{a_i}(fJ) \\ &= {}^{a_i}f^{a_i}(gJ) \Longleftrightarrow^{a_i} (gfJ) \\ &= {}^{a_i}(fgJ).\end{aligned}$$

It is also clear by a similar argument that the operation ^{h} is well defined for fractional ideals of R_i, for all $i = 0, \dots, n$.

Observe that, if J is a homogeneous fractional ideal of S, then ${}^{a_i}J$ is a fractional ideal of R_i. Conversely, given a fractional ideal I of R_i, ${}^{h}I$ is a homogeneous fractional ideal of S.

Definition 3.4. Let $\overline{\mathcal{H}}(S)$ denote the set of nonzero homogeneous fractional ideals of S. A *homogeneous star operation* on S is a mapping:

$$\begin{aligned} \star : \overline{\mathcal{H}}(S) &\longrightarrow \overline{\mathcal{H}}(S) \\ I &\longmapsto I^\star \end{aligned}$$

such that, for every nonzero homogeneous rational function f (i.e., $f = g/h$ with $0 \neq h$ and g homogeneous polynomials in S) in the quotient field of S and every $I, J \in \overline{\mathcal{H}}(S)$ the following conditions are satisfied:

(a) $(f)^\star = (f)$, $(fI)^\star = fI^\star$;

(b) $I \subseteq I^\star$ and if $I \subseteq J$ then $I^\star \subseteq J^\star$;

(c) $I^{\star\star} := (I^\star)^\star = I^\star$.

Remark 3.5. If $I \mapsto I^\star$ is a homogeneous star operation on S, it is clear that $S = (1) = (1)^\star = S^\star$, and if I is a homogeneous ideal of S, then $I \subseteq I^\star \subseteq S^\star = S$. Hence, each homogeneous star operation on S induces a map $I \mapsto I^\star$ from $\mathcal{H}(S)$, the set of homogeneous ideals of S, into $\mathcal{H}(S)$ satisfying conditions (a), (b) and (c). Moreover, for each operation $\star$ from $\mathcal{H}(S)$ onto $\mathcal{H}(S)$ satisfying conditions (a), (b) and (c), if $J \in \overline{\mathcal{H}}(S)$, then there is a homogeneous element $f \in S$ such that $fJ =: I$ is a homogeneous ideal of S. Set $J^\star = (1/f)I^\star$. It is clear that $\star$ is well defined and is a homogeneous star operation on S. From now on, we consider a homogeneous star operation on S as a map from $\mathcal{H}(S)$ onto $\mathcal{H}(S)$ satisfying conditions $(a), (b)$ and (c) (in condition (a), take f to be a homogeneous element of S).

Furthermore it is easily seen that a star operation on S which preserves homogeneous ideals is a homogeneous star operation, but, as expected, not every star operation on S is homogeneous preserving.

Next we provide some examples of homogeneous star operations (part (a)) and an example of a star operation on S that is not a homogeneous star operation (part (b)).

Example 3.6. (a) The identity is clearly, by definition, a homogeneous star operation. We saw earlier that saturation is also a homogeneous star operation (Proposition 2.7 (c), (d), (e) and (f)). We will see that the b-operation and the v-operation (whose definitions are recalled later) are homogeneous star operations on S as well.

(b) Let I be an ideal of S. Since S is a Noetherian integrally closed domain, S is completely integrally closed, so $S = (I : I)$ for each nonzero ideal I of S (see [**7**, Theorem 34.3]). So, by [**10**, Proposition 3.2], the application $v(I) : \mathfrak{F}(S) \to \mathfrak{F}(S)$, $J \mapsto (I : (I : J))$ is a star operation on S for each ideal I.

Consider the nonhomogeneous maximal ideal $M := (X_0 - 1, X_1, \dots, X_n)$ of S and the homogeneous ideal $I := (X_0, \dots, X_{n-1})$. We shall prove that $I^{v(M)} := (M : (M : I))$ is not homogeneous, and hence $v(M)$ cannot be restricted to a homogeneous star operation.

By [**10**, Lemma 3.1], $I^{v(M)} = \cap_{I \subseteq qM} qM$ with q in the quotient field of S. First of all we observe that $I^{v(M)} \supsetneq I$. Suppose by contradiction that $I = I^{v(M)}$. Then, since S is Noetherian, the ideal $(M : I) = (r_1, \dots, r_n)S$ for some finite set $\{r_1, \dots, r_n\}$ of the quotient field of S, and $(M : (M : I)) = (M : (r_1, \dots, r_n)S) = \cap_{i=1}^n r_i^{-1} M$. By setting $q_i := r_i^{-1}$:

$$I = \bigcap_{I \subseteq qM} qM = \bigcap_{i=1}^n q_i M = \bigcap_{i=1}^n q_i M \cap S \subseteq \bigcap_{i=1}^n q_i S \cap S = S,$$

where the last equality holds because I is a prime ideal of height greater than 1 in an integrally closed Noetherian domain, hence, by [**7**, Corollary 44.8]:

$$S = \bigcap_{I \subseteq qS} qS \subseteq \bigcap_{i=1}^n q_i S \cap S \subseteq S.$$

Then, for each i, $q_i S \cap S = S$ and $r_i := q_i^{-1} \in S$. Therefore, $I = (1/r_1)M \cap \cdots \cap (1/r_n)M \cap S$. We can assume without loss of generality that, for all i, $r_i \in S \setminus M$. For, if $r_i \in M$ for some i, $(1/r_i)M = S$ and there is no contribution in the intersection. We have then that $(r_1 \cdots r_n)I = (r_2 \cdots r_n)M \cap \cdots \cap (r_1 \cdots r_{n-1})M \cap (r_1 \cdots r_n)S$. Thus, $I_M = MS_M$ (for all i, $r_i \notin M$), which is a contradiction because I is a prime ideal properly contained in M. So $I \subsetneq I^{v(M)} \subseteq M^{v(M)} = M$.

We prove now that I is maximal among the homogeneous ideals of S contained in M. Suppose that a homogeneous ideal J of S exists such that $I \subsetneq J \subsetneq M$. Then the set

$$\mathfrak{F} := \{J : J \text{ is homogeneous and } I \subseteq J \subsetneq M\}$$

is nonempty and, since S is Noetherian, each ascending chain in the set $\mathfrak{F}$ stabilizes. By Zorn's lemma, $\mathfrak{F}$ has a maximal element P. Suppose P is not prime. Then, by Proposition 2.1, $f, g \in S \setminus P$ homogeneous exist such that $fg \in P$. We can suppose f is in M because M is prime, so we have $P \subsetneq (P, f) \subsetneq M$, because M is not homogeneous, and this contradicts the maximality of P in $\mathfrak{F}$. Hence, P is prime and

$$(0) \subsetneq (X_0) \subsetneq (X_0, X_1) \subsetneq \cdots \subsetneq (X_0, \dots, X_{n-1}) = I \subsetneq P \subsetneq M$$

is a chain of distinct primes of length $n+2 > \dim(S) = n+1$, which is impossible. Therefore, I is maximal in $\mathfrak{F}$ and, since $I \subsetneq I^{v(M)} \subseteq M$, $I^{v(M)}$ is not homogeneous.

We next turn our attention to the "dehomogenization" of a homogeneous star operation. In other words, given a homogeneous star operation $\star$ on S, we construct star operations $\star_i$ on R_i for each $i = 0, \dots, n$.

Proposition 3.7. *Let $\star$ be a homogeneous star operation on S. Then the map $\star_i : \mathfrak{I}(R_i) \to \mathfrak{I}(R_i)$, $I \mapsto I^{\star_i} := {}^{a_i}(({}^hI)^\star)$, where $\mathfrak{I}(R_i)$ denotes the set of ideals of R_i, is a star operation on R_i for each $i = 0, \dots, n$.*

Proof. We want to prove that conditions $(\star_1)$, $(\star_2)$ and $(\star_3)$ defined for star operation in the introduction hold. Let $g \in R_i$ and I be an ideal of R_i;

$$(gI)^{\star_i} = {}^{a_i}({}^h(gI))^\star = {}^{a_i}({}^hg\,{}^hI)^\star = g\,{}^{a_i}({}^hI)^\star = gI^{\star_i}.$$

Since ${}^hR_i = S$ for each i, the first condition $(\star_1)$ holds. Condition $(\star_2)$ is straightforward. The fact that $(I^{\star_i})^{\star_i} \supseteq I^{\star_i}$ follows from $(\star_2)$, and we prove that the reverse inclusion holds too. By $(H3)$ we have $X_i^{mha_i}(({}^hI)^\star) \subseteq ({}^hI)^\star$ for some $m \geq 1$. Since $\star$ is a homogeneous star operation:

$$X_i^m \left[{}^{ha_i}\left(({}^hI)^\star\right)\right]^\star = \left[X_i^{mha_i}\left(({}^hI)^\star\right)\right]^\star \subseteq ({}^hI)^{\star\star} = ({}^hI)^\star.$$

Now, as ^{a_i} preserves inclusion and ${}^{a_i}(X_i^m) = 1$, we have

$$(I^{\star_i})^{\star_i} = {}^{a_i}\left(\left[{}^{ha_i}\left(({}^hI)^\star\right)\right]^\star\right) = {}^{a_i}\left(X_i^m\left[{}^{ha_i}\left(({}^hI)^\star\right)\right]^\star\right) \subseteq {}^{a_i}({}^hI)^\star = I^{\star_i}.$$

Then $\star_i$ is a star operation on R_i for each $i = 0, \dots, n$. ∎

We call the process described in Proposition 3.7 *dehomogenization* of a homogeneous star operation. Our next aim is to reverse this process. So, first of all, we investigate the properties of the set of star operations obtained by dehomogenizing a homogeneous star operation.

Proposition 3.8. *Let $\star$ be a homogeneous star operation on S, and let $\{\star_0, \dots, \star_n\}$ be the star operations obtained by dehomogenizing $\star$. Then ${}^{a_i}(I^\star) = ({}^{a_i}I)^{\star_i}$ for each homogeneous ideal I of S and each $i = 0, \dots, n$.*

Proof. For each $i = 0, \dots, n$, $({}^{ha_i}I)^\star \supseteq I^\star$, then $({}^{a_i}I)^{\star_i} = {}^{a_i}({}^{ha_i}I)^\star \supseteq {}^{a_i}(I^\star)$. Conversely, by (H3) of Remark 2.3, some $m \geq 1$ exists such that $I^\star \supseteq X_i^m({}^{ha_i}I)^\star$, so that ${}^{a_i}(I^\star) \supseteq {}^{a_i}(X_i^m({}^{ha_i}I)^\star) = ({}^{a_i}I)^{\star_i}$ for each $i = 0, \dots, n$. ◻

Corollary 3.9. *Let $\star$ be a homogeneous star operation on S, and let $\{\star_0, \dots, \star_n\}$ be the star operations obtained by dehomogenizing $\star$. Then*

$$ {}^{sat}(I^\star) = {}^{h}\left(({}^{a_0}I)^{\star_0}\right) \cap \cdots \cap {}^{h}\left(({}^{a_n}I)^{\star_n}\right). $$

Proof. By Proposition 2.6,

$$ \begin{aligned} {}^{sat}(I^\star) &= {}^{ha_0}(I^\star) \cap \cdots \cap {}^{ha_n}(I^\star) \\ &= {}^{h}\left(({}^{a_0}I)^{\star_0}\right) \cap \cdots \cap {}^{h}\left(({}^{a_n}I)^{\star_n}\right). \end{aligned} $$

The last equality is by Proposition 3.8. ◻

The lemma below suggests a "star" version for the properties (H1), (H2) and (H3), mentioned in Remark 2.3.

Lemma 3.10. *Let $\star$ be a homogeneous star operation on S, and let $\{\star_0, \dots, \star_n\}$ be the set of star operations on $R_0, \dots, R_n$ obtained as in Proposition 3.7. Then, for each homogeneous ideal I of S,*

(i) $({}^{a_j}I)^{\star_j} \subseteq {}^{a_j}[{}^{h}(({}^{a_i}I)^{\star_i})]$ *for all $j = 0, \dots, n$ and $i = 0, \dots, n$,*

(ii) *For each $i = 0, \dots, n$ a nonnegative integer m_i exists such that $X_i^{m_i a_j}[{}^{h}(({}^{a_i}I)^{\star_i})] \subseteq ({}^{a_j}I)^{\star_j}$ for all $j = 0, \dots, n$.*

Proof. For (i), for each $i = 0, \ldots, n$, $({}^{ha_i}I)^\star \supseteq I^\star$, i.e., for each $j = 0, \ldots, n$, for each $i = 0, \ldots, n$, ${}^{a_j}[({}^{ha_i}I)^\star] \supseteq {}^{a_j}(I^\star)$. By Proposition 3.8, $({}^{a_j}I)^{\star_j} \subseteq {}^{a_j}[{}^{h}(({}^{a_i}I)^{\star_i})]$ for all $j = 0, \ldots, n$ and $i = 0, \ldots, n$.

For (ii), a similar argument as for (i) works by using the inclusion $X_i^{m_i h a_i}(I^\star) \subseteq I^\star$ for some $m_i \geq 1$ for each $i = 0, \ldots, n$. $\square$

By homogenization and dehomogenization we can "move" an ideal of R_i to any of the other R_j's. Condition (i) in Lemma 3.10 suggests that, just as in the case of the identity operation, the behavior of an ideal of R_i under the star operation $\star_i$ reflects the behavior of that same ideal moved into R_j under $\star_j$. Since a homogeneous ideal of S collects together the behaviors of its dehomogenized components, if we want to glue together a collection of star operations on different R_i's, we define two star operations to be compatible if we can move ideals from R_i to R_j, through S, preserving the behavior of the given star operations. This compatibility has to be satisfied by any pair of star operations that we want to "glue" together into a homogeneous star operation.

In particular, it will not be possible to glue together star operations of very different kinds (cf. Example 4.6).

Definition 3.11. Let $\star_0, \ldots, \star_n$ be star operations on $R_0, \ldots, R_n$, respectively. We say that $\star_0, \ldots, \star_n$ are *pairwise compatible* if $({}^{a_j}I)^{\star_j} \subseteq {}^{a_j}[{}^{h}(({}^{a_i}I)^{\star_i})]$ for all $i, j = 0, \ldots, n$ and all homogeneous ideals I of S.

Proposition 3.12. *Let $\{\star_0, \ldots, \star_n\}$ be a set of pairwise compatible star operations on $R_0, \ldots, R_n$. Then the map*:

$$\star : \mathcal{H}(S) \longrightarrow \mathcal{H}(S)$$
$$I \longmapsto I^\star := {}^{sat}\left[\bigcap_{i=0}^{n} {}^{h}\left(({}^{a_i}I)^{\star_i}\right)\right]$$

is a homogeneous star operation on S. Moreover, if $\{\star_0, \ldots, \star_n\}$ are the dehomogenization of a homogeneous star operation $\star'$ on S, then $\star = sat \circ \star'$.

Proof. We need to prove that $\star$ satisfies conditions (a), (b) and (c) of Definition 3.4. It is easily seen that $S^\star = S$. Moreover, saturation, homogenization and dehomogenization preserve inclusions. This is enough to prove (b).

Now suppose that f is a homogeneous element in S. We claim that $(fI)^\star = fI^\star$. We have:

$$\begin{aligned} {}^{a_j}[(fI)^\star] &\subseteq {}^{a_j sat}\left({}^h[({}^{a_j}fI)^{\star_j}]\right), \text{ (by Definition 3.4)} \\ &= {}^{a_j h}[({}^{a_j}fI)^{\star_j}], \text{ (by Proposition 2.7 (a))} \\ &= ({}^{a_j}fI)^{\star_j} \\ &= {}^{a_j}f({}^{a_j}I)^{\star_j}, \text{ (since } \star_j \text{ is a star operation)} \\ &\subseteq {}^{a_j}f^{a_j}[{}^h(({}^{a_i}I)^{\star_i})], \text{ (by the compatibility of the } \star_i\text{'s)} \\ &= {}^{a_j}[f^h(({}^{a_i}I)^{\star_i})]. \end{aligned}$$

Hence, by Proposition 2.7 (b), we have ${}^{sat}[(fI)^\star] \subseteq f^{sat}[{}^h(({}^{a_i}I)^{\star_i})]$ for all $i = 0, \ldots, n$. Thus,

$$(fI)^\star \subseteq {}^{sat}[(fI)^\star] \subseteq f^{sat}\left[\bigcap_{i=0}^{n} {}^h(({}^{a_i}I)^{\star_i})\right] = fI^\star.$$

For the other inclusion, we have

$${}^{a_j}(fI^\star) = {}^{a_j}f^{a_j}(I^\star) \subseteq {}^{a_j}f^{a_j h}[({}^{a_j}I)^{\star_j}] = {}^{a_j}f({}^{a_j}I)^{\star_j} = [{}^{a_j}(fI)]^{\star_j}.$$

By compatibility of the $\star_i$'s,

$${}^{a_j}(fI^\star) \subseteq {}^{a_j}[{}^h(({}^{a_i}fI)^{\star_i})], \text{ for all } i = 0, \ldots, n,$$

i.e., ${}^{sat}(fI^\star) \subseteq {}^{sat\,h}[(({}^{a_i}fI)^{\star_i})]$ for all $i = 0, \ldots, n$, by Proposition 2.7 (b). Hence,

$$fI^\star = {}^{sat}(fI^\star) \subseteq \bigcap_{i=0}^{n} {}^{sat}\left({}^h(({}^{a_i}fI)^{\star_i})\right) = (fI)^\star.$$

So $(fI)^\star = fI^\star$.

For the last condition (c) left, it is clear that $I^\star \subseteq I^{\star\star}$ on one hand. On the other hand,

$$\begin{aligned} (I^\star)^\star &= {}^{sat}\bigcap_{j=0}^{n} {}^h\left(({}^{a_j}(I^\star)^{\star_j})\right) \subseteq {}^{sat}\bigcap_{j=0}^{n} {}^h\left(({}^{a_j}I)^{\star_j\star_j}\right) \\ &= {}^{sat}\bigcap_{j=0}^{n} {}^h\left(({}^{a_j}I)^{\star_j}\right) = I^\star. \end{aligned}$$

If $\{\star_0, \dots, \star_n\}$ are the dehomogenization of a homogeneous star operation $\star'$, it follows directly by Corollary 3.9 that $\star = sat \circ \star'$. □

We call this process *homogenization* of a set of pairwise compatible star operations $\{\star_0, \dots, \star_n\}$.

Remark 3.13. Given a homogeneous star operation $\star$ on S, we can build, by dehomogenization, star operations $\star_i$ on R_i for each $i = 0, \dots, n$. These star operations $\star_i$'s are pairwise compatible. Therefore, by applying Proposition 3.12 to the $n + 1$ star operations obtained, we get a homogeneous star operation that differs from $\star$ by saturation, i.e., it is of the form $sat \circ \star$.

Reciprocally, starting from a set of pairwise compatible star operations $\{\star_0, \dots, \star_n\}$ on $R_0, \dots, R_n$, then we can build, by homogenization, a homogeneous star operation $\star$ on S such that $\star = sat \circ \star$.

By Proposition 3.16 there is a bijection between the set of $(n + 1)$-tuples of compatible star operations, each defined on one of the R_i's and the set of homogeneous star operations star operations on S enjoying the additional property that $\star = sat \circ \star$. This motivates our next definition.

Definition 3.14. A *projective star operation* on S is a homogeneous star operation $\star$ on S such that $sat \circ \star = \star$.

It is clear that a homogeneous star operation $\star$ on S is projective if and only if $\star = sat \circ \star'$ for some homogeneous star operation $\star'$ on S. Consequently, the homogeneous star operation built in Proposition 3.12 is a projective star operation.

To keep a standard notation and avoid confusion between star operations defined on different domains, we shall denote on S the identity, the integral closure of ideals and the divisorial closure of ideals by d, b and v, respectively (the definitions are recalled in Section 4). The same star operations referred to R_i will be denoted by d_i, b_i and v_i, respectively.

Example 3.15. Clearly, the identity operation d on S is a homogenous star operation on S that is not projective. The identities d_i's on the R_i's satisfy the compatibility conditions, by Remark 2.3. The homogenization of $\{d_0, \dots, d_n\}$ is, by Proposition 2.6, the saturation *sat*. On the other hand, by the same arguments, the saturation *sat* is a projective star operation on S.

More examples will be given in the next section, where the b- and v-operations are studied in detail in the context of homogeneous ideals.

We conclude the section with some more properties of projective star operations.

Proposition 3.16. *Let $\{\star_0, \dots, \star_n\}$ be a set of pairwise compatible star operations on $R_0, \dots, R_n$, and let $\star$ be the homogenization of $\{\star_0, \dots, \star_n\}$. Then, for each $i = 0, \dots, n$ and for each ideal I of R_i, we have $I^{\star_i} = {}^{a_i}[({}^hI)^\star]$.*

Proof. For each $i = 0, \dots, n$, and each ideal I of R_i, we have:

$$\begin{aligned} {}^{a_i}[({}^hI)^\star] &= {}^{a_i}sat\left[\bigcap_{k=0}^n {}^h({}^{a_k h}I)^{\star_k}\right] \\ &= \bigcap_{k=0}^n {}^{a_i h}({}^{a_k h}I)^{\star_k} \\ &= I^{\star_i} \cap \bigcap_{\substack{k \neq i \\ k=0}}^n {}^{a_i h}({}^{a_k h}I)^{\star_k}. \end{aligned}$$

Hence,

$$I^{\star_i} \supseteq I^{\star_i} \cap \bigcap_{\substack{k \neq i \\ k=0}}^n {}^{a_i h}({}^{a_k h}I)^{\star_k} \supseteq I^{\star_i},$$

by compatibility of the $\star_i$'s. So $I^{\star_i} = {}^{a_i}[({}^hI)^\star]$. □

We next prove that the same property as in Proposition 3.8 holds when we start with a set of pairwise compatible star operations.

Proposition 3.17. *Let $\{\star_0, \dots, \star_n\}$ be a set of pairwise compatible star operations on $R_0, \dots, R_n$ and $\star$ the homogenization of*

$\{\star_0, \ldots, \star_n\}$. *Then. for any homogeneous ideal I of S, we have* $({}^{a_i}I)^{\star_i} = {}^{a_i}(I^\star)$ *for all* $i = 0, \ldots, n$.

Proof. Let $i \in \{0, \ldots, n\}$. By Proposition 3.16, we have $({}^{a_i}I)^{\star_i} = {}^{a_i}[({}^{ha_i}I)^\star]$. On the other hand, $({}^{ha_i}I)^\star \supseteq I^\star$, which implies $({}^{a_i}I)^{\star_i} = {}^{a_i}({}^{ha_i}I)^\star \supseteq {}^{a_i}(I^\star)$.

Conversely, $I \supseteq X_i^m({}^{ha_i}I)$ for some $m \geq 1$. Since $\star$ is a projective star operation, we have $I^\star \supseteq X_i^m({}^{ha_i}I)^\star$. Therefore, ${}^{a_i}(I^\star) \supseteq {}^{a_i}(X_i^m({}^{ha_i}I)^\star) = {}^{a_i}[({}^{ha_i}I)^\star]$. $\square$

4. Projective b-operation and v-operation. In this section we study the b-operation and the v-operation. First of all we observe that they are homogeneous star operations. We prove that the dehomogenization of b is the set $\{b_0, \ldots, b_n\}$, and similarly the v-operation dehomogenizes at $\{v_0, \ldots, v_n\}$. We show that the v-operation is projective, whilst the b-operation is not in general.

Let L be a field and D a subring (possibly a subfield) of L; we denote by $\Sigma(L/D) := \{V : V$ valuation rings of $L, D \subseteq V\}$, that is, the *Zariski-Riemann space* of L over D.

Recall that, if D is an integral domain with quotient field L and I is any nonzero fractional ideal of D:

(i) the b-operation is defined by the mapping $I \mapsto I^b := \cap_{V \in \Sigma(L/D)} IV$;

(ii) the v-operation is defined by the mapping $I \mapsto I_v := (D : (D : I))$.

For details on the b- and v-operations, the reader is referred to [**7**, Section 32, Section 34].

It is known that, if I is a homogeneous ideal of S, then I^b is a homogeneous ideal of S [**16**, Corollary 5.2.3]. So the b-operation is a homogeneous star operation on S. A natural question is whether the b-operation is projective in general. The example below is negative.

Example 4.1. Consider $S = K[X_0, X_1, X_2]$ and $I = (X_0^2X_1^2, X_0^2X_2^2, X_1^2X_2^2)$ a homogeneous ideal of S. It is enough to see that I has height 2 and no embedded components, so it is saturated. On the other hand, by [**12**, Example 2.6], the integral closure I^b of I has $\mathfrak{m} = (X_0, X_1, X_2)$

as an embedded prime, and hence it is not saturated, by Remark 2.5. So ${}^{sat}(I^b) \neq I^b$ and the b-operation in this case is not projective.

Now we prove that the star operations b_i on the R_i's satisfy the compatibility condition.

Lemma 4.2. *Let I be a homogeneous ideal of S. Then ${}^{a_i}(I^b)$ is an integrally closed ideal of R_i for all $i = 0, \dots, n$.*

Proof. For each i, we have $R_i = S_{M_i} \cap F$ where M_i is the multiplicatively closed subset of S consisting of powers of X_i and F is the quotient field of R_i. Now, for I a homogeneous ideal of S, ${}^{a_i}(I^b) = I^b S_{M_i} \cap F = (IS_{M_i})^b \cap F$ [**17**, VII, Section 5.(10')] and [**16**, Proposition 1.1.4]. Since $(IS_{M_i})^b$ is an integrally closed ideal in S_{M_i}, $(IS_{M_i})^b \cap F$ is integrally closed in $S_{M_i} \cap F = R_i$. So ${}^{a_i}(I^b)$ is integrally closed in R_i, for each $i = 0, \dots, n$. $\square$

Lemma 4.3. *Let I be a homogeneous ideal of S. Then ${}^{a_i}(I^b) = ({}^{a_i}I)^{b_i}$ for all $i = 0, \dots, n$ and the b_i's are therefore pairwise compatible.*

Proof. Let I be a homogeneous ideal of S. We have $I \subseteq I^b$. Since ^{a_i} preserves inclusions for each i, we have ${}^{a_i}I \subseteq {}^{a_i}(I^b)$. But, by Lemma4.2, ${}^{a_i}(I^b)$ is integrally closed. Therefore, $({}^{a_i}I)^{b_i} \subseteq {}^{a_i}(I^b)$.

For the reverse inclusion, let $x \in {}^{a_i}(I^b)$. Then we can write $x = {}^{a_i}r$ for some element $r \in I^b$. Thus, r satisfies an equation of integral dependence of r over I of the form $r^s + c_1 r^{s-1} + \cdots + c_{s-1} r + c_s = 0$ for some positive integer s and $c_j \in I^j$ for each $j = 1, \dots, s$. Since ^{a_i} is a homomorphism, we have $({}^{a_i}r)^s + {}^{a_i}c_1 ({}^{a_i}r)^{s-1} + \cdots + {}^{a_i}c_{s-1} {}^{a_i}r + {}^{a_i}c_s = 0$ with ${}^{a_i}c_j \in ({}^{a_i}I)^j$ for each $j = 1, \dots, s$. Thus, $x = {}^{a_i}r \in ({}^{a_i}I)^{b_i}$. Hence, ${}^{a_i}(I^b) \subseteq ({}^{a_i}I)^{b_i}$. So ${}^{a_i}(I^b) = ({}^{a_i}I)^{b_i}$.

We just proved that the dehomogenization of the b-operation on S are exactly the b_i-operations on the R_i's. Hence, by Lemma 3.10, the b_i-operations are pairwise compatible. $\square$

Remark 4.4. By Lemma 4.3, it is clear that if we start with the b-operation (a homogeneous star operation) on S and dehomogenize it on the R_i's, the $\star_i$'s are exactly the b_i-operations on R_i's. Conversely, if we start with the set of the b_i-operations on the R_i's, we can homogenize it into a projective star operation on S, which is, by Proposition3.12, $sat \circ b$. By Example 4.1, $sat \circ b$ may differ from the b-operation.

Suppose I is a homogeneous ideal of S. Then $I_v := (S : (S : I))$ is homogeneous too (see [**17**, VII, Section 2, Theorem 8]), so if we restrict the divisorial closure in S to $\mathcal{H}(S)$, we get a homogeneous star operation on S that we keep denoting by v. From the general theory on star operations we have that, for any star operation $\star$ on S, $\star$ is less than or equal to v, i.e., $(I^\star)_v = (I_v)^\star = I_v$, for each $I \in \mathfrak{F}(S)$ (cf. [**7**, Theorem 34.1 (4)]).

Proposition 4.5. *For every $i = 0, \dots, n$, the dehomogenization of v to R_i is the divisorial closure, v_i, so in particular v_i and v_j are pairwise compatible for each $i, j = 0, \dots, n$. Furthermore, the homogenization of $\{v_0, \dots, v_n\}$ is exactly v. Hence, $sat \circ v = v$ and v is a projective star operation.*

Proof. For each $i = 0, \dots, n$, let J be an ideal of R_i, and let $J^{\star_i} := {}^{a_i}(({}^h J)_v)$. We will prove that $\star_i = v_i$.

$$\begin{aligned}
J^{\star_i} &= {}^{a_i}(({}^h J)_v) \text{ (by definition of } \star_i) \\
&= {}^{a_i}\left(S : (S : {}^h J)\right) \text{ (by definition of } v) \\
&= \left({}^{a_i}S : ({}^{a_i}S : {}^{a_i h}J)\right) \ (a_i \text{ commutes with colon}) \\
&= (R_i : (R_i : J)) \text{ (since } {}^{a_i}S = R_i) \\
&= J_{v_i} \text{ (by definition of } v_i).
\end{aligned}$$

Hence, the dehomogenization of v is the divisorial closure v_i on R_i and $\{v_0, \dots, v_n\}$ are pairwise compatible by Lemma 3.10 (i).

If $\star$ is the homogenization of $\{v_0, \dots, v_n\}$, we have that $\star = sat \circ v$, but as discussed before sat is less than or equal to v. So, for every $I \in \mathcal{H}(S)$, $({}^{sat}I)_v = {}^{sat}(I_v) = I_v$, and $\star = v$. $\square$

Example 4.6 (Non-compatible star operations). Let $S = K[X_0, X_1, X_2, X_3]$ and v_0 be the v-operation on $R_0 = K[(X_1/X_0), (X_2/X_0), (X_3/X_0)]$, and consider the b-operation b_i on R_i, for $i = 1, 2, 3$. We show that v_0 is not compatible with b_1.

Let $P := (X_2, X_3)$ be a homogeneous prime ideal of S. Then the ideal ${}^{a_0}P = ((X_2/X_0), (X_3/X_0))$ of R_0 is prime and has height 2; hence, by [**7**, Corollary 44.8], $({}^{a_0}P)_{v_0} = R_0$. The dehomogenization ${}^{a_1}P$ in R_1 is also a prime ideal and so it is integrally closed, i.e., $({}^{a_1}P)^{b_1} = {}^{a_1}P$. Note that ${}^{ha_1}P = P$. Thus, ${}^{a_0}({}^{h}[({}^{a_1}P)^{b_1}]) = {}^{a_0}({}^{ha_1}P) = {}^{a_0}P \subsetneq R_0 = ({}^{a_0}P)_{v_0}$. Therefore, condition (i) of Lemma 3.10 is not satisfied and v_0 is not compatible with b_1. By a similar argument, it is possible to show that v_0 is not compatible with b_i, for $i \in \{2, 3\}$.

5. Projective Kronecker function rings. In this section we associate a projective Kronecker function ring to a projective star operation on S and show how this is related to Kronecker function rings of the R_i's. Furthermore, we show that, by using projective star operations, we can build some F-function rings, called projective Kronecker function rings, that are not Kronecker function rings of a domain.

A star operation $\star$ on an integral domain D is *e.a.b.* if, for each finitely generated ideal I, J and N of D, the following cancelation property holds:

$$(IN)^\star \subseteq (JN)^\star \Longrightarrow I^\star \subseteq J^\star.$$

Recall also that we can represent the Kronecker function ring of D with respect to an e.a.b. star operation $\star$, in terms of valuation overrings of D (cf. [**7**, Theorem 32.12]): $\mathrm{Kr}\,(D, \star) = \cap_{V \in \Sigma} V^b$, where Σ is a subset of $\Sigma(L/D)$ and $V^b = V(T)$ is the trivial extension of V to $L(T)$ (cf. [**4**, Theorem 2.2.1]).

Halter-Koch introduced in [**8**] the notion of an L-function ring as a generalization of the Kronecker function ring.

Let L be a field and T an indeterminate for L. A subring $R \subseteq L(T)$ is an *L-function ring* if the following two axioms are satisfied:

(Ax1) $T, T^{-1} \in R$,

(Ax2) For any $f \in L[T]$, $f(0) \in fR$.

It is easily seen that, given a valuation ring of L, V^b is an L-function ring and an arbitrary intersection of L-function rings is an L-function ring. It follows that every Kronecker function ring of a domain D with quotient field L is an L-function ring, but there are L-function rings that cannot be constructed by using star operations (for instance, rings of the form $\cap_{V\in\Sigma(L_1/L_2)}V^b$ where L_1/L_2 is a transcendental field extension, which are studied by Heubo-Kwegna in [**11**]).

We start by defining the e.a.b. cancelation property for projective star operation.

Definition 5.1. A projective star operation $\star$ on S is *e.a.b.* if the dehomogenization of $\star$ consists of e.a.b. star operations on the R_i's.

It is clear by the preceding definition that if we built a projective star operation $\star$ on S by homogenizing a set of e.a.b. star operations $\star_i$'s on the R_i's, then $\star$ is e.a.b. We have, in particular, that $sat\circ b$ is an e.a.b. projective star operation on S.

Lemma 5.2. *Let $\star$ be an e.a.b. projective star operation on S. Then, for each finitely generated homogenous ideal I, J and N of S,*

$$(IN)^\star \subseteq (JN)^\star \Longrightarrow I^\star \subseteq J^\star.$$

Proof. Let I, J, N be finitely generated ideals of S, and suppose that $(IN)^{sat\circ\star} \subseteq (JN)^{sat\circ\star}$. Then we have by Proposition 2.7 (b): for each $i = 0, \dots, n$,

$$\begin{aligned} {}^{a_i}\left((IN)^\star\right) \subseteq {}^{a_i}\left((JN)^\star\right) &\Longleftrightarrow \left({}^{a_i}I^{a_i}N\right)^{\star_i} \subseteq \left({}^{a_i}J^{a_i}N\right)^{\star_i} \\ &\Longrightarrow \left({}^{a_i}I\right)^{\star_i} \subseteq \left({}^{a_i}J\right)^{\star_i} \\ &\Longrightarrow {}^{a_i}(I^\star) \subseteq {}^{a_i}(J^\star). \end{aligned}$$

Thus, by Proposition 2.7 (b), $I^{sat\circ\star} \subseteq J^{sat\circ\star}$. The result follows since $\star = sat\circ\star$. ◻

We investigate some properties of the notion of the content ideal of a homogeneous polynomial of $S[T]$, where T is a variable over S. In particular, we focus on dehomogenization and homogenization of content ideals.

Note that if $f = f_0 + f_1T + \cdots + f_sT^s$ is a homogenous polynomial of $S[T] = K[X_0, \dots, X_n, T]$ in $n+2$ variables, that forces its coefficients to be homogeneous elements of S. Then the content of f is a homogeneous ideal of S.

We will use the notation $C_D(h)$ to indicate the ideal of D which is the content of the polynomial $h(T) \in D[T]$.

Remark 5.3. Let $f = f_0 + f_1 T + \cdots + f_s T^s \in S[T]$ be homogeneous of degree m in $K[X_0, \ldots, X_n, T]$. Then

$$\frac{f}{X_i^m} = \frac{f_0}{X_i^m} + \frac{f_1}{X_i^{m-1}} \frac{T}{X_i} + \cdots + \frac{f_s}{X_i^{m-s}} \left(\frac{T}{X_i}\right)^s \in R_i\left[\frac{T}{X_i}\right],$$

for each $i = 0, \ldots, n$.

We also have, for each $i = 0, \ldots, n$:

$$ {}^{a_i}C_S(f) = ({}^{a_i}f_0, {}^{a_i}f_1, \ldots, {}^{a_i}f_s) = C_{R_i}\left(\frac{f}{X_i^m}\right). \tag{1}$$

Now set

$$F' := \left\{\frac{f}{g} : f, g \text{ homogeneous of same degree in } S[T] \text{ and } g \neq 0\right\}.$$

It is clear that F' is a field, and it is not hard to see that F' is in fact the field $K((X_0/T), \ldots, (X_n/T))$. Let $\star$ be an e.a.b. projective star operation on S. Let

$$\begin{aligned}\mathrm{PKr}\,(S, \star) : = \Big\{\frac{f}{g} &: f, 0 \neq g \text{ homogeneous of same degree in } S[T], \\ &\qquad C(f)^\star \subseteq C(g)^\star\Big\} \\ = \Big\{\frac{f}{g} &\in F' : C(f)^\star \subseteq C(g)^\star\Big\}.\end{aligned}$$

We can immediately note by Lemma 5.2 that the set $\mathrm{PKr}\,(S, \star)$ is well defined using the fact that, for all $f, g \in S[T] \setminus \{0\}$, $C(fg)^\star = (C(f)C(g))^\star$ (cf. [**7**, Lemma 32.6]). We also note that $\mathrm{PKr}\,(S, \star)$ "looks" quite like the classical Kronecker function ring, but contrary to the classical one $S \not\subseteq \mathrm{PKr}\,(S,\star)$. In fact, X_i is not in $\mathrm{PKr}\,(S,\star)$ for any $i = 0,\dots,n$. A natural question is whether $\mathrm{PKr}\,(S,\star)$ is a ring. We give an answer in the next proposition:

Proposition 5.4. *Let $\star$ be an e.a.b. projective star operation on S. $\mathrm{PKr}\,(S,\star)$ is a domain with quotient field $F' = K((X_0/T),\dots,(X_n/T))$ (we do have $K[(X_0/T),\dots,(X_n/T)] \subseteq \mathrm{PKr}\,(S,\star) \subseteq K((X_0/T),\dots,(X_n/T)))$.*

Proof. The fact that $\mathrm{PKr}\,(S,\star)$ is a domain is proved using the same argument as the one of the classical Kronecker function ring [**7**, Proof of Theorem 32.7 (a)]. It is clear that, for each $i = 0,\dots,n$, $(T/X_i) \in \mathrm{PKr}\,(S,\star) \subseteq K((X_0/T),\dots,(X_n/T))$. Since $K \subseteq \mathrm{PKr}\,(S,\star)$, the result follows. $\square$

We next make a connection between the ring $\mathrm{PKr}\,(S,\star)$ and the classical Kronecker function rings $\mathrm{Kr}\,(R_i,\star_i)$, $0 \le i \le n$, when the $\star_i$'s are pairwise compatible e.a.b. star operations on R_i's and $\star$ is the homogenization of the $\star_i$'s. Note in this case that $\star$ is an e.a.b. projective star operation on S (cf. Proposition 3.12).

Theorem 5.5. *Let $\star_0,\dots,\star_n$ be $n+1$ pairwise compatible e.a.b. star operations on $R_0,\dots,R_n$, respectively. Let $\star$ be the homogenization of $\star_0,\dots,\star_n$; hence, $\star$ is an e.a.b. projective star operation on S. Then $\mathrm{PKr}\,(S,\star) = \cap_{i=0}^{n}\mathrm{Kr}\,(R_i,\star_i)$.*

Proof. First we note that the quotient field of $\mathrm{PKr}\,(S,\star)$ is $F' = K((X_0/T),\dots,(X_n/T))$ and the quotient field of each $\mathrm{Kr}\,(R_i,\star_i)$, $0 \le i \le n$, is $K((X_0/X_i),\dots,(X_n/X_i),(T/X_i)) = K((X_0/T),\dots,(X_n/T)) = F'$. So we consider the Kronecker function ring $\mathrm{Kr}\,(R_i,\star_i)$ with respect to the variable T/X_i. We recall that, for each homogeneous element $f \in S[T]$ of degree m, ${}^{a_i}C_S(f) = C_{R_i}(f/X_i^m)$, $0 \le i \le n$ (see relation (1) in Remark 5.3). That way, we have:

$$\begin{aligned} X \in \mathrm{PKr}\,(S,\star) &\Longleftrightarrow X = \frac{f}{g}, f,g \in F', C_S(f)^\star \subseteq C_S(g)^\star \\ &\Longleftrightarrow X = \frac{f}{g}, f,g \in F', {}^{a_i}[C_S(f)^\star] \subseteq {}^{a_i}[C_S(g)^\star] \\ &\qquad \text{for all } i = 0,\dots,n, \end{aligned}$$

$$\begin{aligned}
&\Longleftrightarrow X = \frac{f}{g}, f, g \in F', [{}^{a_i}C_S(f)]^{\star_i} \subseteq [{}^{a_i}C_S(g)]^{\star_i} \\
&\qquad \text{for all } i = 0, \dots, n, \\
&\Longleftrightarrow X = \frac{f}{g}, f, g \in F', C_{R_i}\left(\frac{f}{X_i^m}\right)^{\star_i} \subseteq C_{R_i}\left(\frac{g}{X_i^m}\right)^{\star_i} \\
&\qquad \text{for all } i = 0, \dots, n \\
&\Longleftrightarrow X = \frac{f}{g} \in \bigcap_{i=0}^{n} \mathrm{Kr}\,(R_i, \star_i). \qquad \square
\end{aligned}$$

Definition 5.6. If $\star$ is an e.a.b. projective star operation on S, then $\mathrm{PKr}\,(S, \star)$ is called the *projective Kronecker function ring* of S with respect to $\star$.

Recall that we denote by F the quotient field of domain R_i, $i = 0, \dots, n$. Our next goal is to prove that $\mathrm{PKr}\,(S, \star)$ is an F-function ring with $\star$ an e.a.b. projective star operation on S.

It is easily seen that $\cap_{i=0}^{n}\mathrm{Kr}\,(R_i, \star_i)$ is an F-function ring, as an intersection of F-function rings.

By Theorem 5.5, we have:

Corollary 5.7. *Let $\star$ be an e.a.b. projective star operation on S. Then* $\mathrm{PKr}\,(S, \star)$ *is an F-function ring.*

Although $\mathrm{Kr}\,(F/K) := \cap_{V \in \Sigma(F/K)} V^b$ is not a Kronecker function ring in the classical sense, we prove that it is a projective Kronecker function ring of S, with respect to a suitable projective star operation $\star$ on S.

Proposition 5.8. *The F-function ring* $\mathrm{Kr}\,(F/K) = \cap_{i=0}^{n}\mathrm{Kr}\,(R_i, b_i)$.

Proof. We first remark that $\mathrm{Kr}\,(R_i, b_i) = \cap_{V \in \Sigma(F/R_i)} V^b$. Let $V \in \Sigma(F/R_i)$. Then $K \subseteq R_i \subseteq V \subseteq F$. Hence, $V \in \Sigma(F/K)$. Thus, $\mathrm{Kr}\,(F/K) \subseteq \mathrm{Kr}\,(R_i, b_i)$, for all $i = 0, \dots, n$. Hence, $\mathrm{Kr}\,(F/K) \subseteq \cap_{i=0}^{n}\mathrm{Kr}\,(R_i, b)$.

Now let $V \in \Sigma(F/K)$. We have $F \subseteq K' := K(X_0, \dots, X_n)$.

Let w be a valuation that extends the valuation v to K'. Pick j such that $w(X_j) = \min\{w(X_i) : 0 \leq i \leq n\}$. Then $w((X_i/X_j)) \geq 0$ for all $i = 0, \dots, n$. Hence, $R_j \subseteq W \cap F = V \subseteq F$. So $V \in \Sigma(F/R_j)$ for some j. Thus, $\cap_{V \in \Sigma(F/R_i)} V^b \subseteq \mathrm{Kr}\,(R_j, b) \subseteq V^b$. Hence, $\cap_{i=0}^n \mathrm{Kr}\,(R_i, b) \subseteq \mathrm{Kr}\,(F/K)$. $\square$

By Lemma 4.3, the b_i's are pairwise compatible e.a.b. star operations on the R_i's, and their homogenization is $sat \circ b$ which is projective and e.a.b. By Theorem 5.5, $\mathrm{PKr}\,(S, sat \circ b) = \cap_{i=0}^n \mathrm{Kr}\,(R_i, b_i) = \mathrm{Kr}\,(F/K)$. Thus:

Corollary 5.9. *The F-function ring* $\mathrm{Kr}\,(F/K) = \mathrm{PKr}\,(S, sat \circ b)$ *and is a projective Kronecker function ring.*

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