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ON THE EQUALITY OF ORDINARY AND SYMBOLIC POWERS OF IDEALS

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ABSTRACT. We consider the following question concerning the equality of ordinary and symbolic powers of ideals. In a regular local ring, if the ordinary and symbolic powers of a prime ideal are the same up to its height, then are they the same for all powers? We provide supporting evidence of a positive answer for classes of prime ideals defining monomial curves or rings of low multiplicities.

1. Introduction. Let R be a Noetherian local ring of dimension d, and let P be a prime ideal of R. For a positive integer n, the nth symbolic power of P, denoted by $P^{(n)}$, is defined as

$$P^{(n)} := P^n R_P \cap R = \{x \in R \mid \text{there exists an } s \in R \setminus P,\ sx \in P^n\}.$$

One readily sees from the definition that $P^n \subseteq P^{(n)}$ for all n, but they may not be equal in general. Comparing the ordinary and symbolic powers of ideals is a subject of interest in both commutative algebra and algebraic geometry, see for instance [**2, 8, 10–13, 15, 21**]. In this paper, we are interested in criteria for the equality. In particular, we would like to know if $P^n = P^{(n)}$ for all n up to some value implies that they are equal for all n. The following question was posed by Huneke in this regard.

Question 1.1. Let R be a regular local ring of dimension d, and let P be a prime ideal of height $d-1$. If $P^n = P^{(n)}$ for all $n \leq d-1$, then is $P^n = P^{(n)}$ for all n?

An affirmative answer to Question 1.1 is equivalent to P being generated by a regular sequence [**7**]. Furthermore, it is equivalent to showing that if $P^n = P^{(n)}$ for all $n \leq d-1$, then the analytic spread of P is $d-1$. This is not known even for the defining ideals of monomial

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curves $\mathsf{k}[[t^{a_1}, \dots, t^{a_d}]]$ in embedding dimension 4. Huneke answered Question 1.1 positively in dimension 3, and in dimension 4 if R/P is Gorenstein [**14**, Corollaries 2.5, 2.6]. One would like to remove the Gorenstein assumption. There are supporting examples showing that the Gorenstein property of R/P might follow from $P^2 = P^{(2)}$. In fact, this is very close to a conjecture by Vasconcelos which states that if P is syzygetic and R/P and P/P^2 are Cohen-Macaulay, then R/P is Gorenstein [**23**]. Note that, if P has height $d-1$, then R/P is Cohen-Macaulay, and P/P^2 being Cohen-Macaulay is equivalent to $P^2 = P^{(2)}$. Therefore, one is tempted to ask the following question.

Question 1.2. Let R be a regular local ring of dimension d and P a prime ideal of height $d-1$. If $P^2 = P^{(2)}$, then is R/P Gorenstein?

Note that, by Huneke's result, [**14**, Corollary 2.6], if Question 1.2 has an affirmative answer, then so does Question 1.1 when dimension of R is 4. The converse of Question 1.2 is true in dimension 4 by Herzog [**9**]. Also, Question 1.2 has been answered positively for some classes of algebras [**18**], but it is not true in general (see, for instance, [**18**, 6.1]). In this paper, we consider the case where P is the defining ideal of a monomial curve $\mathsf{k}[[t^{a_1}, \dots, t^{a_d}]]$ and we give an affirmative answer to Questions 1.1 and 1.2 when $d = 4$ and $\{a_i\}$ forms an arithmetic sequence. In higher dimensions, if $\{a_i\}$ contains an arithmetic subsequence of length 5 in which the terms are not necessarily consecutive, we observe that $P^2 \neq P^{(2)}$, hence we have a positive answer to Questions 1.1 and 1.2. We extend these results to certain modifications of arithmetic subsequences. We also consider one-dimensional prime ideals P of a regular local ring R in general and we show that if R/P has low multiplicity, then Question 1.1 has a positive answer.

2. Monomial curves. Let $a_1, \dots, a_d$ be an increasing sequence of positive integers with $\gcd(a_1, \dots, a_d) = 1$. Assume that the a_i's generate a numerical semigroup non-redundantly. Consider the monomial curve

$$A = \mathsf{k}[[t^{a_1}, \dots, t^{a_d}]] \subset \mathsf{k}[[t]]$$

over a field k, with maximal ideal $\mathfrak{m}_A := (t^{a_1}, \dots, t^{a_d})A$. Let $R = \mathsf{k}[[x_1, \dots, x_d]]$ be a formal power series ring with maximal ideal $\mathfrak{m} =$

$(x_1, \dots, x_d)R$, and let P be the kernel of the homomorphism

$$\mathsf{k}\,[[x_1, \dots, x_d]] \longrightarrow \mathsf{k}\,[[t^{a_1}, \dots, t^{a_d}]]$$

obtained by mapping x_i to t^{a_i} for all i. Therefore, A is isomorphic to R/P. Note that $P \subset \mathfrak{m}^2$ because of the non-redundancy assumption on the a_i's. We state the following well-known properties about monomial curves.

Lemma 2.1. *In the above setting,*

(1) *The ideal $t^{a_1}A$ is a minimal reduction of $\mathfrak{m}_A$.*

(2) *The Hilbert-Samuel multiplicity $e(\mathfrak{m}_A, A)$ of A is a_1.*

(3) *The multiplicity $e(\mathfrak{m}_A, A)$ is at least d, i.e., $a_1 \geq d$.*

For the third property above, we may assume that k is an infinite field. Then by [**1**, Fact (1)], we have $e(\mathfrak{m}_A, A) \geq \lambda(\mathfrak{m}_A/\mathfrak{m}_A^2)$, where $\lambda(-)$ denotes the length, and observe that $\lambda(\mathfrak{m}_A/\mathfrak{m}_A^2) = d$, by the non-redundancy condition on the a_i's. Note that the third property also follows from Theorem 3.1. We begin with the following result that describes the generators of P when $d = 4$ and the set of exponents $\{a_i\}$ forms an arithmetic sequence.

Proposition 2.2. *Let A be the monomial curve*

$$\mathsf{k}\,[\![t^a, t^{a+r}, t^{a+2r}, t^{a+3r}]\!],$$

where a and r are positive integers that are relatively prime. Consider A as R/P, where $R = \mathsf{k}\,[[x, y, z, w]]$ and P is the defining ideal of A. Then P is minimally generated by

$$\begin{cases} z^2 - yw,\ yz - xw,\ y^2 - xz,\ x^{k+r} - w^k, & \text{if } a = 3k \\[2ex] z^2 - yw,\ yz - xw,\ y^2 - xz,\ x^{k+r}z - w^{k+1}, & \\ x^{k+r}y - zw^k,\ x^{k+r+1} - yw^k, & \text{if } a = 3k+1 \\[2ex] z^2 - yw,\ yz - xw,\ y^2 - xz,\ x^{k+r+1} - zw^k, & \\ x^{k+r}y - w^{k+1}, & \text{if } a = 3k+2, \end{cases}$$

where k is a positive integer.

Proof. Since the numerical semigroup is non-redundantly generated, a is greater than or equal to 4 by Lemma 2. Thus $k \geq 2$ if $a = 3k$ and $k \geq 1$ if $a = 3k+1$ or $3k+2$. In each case, let I be the ideal generated by the above-listed elements and $\mathfrak{m}$ the maximal ideal (x, y, z, w) of R. One can directly check that $I \subset P$. For all cases, we will use the following method to show that $I = P$. First, we show that $(P, x) = (I, x)$. Then it follows that $P = I + x(P : x)$. But $(P : x) = P$, since x is not in P. Thus, $P = I + xP$, which implies $P = I$, by Nakayama's lemma. To show $(P, x) = (I, x)$, let $\widetilde{I} = (I, x)$. The short exact sequence

$$0 \longrightarrow R/(\widetilde{I} : y) \xrightarrow{\cdot y} R/\widetilde{I} \longrightarrow R/(\widetilde{I}, y) \longrightarrow 0$$

yields the length equation $\lambda_R(R/\widetilde{I}) = \lambda_R(R/(\widetilde{I} : y)) + \lambda_R(R/(\widetilde{I}, y))$. Since R/P is Cohen-Macaulay and the image of the ideal (x) in R/P is a minimal reduction of $\mathfrak{m}/P$ by Lemma 2.1, we have

$$a = e(\mathfrak{m}, R/P) = \lambda_R(R/(P, x)) \leq \lambda_R(R/\widetilde{I}).$$

Thus, it is enough to show

$$\lambda_R(R/(\widetilde{I} : y)) + \lambda_R(R/(\widetilde{I}, y)) \leq a.$$

If $a = 3k$, then $\widetilde{I} = (x, z^2 - yw, yz, y^2, w^k)$. Therefore, $(\widetilde{I}, y) = (x, y, z^2, w^k)$ and the ideal $\widetilde{I} : y$ contains the ideal (x, y, z, w^k). Thus,

$$\begin{aligned}\lambda_R(R/(\widetilde{I} : y)) + \lambda_R(R/(\widetilde{I}, y))& \\ &\leq \lambda_R(R/(x, y, z, w^k)) + \lambda_R(R/(x, y, z^2, w^k)) \\ &\leq k + 2k = a.\end{aligned}$$

If $a = 3k + 1$, then $\widetilde{I} = (x, z^2 - yw, yz, y^2, w^{k+1}, zw^k, yw^k)$. Hence, $(x, y, z, w^k) \subset \widetilde{I} : y$ and $(\widetilde{I}, y) = (x, y, z^2, zw^k, w^{k+1})$. Note that $\lambda_R(R/(x, y, z^2, zw^k, w^{k+1})) = 2k + 1$ and $\lambda_R(R/(x, y, z, w^k)) = k$. Thus,

$$\lambda_R(R/(\widetilde{I} : y)) + \lambda_R(R/(\widetilde{I}, y)) \leq k + (2k + 1) = a.$$

If $a = 3k + 2$, then $\widetilde{I} = (x, z^2 - yw, yz, y^2, zw^k, w^{k+1})$. Therefore, $(x, y, z, w^{k+1}) \subset \widetilde{I} : y$ and $(\widetilde{I}, y) = (x, y, z^2, zw^k, w^{k+1})$. Similar to the previous case, $\lambda_R(R/(x, y, z^2, zw^k, w^{k+1}))$ is $2k + 1$ and $\lambda_R(R/(x, y, z, w^{k+1})) = k + 1$. Hence, we obtain

$$\lambda_R(R/(\widetilde{I} : y)) + \lambda_R(R/(\widetilde{I}, y)) \leq (k + 1) + (2k + 1) = a.$$

To show that P is minimally generated by the listed elements in each case, we can compute $\mu(P) = \lambda_R(P/\mathfrak{m}P)$. In fact, if we let $\overline{R} = R/xR$, then

$$\mu(P\overline{R}) = \lambda_R(P\overline{R}/\mathfrak{m}P\overline{R}) = \lambda_R(P + (x)/\mathfrak{m}P + (x)) \\ = \lambda_R(P/\mathfrak{m}P + P \cap (x)).$$

But $P \cap (x) = x(P : x) = xP \subset \mathfrak{m}P$. Thus $\mu(P\overline{R}) = \mu(P)$. Therefore, to compute the minimal number of generators in each case, we can go modulo (x) first. If $a = 3k$, we will show in Theorem 2.3 that $P^2 \neq P^{(2)}$; hence, P is not a complete intersection ideal. Thus, it cannot have a fewer number of generators than 4. If $a = 3k + 2$, we will show in Theorem 2.3 that P is generated by the 4 by 4 Pfaffians of a 5 by 5 skew-symmetric matrix. Hence by the Buchsbaum-Eisenbud structure theorem for height 3 Gorenstein ideals in [**6**], P is minimally generated by the listed elements in this case. Thus, we only need to deal with the case $a = 3k + 1$, where one can check directly that the ideal $P\overline{R}$ is minimally generated by $z^2 - yw$, yz, y^2, w^{k+1}, zw^k, yw^k. □

Theorem 2.3. *Let A be the monomial curve $\mathsf{k}[[t^a, t^{a+r}, t^{a+2r}, t^{a+3r}]]$, where a and r are positive integers that are relatively prime. Regard A as R/P, where $R = \mathsf{k}[[x, y, z, w]]$ and P is the defining ideal of A.*

(1) *If $a = 3k$ or $3k + 1$, then R/P is not Gorenstein and $P^2 \neq P^{(2)}$.*

(2) *If $a = 3k+2$, then R/P is Gorenstein, $P^2 = P^{(2)}$ and $P^3 \neq P^{(3)}$.*

Proof. If $a = 3k$, then one can see that P contains the 2 by 2 minors of

$$M = \begin{bmatrix} x & y & z \\ y & z & w \\ zw^{k-1} & x^{k+r} & yx^{k+r-1} \end{bmatrix}.$$

Let $D = \det(M)$. Note that $D \notin P^2$, since D is not in P^2 modulo (x, y). We will show that $D \in P^{(2)}$. We have $\det(\operatorname{adj}(M)) \cdot D = D^3$, where $\operatorname{adj}(M)$ is the adjoint matrix of M. Note that $D \neq 0$; for example, it is not zero modulo (x, y). Thus $D^2 = \det(\operatorname{adj}(M))$. But $\det(\operatorname{adj}(M)) \in P^3$, since the entries of $\operatorname{adj}(M)$ are in P. Hence, $D^2 \in P^3$. Therefore, the image of D^2 in the associated graded ring $G_P := \operatorname{gr}_{PR_P}(R_P)$ is zero. Note that G_P is a domain as R_P is a regular local ring. Hence, the image of D is zero in G_P, which shows that the image of D in the localization R_P is in P^2R_P, i.e., $D \in P^{(2)}$. One could also directly show that $w \cdot \det(M) \in P^2$; hence, $\det(M) \in P^{(2)}$, as w is not in P. Now by Herzog's theorem [**9**, Satz 2.8], we conclude that R/P is not Gorenstein. Since, in Proposition 2.2, we have shown that P is minimally generated by 4 elements, we could also use the Buchsbaum-Eisenbud structure theorem for height 3 Gorenstein ideals in [**6**], or Bresinsky's result in [**3**] which states that if a monomial curve in dimension 4 is Gorenstein, then P is minimally generated by 3 or 5 elements.

If $a = 3k + 1$, then P contains the 2 by 2 minors of

$$M = \begin{bmatrix} x & y & z \\ y & z & w \\ zw^{k-1} & w^k & x^{k+r} \end{bmatrix}.$$

With a similar argument as in the previous case, one can show that $\det(M) \in P^{(2)} \setminus P^2$. Thus by Herzog's result, R/P is not Gorenstein.

If $a = 3k+2$, then by Proposition 2.2, one can see that P is generated by the 4×4 Pfaffians of

$$M = \begin{bmatrix} 0 & -w^k & 0 & x & y \\ w^k & 0 & x^{k+r} & y & z \\ 0 & -x^{k+r} & 0 & z & w \\ -x & -y & -z & 0 & 0 \\ -y & -z & -w & 0 & 0 \end{bmatrix}.$$

Thus, by the Buchsbaum-Eisenbud structure theorem for height 3 Gorenstein ideals in [**6**], we obtain that R/P is Gorenstein and P is minimally generated by the 5 listed elements in Proposition 2.2. Hence, $P^2 = P^{(2)}$ by Herzog's result [**9**, Satz 2.8], and $P^3 \neq P^{(3)}$ by Huneke's result [**14**, Corollary 2.6], as P is not a complete intersection ideal. □

Corollary 2.4. *Question* 1.1 *and Question* 1.2 *have affirmative answers for monomial curves as in Theorem* 2.3.

Now we consider monomial curves in higher dimensions.

Theorem 2.5. *Let A be the monomial curve $\mathsf{k}[[t^{a_1}, \dots, t^{a_d}]]$. Consider A as R/P, where $R = \mathsf{k}[[x_1, \dots, x_d]]$ and P is the defining ideal* *of A. If $\{a_i\}$ has an arithmetic subsequence of length 5, whose terms are not necessarily consecutive, then $P^2 \neq P^{(2)}$.*

Proof. If $\{a_i\}$ has an arithmetic subsequence $\{b_1, \dots, b_5\}$ of length 5, without loss of generality we may assume that $x_1, \dots, x_5$ correspond to $t^{b_1}, \dots, t^{b_5}$. Then, one can see that P contains the 2 by 2 minors of

$$M = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \\ x_3 & x_4 & x_5 \end{bmatrix}.$$

We observe that $\det(M) \notin P^2$, since $\det(M)$ is a polynomial of degree 3 and the generators of P^2 have degree at least 4 as $P \subset \mathfrak{m}^2$. Also note that $\det(M) \neq 0$; for example, it is not zero modulo (x_2, x_3). Thus, by a similar argument as in the proof of Theorem 2.3, one can show that $\det(M) \in P^{(2)}$. □

Corollary 2.6. *Question* 1.1 *and Question* 1.2 *have positive answers for monomial curves as in Theorem* 2.5.

Using a result of Morales [**20**, Lemma 3.2], we can extend Theorems 2.3 and 2.5 to a larger class of monomial curves. As before, let A be the monomial curve $\mathsf{k}[[t^{a_1}, \dots, t^{a_d}]]$. In the following, we will not assume any particular order on the a_i's. Write A as R/P, where R is $\mathsf{k}[[x_1, \dots, x_d]]$ and P is the defining ideal of A. For a positive integer c, relatively prime to a_1, let $\widetilde{A}$ be the modified monomial curve $\mathsf{k}[[t^{a_1}, t^{ca_2}, \dots, t^{ca_d}]]$. Note that $a_1, ca_2, \dots, ca_d$ non-redundantly generate their numerical semigroup too. Write $\widetilde{A}$ as $\widetilde{R}/\widetilde{P}$, where $\widetilde{R}$ denotes $\mathsf{k}[[x_1, \dots, x_d]]$ and $\widetilde{P}$ is the defining ideal of $\widetilde{A}$. Consider $\widetilde{R}$ as an R-module via the map $\phi : R \to \widetilde{R}$ that sends x_1 to x_1^c and fixes x_i for all $i \neq 1$. For a polynomial $f(x_1, \dots, x_d) \in R$, let $\widetilde{f}$ be the polynomial $f(x_1^c, \dots, x_d)$.

Lemma 2.7 [**20**]. *$\widetilde{R}$ is a faithfully flat extension of R. Moreover, $P\widetilde{R} \cap R = P$ and $\widetilde{P} = P\widetilde{R}$. In fact, $f \in P$ if and only if $\widetilde{f} \in \widetilde{P}$, and if $\{g_i\}$ is a minimal generating set for P, then $\{\widetilde{g_i}\}$ is a minimal generating set for $\widetilde{P}$. In addition, for all positive integers k, $f \in P^k$*

if and only if $\widetilde{f} \in \widetilde{P}^k$, *and* $f \in P^{(k)}$ *if and only if* $\widetilde{f} \in \widetilde{P}^{(k)}$, *i.e.*, $\widetilde{P}^k \cap R = P^k$ *and* $\widetilde{P}^{(k)} \cap R = P^{(k)}$.

Using Lemma 2.7, we obtain the following extension of Theorems 2.3 and 2.5.

Corollary 2.8. *If Question* 1.1 *has an affirmative answer for a monomial curve A, then it also has an affirmative answer for the monomial curve $\widetilde{A}$. In particular, Question* 1.1 *has an affirmative answer for successive modifications of the monomial curves as in Theorems* 2.3 *and* 2.5 *in the sense of Morales.*

Proof. If $\widetilde{P}^n = \widetilde{P}^{(n)}$ for all positive integers $n \leq d-1$, then by Lemma 2.7, we obtain that $P^n = P^{(n)}$ for all $n \leq d-1$. Thus, by hypothesis, P is a complete intersection and hence, by Lemma 2.7, we obtain that $\widetilde{P}$ is a complete intersection. □

3. Low multiplicities. Let R be a regular local ring with maximal ideal $\mathfrak{m}$ and of dimension d. Let P be a prime ideal of height $d-1$. We will show that Question 1.1 has an affirmative answer when the Hilbert-Samuel multiplicity $e(R/P)$ is sufficiently small.

Theorem 3.1. *Let R be a regular local ring with maximal ideal $\mathfrak{m}$ and of dimension d. Assume P is a prime ideal of height $d-1$ such that $P \subset \mathfrak{m}^2$. Then $P^n \neq P^{(n)}$ for a positive integer n, if*

$$e(R/P) < \prod_{r=0}^{d-2} \frac{2n+r}{n+r}.$$

Proof. We may assume the residue field of R is infinite, see for instance [**16**, Lemma 8.4.2]. Thus, as R/P has dimension one, there exists an $x \in R$ whose image in R/P is a minimal reduction of $\mathfrak{m}/P$. Note that x cannot be in $\mathfrak{m}^2$ by Nakayama's lemma; hence, $R/(x)$ is regular. Recall that, in a regular local ring S with maximal ideal $\mathfrak{n}$ and of dimension k, $\lambda_S(S/\mathfrak{n}^n) = \binom{n+k-1}{k}$ for all positive integers n. Therefore, since $P^n \subset \mathfrak{m}^{2n}$, we have $\lambda_R(R/(P^n, x)) \geq \lambda_R(R/(\mathfrak{m}^{2n}, x)) = \binom{2n+d-2}{d-1}$. On the other hand, since R/P is a one-dimensional Cohen-Macaulay ring, using the associativity formula for multiplicities, we obtain

$$\begin{aligned}\lambda_R(R/(P^{(n)},x)) &= e((x),R/P^{(n)})\\ &= \lambda_{R_P}(R_P/P^nR_P)\cdot e((x),R/P)\\ &= \binom{n+d-2}{d-1}\cdot e(R/P).\end{aligned}$$

The multiplicity bound in the statement is equivalent to

$$\binom{n+d-2}{d-1}\cdot e(R/P) < \binom{2n+d-2}{d-1}.$$

Therefore, $\lambda_R(R/(P^{(n)},x)) < \lambda_R(R/(P^n,x))$. Thus, P^n and $P^{(n)}$ cannot be the same. □

One can easily observe that the multiplicity bound in Theorem 3.1 is increasing with respect to n. Thus, letting $n = d-1$, we obtain the largest bound that guarantees $P^{d-1} \neq P^{(d-1)}$. Therefore, we have the following corollary.

Corollary 3.2. *Under the assumptions of Theorem* 3.1*, Question* 1.1 *has a positive answer provided*

$$e(R/P) < \prod_{r=0}^{d-2} \frac{2d+r-2}{d+r-1}.$$

Note that the multiplicity bound in Corollary 3.2 grows at least exponentially with respect to d, since each term of the product is greater than 3/2.

The next corollary is an application of Theorem 3.1 in the case of monomial curves in embedding dimension 4.

Corollary 3.3. *Let* $A = \mathsf{k}[[t^{a_1},t^{a_2},t^{a_3},t^{a_4}]]$. *Consider* A *as* R/P, *where* $R = \mathsf{k}[[x,y,z,w]]$ *and* P *is the defining ideal of* A. *If* $a_1 = 4$ *or* 5, *then* $P^3 \neq P^{(3)}$. *Therefore, Question* 1.1 *has a positive answer in this case.*

Proof. Apply Theorem 3.1 for $n = 3$ and $d = 4$. On the one hand $e(R/P) = a_1 \leq 5$, and on the other hand the multiplicity bound reduces to 5.6. Hence, $P^3 \neq P^{(3)}$. □

We remark that, by Corollary 2.8, Question 1.1 has an affirmative answer for successive modifications of the monomial curves as in Corollary 3.3 in the sense of Morales.

4. Remarks. We end this paper with some remarks and observations on equality of the ordinary and symbolic powers of ideals.

Remark 4.1. The multiplicity bound in Theorem 3.1 approaches 2^{d-1} as n tends to infinity. Thus, if $e(R/P) < 2^{d-1}$, then $P^n \neq P^{(n)}$ for n large. Hence, if Question 1.1 has a positive answer and $P^n = P^{(n)}$ for all $n \leq d-1$, then $e(R/P) \geq 2^{d-1}$. This is consistent with the conclusion of Question 1.1, that P is a complete intersection. To see this, suppose P is generated by a regular sequence $a_1, \ldots, a_{d-1}$ and x is a minimal reduction of $\mathfrak{m}/P$ in R/P. Then, by [**19**, Theorem 14.9], we have

$$e(\mathfrak{m}, R/P) = \lambda_R(R/(P,x)) = \lambda_R(R/(a_1, \ldots, a_d)) \geq \prod_{i=1}^{d} \operatorname{ord}_{\mathfrak{m}}(a_i) \geq 2^{d-1},$$

where $a_d = x$. Note that $\operatorname{ord}_{\mathfrak{m}}(x) = 1$ and $\operatorname{ord}_{\mathfrak{m}}(a_i) \geq 2$ for $i = 1, \ldots, d-1$, as we are assuming $P \subset \mathfrak{m}^2$.

Remark 4.2. We know that if $P^n = P^{(n)}$ for n large, then P is a complete intersection [**7**]. The conclusion is also true if $P^n = P^{(n)}$ for infinitely many n, see for instance Brodmann's result on the stability of associated primes of R/P^n in [**4**]. This can also be obtained by using superficial elements, at least when R has infinite residue field and P has positive grade. If $P^n = P^{(n)}$ for infinitely many n, then one can show that $P^n = P^{(n)}$ for n large. To see this, let $x \in P$ be a superficial element, in the sense that $P^{n+1} : x = P^n$ for n large, see [**16**, 8.5.7]. Hence, if there exists an element $b \in P^{(n)} \setminus P^n$, then we have $xb \in P^{(n+1)} \setminus P^{n+1}$ for n large.

Remark 4.3. If $P^n = P^{(n)}$ for n large, then the analytic spread of P is at most $d-1$ [**5**]. We note that this can also be seen via ε-multiplicity for one-dimensional primes. For a prime ideal P of height $d-1$, we have

$$H^0_{\mathfrak{m}}(R/P^n) = P^{(n)}/P^n,$$

where the left hand side is the zero-th local cohomology of R/P^n with support in $\mathfrak{m}$. Thus, if $P^n = P^{(n)}$ for n large, then ε-multiplicity of P is zero, where

$$\varepsilon(P) = \limsup_n \frac{d!}{n^d} \cdot \lambda_R(H^0_{\mathfrak{m}}(R/P^n)).$$

Hence, by [**17**, Theorem 4.7] or [**22**, Theorem 4.2], the analytic spread of P is at most $d-1$.

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