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PLANES OF THE FORM $b(X,Y)Z^n - a(X,Y)$ OVER A DVR

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ABSTRACT. In this paper we extend an epimorphism theorem of Wright to the case of discrete valuation rings. We will show that if (R,t) is a discrete valuation ring, $n \geq 2$ is an integer not divisible by the characteristic of the residue field R/tR, and $g \in R[X,Y,Z]$ is a polynomial of the form $g = b(X,Y)Z^n - a(X,Y)$ such that $R[X,Y,Z]/(g)$ is a polynomial algebra in two variables, then g and Z form a pair of variables in $R[X,Y,Z]$. We will also show that the result holds over any Noetherian domain containing $\mathbf{Q}$.

1. Introduction. For a commutative ring R with unity, let $R^{[n]}$ denote the polynomial ring in n variables. An important question in affine algebraic geometry is the following epimorphism problem:

Question 1. Let K be a field of characteristic 0. Let $g \in K[X,Y,Z]$ $\left(= K^{[3]}\right)$ be such that $K[X,Y,Z]/(g) = K^{[2]}$. Is then $K[X,Y,Z] = K[g]^{[2]}$?

While the problem is open in general, a few special cases have been investigated by Russell [**12**], Russell and Sathaye [**13**], Sathaye [**14**] and Wright [**17**]; in some of these cases, Question 1 has an affirmative answer even when K is a field of positive characteristic. In particular, they considered polynomials of the form $b(X,Y)Z^n - a(X,Y)$ and obtained affirmative answers when

(1) $n = 1$, K a field of characteristic 0 [**14**].

(2) $n = 1$, K a field of any characteristic [**12**].

(3) $n \geq 2$ and K an algebraically closed field of characteristic $p \geq 0$ with $p \nmid n$ [**17**].

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In this paper we shall first show (see Theorem 4.5) that the above result (3) of Wright holds even when K is not necessarily algebraically closed.

We now consider the corresponding question over a discrete valuation ring (to be abbreviated henceforth as DVR).

Question 2. Let (R, t) be a DVR containing $\mathbf{Q}$, and let $g \in R[X, Y, Z]$ $\left(= R^{[3]}\right)$ be such that $R[X, Y, Z]/(g) = R^{[2]}$. Is then $R[X, Y, Z] = R[g]^{[2]}$?

As shown by Bhatwadekar-Dutta in [**7**, Section 4], this problem is closely related to the problem of $\mathbf{A}^2$-fibration over a regular two-dimensional affine spot over a field of characteristic zero. Hence, one could explore Question 2 at least for polynomials like $g = b(X, Y)Z^n - a(X, Y)$ for which the corresponding Question 1 has been settled. For such polynomials, in view of the corresponding results over fields, one could extend the investigation of Question 2 even to the positive characteristic case.

The first investigation in this direction was made by Bhatwadekar-Dutta in [**6**]. They showed [**6**, Theorem 3.5] that Question 2 has an affirmative answer (in any characteristic) when $g = b(X, Y)Z - a(X, Y)$ with $t \nmid b(X, Y)$, thereby partially generalizing Sathaye's theorem on linear planes over a field [**14**].

The main aim of this paper is to show that Question 2 has an affirmative answer for polynomials of the form $g = b(X, Y)Z^n - a(X, Y)$, where $n \geq 2$ is an integer not divisible by the characteristic of R/tR, thereby obtaining a generalization of Wright's theorem [**17**] quoted in Section 2 (Theorem 2.1). More precisely, we will prove the following (see Theorem 5.3):

Theorem A. *Let (R, t) be a* DVR *with field of fractions K and residue field k. Let $g \in R[X, Y, Z]$ $\left(= R^{[3]}\right)$ be of the form $g = bZ^n - a$ where $a, b \in R[X, Y]$ with $b \neq 0$ and n is an integer ≥ 2 such that n is not divisible by the characteristic of k. Suppose that $R[X, Y, Z]/(g) = R^{[2]}$. Then $R[X, Y, Z] = R[g, Z]^{[1]}$, $R[X, Y] = R[a]^{[1]}$ and $b \in R[X_0]$ for some $X_0 \in R[X, Y]$ satisfying $K[X, Y] = K[X_0, a]$.*

The proof of Bhatwadekar-Dutta's theorem on linear planes over a DVR is highly technical. However, in the case of planes of the form $bZ^n - a$ with $n \geq 2$, the proof turns out to be much simpler due to the fact that g is a variable *along with* Z.

Using theorems on residual variables of Bhatwadekar-Dutta [**5**], one can also see that the result for $n \geq 2$ holds over any Noetherian domain containing $\mathbf{Q}$. We shall prove (see Theorem 6.2):

Theorem B. *Let R be a Noetherian domain containing $\mathbf{Q}$. Let $g \in R[X,Y,Z]$ $\left(= R^{[3]}\right)$ be of the form $g = bZ^n - a$ where $a, b \in R[X,Y]$ and n is an integer ≥ 2. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] = R[g,Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$.*

In fact Theorem 6.2 will show that the above result also holds over any Noetherian seminormal domain containing a field of characteristic $p \geq 0$, if $p \nmid n$.

In Section 2, we state some results which will be used subsequently; in Section 3, we review the case $n = 1$; in Sections 4 and 5, we prove our main results over a field and DVR, respectively; and in Section 6, we prove our result for rings containing a field.

2. Preliminaries. Throughout this paper all rings will be commutative with unity. For a ring R, we shall use the notation $A = R^{[n]}$ to mean that A is isomorphic, as an R-algebra, to a polynomial ring in n variables over R; the symbol R^* will denote the group of units of R. For a prime ideal P of R, $k(P)$ will denote the residue field R_P/PR_P. An integral domain R with field of fractions K is called *seminormal* if it satisfies the condition: an element $a \in K$ will belong to R if $a^2, a^3 \in R$.

We now state some results which will be used in our proofs. First we state the result of Wright [**17**, page 95] which we will generalize in Sections 4–6.

Theorem 2.1. *Let k be an algebraically closed field of characteristic $p \geq 0$. Let $g \in k[X,Y,Z]$ $\left(= k^{[3]}\right)$ be of the form $bZ^n - a$ where $a, b \in k[X,Y]$ with $b \neq 0$ and n is an integer ≥ 2 not divisible by p.*

Suppose that $k[X,Y,Z]/(g) = k^{[2]}$. Then there exist variables $\widetilde{X}, \widetilde{Y}$ in $k[X,Y]$ such that $a = \widetilde{Y}$, $b \in k[\widetilde{X}]$ and $k[X,Y,Z] = k[\widetilde{X}, g, Z]$.

We now state a version of the automorphism theorem by Jung [**11**] and van der Kulk [**16**] as presented in [**17**, Appendix, Theorems 2 and 3].

Theorem 2.2. *Let k be a field and $A = k[U,V](= k^{[2]})$. Let $GA_2(k)$ denote the group of k-automorphisms of A, $Af_2(k)$ the subgroup of $GA_2(k)$ defined by $Af_2(k) = \{(U,V) \mapsto (\alpha_1 U + \beta_1 V + \gamma_1, \alpha_2 U + \beta_2 V + \gamma_2) \mid \alpha_i, \beta_i, \gamma_i \in k \text{ and } \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0\}$, $\mathcal{E}_2(k)$ the subgroup of $GA_2(k)$ defined by $\mathcal{E}_2(k) = \{(U,V) \mapsto (\alpha U + h(V), \beta V + \gamma) \mid \alpha, \beta \in k^*, \gamma \in k \text{ and } h(V) \in k[V]\}$ and $Bf_2(k) = Af_2(k) \cap \mathcal{E}_2(k)$. Then $GA_2(k) = Af_2(k) *_{Bf_2(k)} \mathcal{E}_2(k)$. Moreover, if $\sigma \in GA_2(k)$ is of finite order, then there exists a $\tau \in GA_2(k)$ such that either $\tau\sigma\tau^{-1} \in Af_2(k)$ or $\tau\sigma\tau^{-1} \in \mathcal{E}_2(k)$.*

Now we state a result by Sathaye [**14**, Corollary 1] which we will use to prove Lemma 4.2.

Theorem 2.3. *Let L be a separable field extension of k. Assume that there exist $h \in k[X,Y]$ and $f_i \in L[X,Y]$, $1 \leq i \leq s$, such that*

(1) *$L[X,Y]/(f_i) = L^{[1]}$ for each i.*

(2) *$(f_i, f_j)L[X,Y] = L[X,Y]$ for $i \neq j$.*

(3) $h = \prod_{i=1}^{s} {f_i}^{r_i}$, $r_i > 0$.

Then there exist $f \in k[X,Y]$, $\lambda_i \in L^*$ and $\mu_i \in L$ such that $f_i = \lambda_i f + \mu_i$ for each i, $1 \leq i \leq s$.

We will also use the following special case of the result [**8**, Theorem 7].

Theorem 2.4. *Let k be a field, L a separable field extension of k, A a* UFD *containing k and B an A-algebra such that $B \otimes_k L = (A \otimes_k L)^{[1]}$. Then $B = A^{[1]}$.*

We will use the following version of a cancelation theorem due to Abhyankar et al. [**2**, Theorem 3.3].

Theorem 2.5. *Let A be an affine domain over a field k such that k is algebraically closed in A and* tr. $\deg_k(A) = 1$. *Suppose that B is another k-algebra such that $A^{[n]} = B^{[n]}$ for some $n \geq 1$. Then either $B = A$ or $B \cong A = k^{[1]}$.*

We now state a version of the Russell-Sathaye criterion [**13**, Theorem 2.3.1] for a ring to be a polynomial algebra over a subring (see [**6**, Theorem 2.6]).

Theorem 2.6. *Let $R \subset A$ be integral domains with A being finitely generated over R. Suppose that there exist primes $p_1, p_2, \dots, p_n$ in R such that for each i, $1 \leq i \leq n$,*

(1) *p_i remains prime in A,*

(2) *$p_iA \cap R = p_iR$,*

(3) *$A[1/(p_1p_2\cdots p_n)] = R[1/(p_1p_2\cdots p_n)]^{[1]}$ and*

(4) *R/p_iR is algebraically closed in A/p_iA.*

Then $A = R^{[1]}$.

The following result from [**6**, Lemma 2.5] will enable us to apply Theorem 2.6.

Lemma 2.7. *Let R be an integral domain, and let $F \in R[X,Y]$ $\big(= R^{[2]}\big)$ be such that $R[X,Y]/(F) = R^{[1]}$. Then $R[F]$ is algebraically closed in $R[X,Y]$.*

Finally, we state a result on residual variables which will be our main tool to prove Theorem B. It comes as a direct consequence of Theorem 3.1, Theorem 3.2 and Remark 3.4 in [**5**].

Theorem 2.8. *Let R be a Noetherian domain such that either R contains $\mathbf{Q}$ or R is seminormal, A be a polynomial algebra in n variables over R and $W_1, W_2, \dots, W_{n-1} \in A$. Then the following are equivalent:*

1. *$A = R[W_1, W_2, \dots, W_{n-1}]^{[1]}$.*

2. *$A \otimes_R k(P) = (R[W_1, W_2, \dots, W_{n-1}] \otimes_R k(P))^{[1]}$ for every prime ideal P of R.*

3. Planes of the form $bZ - a$. We recall below the earlier result on linear planes over a DVR [**6**, Theorem 3.5].

Theorem 3.1. *Let (R,t) be a* DVR*, and let $g \in R[X,Y,Z]$ $\left(= R^{[3]}\right)$ be of the form $g = bZ - a$ where $a, b \in R[X,Y]$ and $b \notin \mathrm{tR}\,[X,Y]$. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] = R[g]^{[2]}$.*

We now show that the result can be generalized to the case of Dedekind domain in the following form.

Theorem 3.2. *Let R be a Dedekind domain, and let $g \in R[X,Y,Z]$ $\left(= R^{[3]}\right)$ be of the form $g = bZ - a$ where $a, b \in R[X,Y]$ and the coefficients of b generate the unit ideal of R. Suppose that $B = R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] = R[g]^{[2]}$.*

Proof. By Theorem 3.1, $R_{\mathfrak{m}}[X,Y,Z] = R_{\mathfrak{m}}[g]^{[2]}$ for each maximal ideal $\mathfrak{m}$ of R. Hence, by [**3**], it follows that $R[X,Y,Z]$ is $R[g]$-isomorphic to the symmetric algebra $\mathrm{Sym}_{R[g]}(P)$ for some finitely generated projective $R[g]$-module P of rank two. Thus it is enough to show that P is a free $R[g]$-module. Since $R[g]$ is a retract of $R[X,Y,Z]$, it is enough to show that $P \otimes_{R[g]} R[X,Y,Z]$ is a free $R[X,Y,Z]$-module. Note that, since $R[X,Y,Z] \cong \mathrm{Sym}_{R[g]}(P)$, we have $\Omega_{R[g]}(R[X,Y,Z]) = P \otimes_{R[g]} R[X,Y,Z]$. Thus, the proof will be complete if we show that the projective $R[X,Y,Z]$-module $\Omega_{R[g]}(R[X,Y,Z])$ is actually free.

Now consider the exact sequence:

$$\Omega_R(R[g]) \otimes_{R[g]} (R[X,Y,Z]) \xrightarrow{\theta} \Omega_R(R[X,Y,Z]) \longrightarrow \Omega_{R[g]}(R[X,Y,Z]) \longrightarrow 0.$$

Let g_X, g_Y and g_Z denote the partial derivatives of g with respect to X, Y and Z respectively. Now note that $(g_X, g_Y, g_Z)R[X,Y,Z] = R[X,Y,Z]$. Since $\dim R = 1$, by Suslin's theorem [**15**, Theorem 2.6], the unimodular row $[g_X, g_Y, g_Z]$ can be completed to an invertible matrix. Since $\Omega_R(R[X,Y,Z])$ is a free $R[X,Y,Z]$-module of rank three with bases dX, dY and dZ, and since $\mathrm{Im}\,(\theta)$ is generated by $g_X dX + g_Y dY + g_Z dZ$, it now follows that $\Omega_{R[g]}(R[X,Y,Z]) (= \Omega_R(R[X,Y,Z])/\mathrm{Im}\,(\theta))$ is a free $R[X,Y,Z]$ module of rank two. This completes the proof. $\square$

Remark 3.3. Let (R,t) be a DVR containing $\mathbf{Q}$, and let $g = bZ - a$ where $b = tY^2$ and $a = -Y - tY(X+X^2) - t^2X$. Then $R[X,Y,Z]/(g) = R^{[2]}$ (see [**7**, Example 4.13]). In this example, $t \mid b$; and it is not yet known whether $R[X,Y,Z] = R[g]^{[2]}$.

4. Planes of the form $bZ^n - a$ over a field. In this section we will show that Wright's arguments in [**17**] can be modified to show that his result (Theorem 2.1) can be extended over any field. We first prove a few auxiliary results (Lemmas 4.1 and 4.2), then consider the case when the field k contains all nth roots of unity (Proposition 4.4) and finally show that Theorem 2.1 holds over any field (Theorem 4.5). We first record a result on $\mathrm{Aut}_k(k^{[2]})$.

Lemma 4.1. *Let k be a field of characteristic $p \geq 0$ and σ a k-automorphism of $B = k^{[2]}$ of order n such that $p \nmid n$. Suppose that k contains all the nth roots of unity. Then there exist elements $U, V \in B$ and $\alpha, \beta \in k^*$ such that $B = k[U,V]$, $\sigma(U) = \alpha U$ and $\sigma(V) = \beta V$, where $\alpha^n = \beta^n = 1$.*

Proof. By Theorem 2.2, one can choose coordinates U', V' of B such that either $\sigma \in \mathcal{E}_2(k)$ or $\sigma \in Af_2(k)$.

Case $\sigma \in \mathcal{E}_2(k)$. In this case $\sigma(U') = \alpha U' + \mu$ and $\sigma(V') = \beta V' + f_1(U')$, where $\alpha, \beta \in k^*$, $\mu \in k$ and $f_1(U') \in k[U']$. Since σ is of order n, we have $\alpha^n = \beta^n = 1$. Note that if $\alpha = 1$, then $U' = \sigma^n(U') = U' + n\mu$ and hence $\mu = 0$, as $p \nmid n$.

Set

$$U := \begin{cases} U' & \text{if } \alpha = 1. \\ U' + (\mu/(\alpha - 1)) & \text{if } \alpha \neq 1. \end{cases}$$

Then $k[U',V'] = k[U,V']$, $\sigma(U) = \alpha U$ and $\sigma(V') = \beta V' + f(U)$ for some $f(U) \in k[U]$. We will now show that we can choose $g(U) \in k[U]$ such that $\sigma(V' + g(U)) = \beta(V' + g(U))$. Let $f(U) = \sum_{i=0}^{r} a_i U^i$.

First we show that for any i, $1 \leq i \leq r$, if $a_i \neq 0$, then $\alpha^i \neq \beta$. Suppose $\beta = \alpha^i$. Now, from the relation $V' = \sigma^n(V')$, we get

$$\beta^{n-1} f(U) + \beta^{n-2} f(\alpha U) + \cdots + f(\alpha^{n-1} U) = 0,$$

which implies that

$$\beta^{n-1} a_i + \beta^{n-2} \alpha^i a_i + \beta^{n-3} \alpha^{2i} a_i + \cdots + \alpha^{(n-1)i} a_i = 0,$$

i.e., $n\beta^{n-1}a_i = 0$, and hence $a_i = 0$ (as $p \nmid n$ and $\beta \neq 0$). Thus $\alpha^i \neq \beta$ if $a_i \neq 0$.

Now, for each i, $1 \leq i \leq r$, we define b_i as follows:

$$b_i = \begin{cases} 0 & \text{if } a_i = 0. \\ a_i/(\beta - \alpha^i) & \text{if } a_i \neq 0. \end{cases}$$

Let $g(U) = \sum_{i=0}^{r} b_i U^i$ and set

$$V := V' + g(U).$$

Then $\sigma(V) = \beta V$. Thus $k[U', V'] = k[U, V]$, $\sigma(U) = \alpha U$ and $\sigma(V) = \beta V$.

Case $\sigma \in Af_2(k)$. In this case $\sigma(U') = \alpha_1 U' + \beta_1 V' + \gamma_1$ and $\sigma(V') = \alpha_2 U' + \beta_2 V' + \gamma_2$ for some $\alpha_i, \beta_i, \gamma_i \in k$ $(i = 1, 2)$ with $\alpha_1\beta_2 \neq \beta_1\alpha_2$. Let $\overline{k}$ be an algebraic closure of k. Choose $\lambda \in \overline{k}$ such that $(\alpha_1 - \lambda)(\beta_2 - \lambda) - \alpha_2\beta_1 = 0$. Then λ is an eigenvalue of the linear transformation $(X, Y) \mapsto (\alpha_1 X + \alpha_2 Y, \beta_1 X + \beta_2 Y)$ of $\overline{k}^2$. Let $(\nu_1, \nu_2) \in \overline{k}^2$ be a non-zero eigenvector corresponding to the eigenvalue λ. Then we have

$$\alpha_1\nu_1 + \alpha_2\nu_2 = \lambda\nu_1$$
$$\beta_1\nu_1 + \beta_2\nu_2 = \lambda\nu_2.$$

Therefore, $\sigma(\nu_1 U' + \nu_2 V') = \lambda(\nu_1 U' + \nu_2 V') + \mu$ where $\mu = \nu_1\gamma_1 + \nu_2\gamma_2$. Since σ is of order n, we have $\lambda^n = 1$, and hence $\lambda \in k^*$. Thus, we may choose $\nu_1, \nu_2 \in k$. Therefore, setting $U := \nu_1 U' + \nu_2 V'$, we have $\sigma(U) = \lambda U + \mu$, and hence $\sigma(V') = \kappa V' + h(U)$ for some $\kappa \in k^*$ and $h(U) \in k[U]$. Now, by taking U and V' to be the coordinates for B, the problem reduces to the previous case: $\sigma \in \mathcal{E}_2(k)$.

Thus, in both the cases we get $U, V \in B$ and $\alpha, \beta \in k^*$ such that $B = k[U, V]$, $\sigma(U) = \alpha U$ and $\sigma(V) = \beta V$. This completes the proof. □

We now record a consequence of Sathaye's result (Theorem 2.3).

Lemma 4.2. *Let k be a field, $B = k^{[2]}$ and $b \in B \setminus k$. Suppose that there exist a separable algebraic extension $E|_k$ and an element $X' \in B \otimes_k E$ such that $B \otimes_k E = E[X']^{[1]}$ and $b \in E[X']$. Then there exists an $X \in B$ such that $b \in k[X]$, $B = k[X]^{[1]}$ and $E[X'] = E[X]$.*

Proof. Without loss of generality, we assume $E|_k$ to be a finite Galois extension. Let $B = k[X_1, Y_1]$. Then $B \otimes_k E = E[X_1, Y_1] = E[X']^{[1]}$. Let $X' = \phi(X_1, Y_1)$. Interchanging X_1 and Y_1, if necessary, we may assume that the X_1-degree of $\phi(X_1, Y_1)$ is positive. Hence, the leading coefficient of X_1 in $\phi(X_1, Y_1)$ is a non-zero element $\lambda \in E$ [**1**, Proposition 11.12, page 85]. Let $X'' = X'/\lambda$.

Let $G = \{\sigma_i \mid i = 1, 2, \ldots, m\}$ be the group of k-automorphisms of $E|_k$. We extend each $\sigma \in G$ to a B-automorphism of $B \otimes_k E$. Let $\overline{k}$ be an algebraic closure of k containing E and $b = \prod_{i=1}^{s}(\lambda_i X'' + \mu_i)^{n_i}$ be the prime decomposition of b in $\overline{k}[X'']$, where $\lambda_i \in \overline{k}^*$, $\mu_i \in \overline{k}$ and $n_i \in \mathbf{N}$, $1 \le i \le s$. Since $\sigma(b) = b$ for each $\sigma \in G$, $b = \prod_{i=1}^{s}(\sigma(\lambda_i)\sigma(X'') + \sigma(\mu_i))^{n_i}$ is also a prime decomposition of b in $\overline{k}[X'']$. This shows that for each $\sigma \in G$, there exists an $\alpha \in \overline{k}^*$ and $\beta \in \overline{k}$ such that $\sigma(X'') = \alpha X'' + \beta$. Since X'' and $\sigma(X'')$ are both monic in X_1, it follows that $\alpha = 1$.

Since X'' is a variable of $B \otimes_k E$, we have $(B \otimes_k E)/(\sigma(X'')) = E^{[1]}$ for each $\sigma \in G$. It is also easy to see that if $\sigma_i(X'') \neq \sigma_j(X'')$ for $\sigma_i, \sigma_j \in G$, then $\sigma_i(X'')$ and $\sigma_j(X'')$ are comaximal in $B \otimes_k \overline{k}$ and hence comaximal in $B \otimes_k E$. Let $f_1, \ldots, f_t$ be the distinct elements of the set $\{\sigma(X'') \mid \sigma \in G\}$. Then, for each i, $1 \le i \le t$, there exists an $m_i \in \mathbf{N}$ such that $\prod_{\sigma \in G} \sigma(X'') = \prod_{i=1}^{t} f_i^{m_i} \in B$, $(B \otimes_k E)/(f_i) = E^{[1]}$, and for $i \neq j$, f_i and f_j are comaximal in $B \otimes_k L$. Since $B = k^{[2]}$, applying Theorem 2.3, we get that for each $\sigma \in G$ there exist $\lambda \in E^*$ and $\mu \in E$ such that $\lambda\sigma(X'') + \mu \in B$. Fix $\sigma \in G$ and let $X = \lambda\sigma(X'') + \mu \in B$. Then $E[X''] = E[\sigma(X'')] = E[X]$ and $b \in E[X] \cap B$. Since $B = k^{[2]}$ and $X \in B$, we have $E[X] \cap B = k[X]$. Hence, $b \in k[X] \subset B$. Now since $B = k^{[2]}$ and $B \otimes_k E = E[X'']^{[1]} = E[X]^{[1]}$, by Theorem 2.4, we see that $B = k[X]^{[1]}$. By construction, $E[X] = E[X''] = E[X']$. □

For convenience, we state below a result which follows from a lemma of Sathaye [**14**, Lemma 1].

Lemma 4.3. *Let k be a field, and suppose X' is a variable in $k[X_1, X_2, \ldots, X_n]$ $\left(= k^{[n]}\right)$ which is comaximal with X_1. Then $X' = \alpha X_1 + \beta$ with $\alpha, \beta \in k$, $\alpha \neq 0$.*

Proposition 4.4. *Let k be a field of characteristic $p \ge 0$ containing the nth roots of unity, and let $g \in k[X, Y, Z]$ $\left(= k^{[3]}\right)$ be of the form $bZ^n - a$ where $a, b \in k[X, Y]$ with $b \neq 0$ and n is an integer ≥ 2 not divisible by p. Suppose that $B := k[X, Y, Z]/(g) = k^{[2]}$. Then there* *exist variables U, V in B such that V is the image of Z in B, $b \in k[U]$ and $k[X,Y] = k[U,a] = k^{[2]}$.*

Proof. Let σ be the k-automorphism of B induced by the k-automorphism $\widetilde{\sigma}$ of $k[X,Y,Z]$ defined by $\widetilde{\sigma}((X,Y,Z)) = (X,Y,\omega Z)$ where ω is a primitive nth root of unity. Obviously, σ has order n.

Since $B = k^{[2]}$, by Lemma 4.1, there exist elements $U', V' \in B$ and $\alpha, \beta \in k^*$ such that $B = k[U',V']$, $\sigma(U') = \alpha U'$ and $\sigma(V') = \beta V'$, where $\alpha^n = \beta^n = 1$. Let $\mathfrak{z}$ be the image of Z in B and $A = k[X,Y][a/b]$. Then $\mathfrak{z}^n = a/b$ and $B = A[\mathfrak{z}] = k[X,Y][\mathfrak{z}] = A \oplus \mathfrak{z}A \oplus \mathfrak{z}^2 A \oplus \cdots \oplus \mathfrak{z}^{n-1}A$ so that, for any $x \in B$, $\mathfrak{z} \mid (x - \sigma(x))$. Thus $\mathfrak{z} \mid (1-\alpha)U'$ and $\mathfrak{z} \mid (1-\beta)V'$. But since U' and V' cannot have common (non-unit) factors and $\mathfrak{z} \notin k^*$, we have either $\alpha = 1$ or $\beta = 1$. Interchanging U' and V', if necessary, we assume that $\alpha = 1$. Then the ring of invariants of σ is $A = k[X,Y][a/b] = k[U', a/b] (= k^{[2]})$. Note that V' is a unit multiple of $\mathfrak{z}$. Thus $B = k[U', \mathfrak{z}]$. Set $V := \mathfrak{z}$.

Now we show that we can choose U from $k[X,Y]$ such that $B = k[U,V]$, $b \in k[U]$ and $k[X,Y] = k[U,a]$. If $b \in k^*$, then $k[X,Y] = k[X,Y][a/b] = k[U', a/b]$, so that, in this case, we may set $U := U'$. We now consider the case $b \notin k^*$. Let $p_1, p_2, \ldots, p_m$ be the distinct irreducible factors of b in $A (= k^{[2]})$, and set $\mathfrak{p}_i := k[X,Y] \cap p_i A$. Note that for each $i = 1, 2, \ldots, m$, both b and $a (= b.a/b) \in k[X,Y] \cap bA \subseteq \mathfrak{p}_i$. This shows that $(bZ^n - a)k[X,Y,Z] \subsetneqq \mathfrak{p}_i[Z]$ which implies $\operatorname{ht} \mathfrak{p}_i > 1$. Thus each $\mathfrak{p}_i$ is a maximal ideal of $k[X,Y]$. Let $\overline{k}$ denote an algebraic closure of k, L_i a subfield of $\overline{k}$ isomorphic to $k[X,Y]/\mathfrak{p}_i$, and let L be the subfield of $\overline{k}$ generated by the fields $L_1, L_2, \ldots, L_m$. Then L_i is an algebraic extension of k and $A/\mathfrak{p}_i A = (k[X,Y]/\mathfrak{p}_i)[\zeta_i] = L_i[\zeta_i]$ where ζ_i is the image of a/b in $A/\mathfrak{p}_i A$. Since $\mathfrak{p}_i A \subseteq p_i A$, it follows that ζ_i is transcendental over L_i and $\mathfrak{p}_i A$ is a prime ideal of A. As $\operatorname{ht} p_i A = 1$ and $\mathfrak{p}_i A \neq 0$, we have $p_i A = \mathfrak{p}_i A$. This shows that p_i are pairwise comaximal in A and hence in B.

Let $g(\zeta_i)$ be the image of U' in $A/p_i A = L_i[\zeta_i]$. Then $U' - g(a/b)$ is divisible by p_i in $A \otimes_k L_i$. But $U' - g(a/b) = U' - g(V^n)$ is a variable in both $A \otimes_k L_i$ and $B \otimes_k L_i$. Hence $U' - g(a/b)$ is a constant multiple of p_i. Thus, $A \otimes_k L_i = L_i[p_i, a/b]$, $B \otimes_k L_i = L_i[p_i, V]$, and for $i \neq j$, $(p_i, p_j)B \otimes_k L = B \otimes_k L$. Set $U := p_1$. Using Lemma 4.3, we have $p_i = \lambda_i U + \mu_i$ for $\lambda_i \in L^*$ and $\mu_i \in L$. So, we have

$b \in L[U]$. This shows that U is integral over $L[X,Y]$ and hence over $k[X,Y]$. As $U \in k[X,Y][a/b]$ and $k[X,Y]$ is a normal domain, we have $U \in k[X,Y]$. Since $L|_k$ is faithfully flat, it follows that $B = k[U,V]$ with $U \in k[X,Y], V = \mathfrak{z}$ and $b \in k[U]$. Now, the argument in [**17**, page 98] shows that $k[X,Y] = k[U,a]$. □

Theorem 4.5. *Let k be a field of characteristic $p \geq 0$, and let $g \in k[X,Y,Z]$ be of the form $bZ^n - a$ where $a, b \in k[X,Y]$ with $b \neq 0$ and n is an integer ≥ 2 not divisible by p. Suppose that $B := k[X,Y,Z]/(g) = k^{[2]}$ and identify $k[X,Y]$ with its image in B. Then there exist variables U, V in B such that V is the image of Z in B, $U \in k[X,Y]$, $b \in k[U]$, $k[X,Y] = k[U,a]$ and $k[X,Y,Z] = k[U,g,Z]$.*

Proof. Let E be the field obtained by adjoining all the nth roots of unity to k. Since $p \nmid n$, E is a Galois extension over k. By Proposition 4.4, we get variables U' and V' of $B \otimes_k E$ $\left(= k[X,Y,Z]/(g) = E^{[2]}\right)$ such that V' is the image of Z, $b \in E[U']$ and $E[X,Y] = E[U',a]$. As $E|_k$ is separable, we have $k[X,Y] = k[a]^{[1]}$ by Theorem 2.4. If $b \in k[X,Y] \backslash k$, then, by Lemma 4.2, we get $U \in k[X,Y]$ such that $k[X,Y] = k[U]^{[1]}$, $b \in k[U]$ and $E[U] = E[U']$. Since $E|_k$ is faithfully flat, $E[U',a] = E[U,a]$ and $k[U,a] \subseteq k[X,Y]$, we have $k[U,a] = k[X,Y]$. If $b \in k$, then we choose U to be any complementary variable of a in $k[X,Y]$.

From the relation $k[U,a] = k[X,Y]$, we have

$$k[X,Y,Z] = k[U,a,Z] = k[U, bZ^n - a, Z] = k[U,g,Z].$$

The relation $k[X,Y,Z] = k[U,g,Z]$ shows that B is generated by the images of U and Z. This completes the proof. □

Remark 4.6. Theorem 4.5 does not hold if $p \mid n$. Consider a field k of characteristic $p > 0$ and the polynomial $g = Z^{p^e} - Y - Y^{sp} \in k[Y,Z]$ where $p \nmid s$ and $e \geq 2$. Then $k[Y,Z]/(g) = k^{[1]}$ but $k[Y,Z] \neq k[g]^{[1]}$ (see [**1**, Example 9.12, page 72]). Using a result of Hamann [**10**, Theorem 2.6], it follows that $k[X,Y,Z] \neq k[g]^{[2]}$ although $k[X,Y,Z]/(g) = k^{[2]}$.

5. Planes of the form $bZ^n - a$ over a DVR. In this section we shall prove Theorem A. We first record two results on factorial domains.

Lemma 5.1. *Let R be a* UFD *with field of fractions K. Let $U \in R[X,Y]$ be such that $K[X,Y] = K[U]^{[1]}$. Then $K[U] \cap R[X,Y]$ is an inert subring of $R[X,Y]$ and $K[U] \cap R[X,Y] = R[W]$ $\left(= R^{[1]}\right)$, where W is an element of $R[X,Y]$ such that $K[W] = K[U]$.*

Proof. Let $D = K[U] \cap R[X,Y]$. Clearly, D is an inert subring of $R[X,Y]$ and hence a UFD of transcendence degree one over R. Therefore, by [**2**, Theorem 4.1], $D = R[W]$ $\left(= R^{[1]}\right)$ for some $W \in R[X,Y]$. Clearly, $K[W] = K[U]$. □

Lemma 5.2. *Let R be a* UFD *of characteristic $p \geq 0$ with field of fractions K. Let $g \in R[X,Y,Z]$ $\left(= R^{[3]}\right)$ be of the form $g = bZ^n - a$ where $a, b \in R[X,Y]$ with $b \neq 0$ and n is an integer ≥ 2 such that $p \nmid n$. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then*

(i) *$R[a] = K[a] \cap R[X,Y]$.*

(ii) *$R[a]$ is an inert subring of $R[X,Y]$.*

(iii) *$tR[X,Y] \cap R[a] = tR[a]$ for every $t \in R$.*

Proof. (i) By Theorem 4.5, $K[X,Y] = K[a]^{[1]}$ and, by Lemma 5.1, $K[a] \cap R[X,Y] = R[W]$ for some $W \in R[X,Y]$ satisfying $K[a] = K[W]$. It then follows that $a = \lambda W + \mu$ where $\lambda, \mu \in R$. We claim that $\lambda \in R^*$. Suppose $\lambda \notin R^*$. Let q be a prime factor of λ, and let L denote the algebraic closure of the field of fractions of R/qR. Let $\overline{a}$ and $\overline{b}$ denote the images of a and b, respectively, in $L[X,Y]$. Then we would have $\overline{a}(= \mu) \in L$; in fact, as $L[X,Y,Z]/(g) = L[X,Y,Z]/(\overline{b}Z^n - \overline{a}) = L^{[2]}$, we would have that $\overline{a}$ is a unit in L. Since $L[X,Y] \hookrightarrow L[X,Y,Z]/(\overline{b}Z^n - \overline{a})(= L^{[2]})$, it would follow that $\overline{b} \in L^*$. But then, as $n \geq 2$, $L[X,Y,Z]/(\bar{b}Z^n - \bar{a})$ would not be an integral domain, contradicting that $L[X,Y,Z]/(\bar{b}Z^n - \bar{a}) = L^{[2]}$. Thus $\lambda \in R^*$, and hence $R[a] = R[W] = K[a] \cap R[X,Y]$.

(ii) and (iii) follow from (i). □

We now prove Theorem A.

Theorem 5.3. *Let (R,t) be a* DVR *with residue field k, and let $p(\geq 0)$ be the characteristic of k. Let $g \in R[X,Y,Z]$ $\left(= R^{[3]}\right)$ be of the form $g = bZ^n - a$ where $a, b \in R[X,Y]$ with $b \neq 0$ and n is an* *integer* ≥ 2 *such that* $p \nmid n$. *Suppose that* $R[X,Y,Z]/(g) = R^{[2]}$. *Then* $R[X,Y,Z] = R[g,Z]^{[1]}$, $R[X,Y] = R[a]^{[1]}$ *and* $b \in R[X_0]$ *for some* $X_0 \in R[X,Y]$ *satisfying* $K[X,Y] = K[X_0, a]$.

Proof. Let K and k denote, respectively, the field of fractions and the residue field of (R,t). For any $f \in R[X,Y,Z]$, let $\overline{f}$ denote the image of f in $k[X,Y,Z]$. By hypothesis, $K[X,Y,Z]/(bZ^n - a) = K^{[2]}$ and $k[X,Y,Z]/(\overline{b}Z^n - \overline{a}) = k^{[2]}$. Hence, by Theorem 4.5, $K[X,Y] = K[a]^{[1]}$ and $K[X,Y,Z] = K[Z, bZ^n - a]^{[1]}$.

If $t \nmid b$, then, by Theorem 4.5, $k[X,Y,Z] = k[Z,\overline{g}]^{[1]}$ and $k[X,Y] = k[\overline{a}]^{[1]}$. Hence, by Theorem 2.6, we get $R[X,Y,Z] = R[g,Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$.

We now consider the case $t \mid b$. Now $\overline{g} = \overline{a}$ so that

$$k[X,Y,Z]/(\overline{a})\ (= k[X,Y]/(\overline{a}))^{[1]} \ = k[X,Y,Z]/(\overline{g}) = k^{[2]}.$$

Hence, by Theorem 2.5, $k[X,Y]/(\overline{a}) = k^{[1]}$. Therefore, by Lemma 2.7, we see that $k[\overline{a}]$ is algebraically closed in $k[X,Y]$. Since t is prime in both $R[a](= R^{[1]})$ and $R[X,Y]$, and since a is a generic variable of $R[X,Y]$, using Theorem 2.6, we see that $R[X,Y] = R[a]^{[1]}$. By a similar argument, we have $R[X,Y,Z] = R[g,Z]^{[1]}$.

Now, by Theorem 4.5, one can choose $U \in R[X,Y]$ such that $K[X,Y] = K[U,a]$ and $b \in K[U]$. By Lemma 5.1, $K[U] \cap R[X,Y] = R[X_0]$ for some $X_0 \in R[X,Y]$ satisfying $K[U] = K[X_0]$. Thus, $b \in R[X_0]$ where $K[X_0, a] = K[U,a] = K[X,Y]$. Hence, the result. ◻

Note that, in the case R is a $\mathbf{Q}$-algebra, the hypothesis in Theorem 5.3 regarding n ($p \nmid n$) is automatically satisfied. Thus, in particular, Theorem 5.3 holds when R is a DVR containing $\mathbf{Q}$. In the next section we shall see a generalization of this result (Theorem 6.2).

Remark 5.4. Note that, in the notation of Theorem 5.3, X_0 need not be a variable in $R[X,Y]$. Consider a DVR (R,t). Let $g = bZ^n - a$ where $a = -Y$ and $b = t^2X + tY^2$, and let $X_0 = tX + Y^2$. Then $R[X,Y,Z]/(g) = R^{[2]}$, $b \in R[X_0]$, $K[X,Y] = K[X_0,Y]$ but $R[X,Y] \neq R[X_0]^{[1]}$.

The following example shows that, without the hypothesis $p \nmid n$, Theorem 5.3 need not hold even over a DVR of characteristic 0.

Example 5.5. Let $R = \mathbf{Z}_{(p)}$ where p is a prime in $\mathbf{Z}$, $K = Qt(R) = \mathbf{Q}$ and $k = R/pR = \mathbf{Z}/p\mathbf{Z}$. Let $a = Y^p + Y + pX$ and $g = Z^p - a \in R[X,Y,Z]$. Then $R[X,Y,Z] = R[g]^{[2]}$; in particular, $R[X,Y,Z]/(g) = R^{[2]}$. But $R[X,Y] \neq R[a]^{[1]}$.

Proof. Let $Z' = Z - Y$. Then $R[X,Y,Z] = R[X,Y,Z']$ and $g = Z'^p - pf(Z',Y) - Y - pX$ for some $f \in R[Z',Y]$. Let $D = R[g,Z']$. We have $K[X,Y,Z] = K[g,Y,Z] = K[g,Z']^{[1]}$ and $k[X,Y,Z] = k[\overline{g},X,Z'] = k[\overline{g},Z']^{[1]}$ where $\overline{g}$ denotes the image of g in $k[X,Y,Z]$. Since p is prime in R, p is prime in both $R[X,Y,Z]$ and D. Hence, by Theorem 2.6, $R[X,Y,Z] = D^{[1]} = R[g]^{[2]}$. Let $\overline{a}$ denote the image of a in $k[X,Y]$. Since $k[\overline{a}] = k[Y + Y^p]$ is not algebraically closed in $k[X,Y]$, $\overline{a}$ is not a variable in $k[X,Y]$ and hence a is not a variable in $R[X,Y]$. □

However the next result shows that Theorem 5.3 extends to any DVR (R,t) of characteristic 0 (without assuming that the characteristic of R/tR does not divide n), if the element a is such that $(R/tR)[\overline{a}]$ is algebraically closed in $(R/tR)[X,Y]$.

Proposition 5.6. *Let (R,t) be a* DVR *of characteristic* 0 *with residue field k, and let $g \in R[X,Y,Z]$ $\left(= R^{[3]}\right)$ be of the form $g = bZ^n - a$ where $a,b \in R[X,Y]$, $b \neq 0$ and n is an integer ≥ 2. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$ and $k[\overline{a}]$ is algebraically closed in $k[X,Y]$. Then $R[X,Y] = R[a]^{[1]}$ and $R[X,Y,Z] = R[Z,g]^{[1]}$.*

Proof. We see that $R[1/t][X,Y] = R[1/t][a]^{[1]}$ by Theorem 4.5, t is prime in both $R[a]$ and $R[X,Y]$, $tR[X,Y] \cap R[a] = tR[a]$ by Lemma 5.2 and $(R/tR)[\overline{a}]$ is algebraically closed in $(R/tR)[X,Y]$ by hypothesis. Hence, by Theorem 2.6, $R[X,Y] = R[a]^{[1]}$. Let $B := R[X,Y,Z]/(g)$ $\left(= R^{[2]}\right)$, and denote the image of Z in B by $\mathfrak{z}$. Then $B/(\mathfrak{z}) = R[X,Y,Z]/(Z, bZ^n - a) = R[X,Y]/(a) = R^{[1]}$, and hence, by the generalized epimorphism theorem of Bhatwadekar [**4**, Theorem 3.7], we have $B = R[\mathfrak{z}]^{[1]}$. Let $C = R[Z]$. Identifying the image of Z in B

with Z itself, we have $C[X,Y]/(g) = C^{[1]}$. Since C is a normal domain of characteristic 0, again by Bhatwadekar's result [**4**, Theorem 3.7], we have $C[X,Y] = C[g]^{[1]}$, i.e., $R[X,Y,Z] = R[g,Z]^{[1]}$. □

In view of Example 5.5, we ask:

Question 5.7. Let (R,t) be a DVR of characteristic 0 such that the characteristic of the residue field is positive, say p. Let $g = bZ^{pm} - a \in R[X,Y,Z]$ be such that $R[X,Y,Z]/(g) = R^{[2]}$ where $a, b(\neq 0) \in R[X,Y]$ and $m \geq 1$. Is then $R[X,Y,Z] = R[g]^{[2]}$?

6. Planes of the form $bZ^n - a$ over rings containing a field. In this section we prove a generalized version of Theorem B (Theorem 6.2). We shall essentially follow the approach of Bhatwadekar in [**4**] and then apply the result on residual variables (Theorem 2.8). We first state a result which will be needed in the proof of Theorem 6.2.

Lemma 6.1. *Let R be a Noetherian domain, and let $b\ (\neq 0) \in R$. Then, for each non-zero prime ideal P of R, there exists a discrete valuation ring V with maximal ideal $\mathfrak{m}_V$ together with a homomorphism $\phi : R \longrightarrow V$ such that $\phi(b) \neq 0$, $\phi^{-1}(\mathfrak{m}_V) = P$ and $V/\mathfrak{m}_V$ is algebraic over $k(P)$.*

Proof. Let P be a non-zero prime ideal of R, and let n be the height of P. Since R is a Noetherian domain, there exists a prime ideal Q of R of height $n-1$ such that $Q \subsetneqq P$ and $b \notin Q$. Let $D = R/Q$ and $\mathfrak{p} = P/Q$ be the image of P in D. Let C be the normalization of D and $\mathcal{P}$ a prime ideal of C lying over $\mathfrak{p}$. Set $V := C_{\mathcal{P}}$ and $\mathfrak{m}_V := \mathcal{P}V$, the maximal ideal of the local ring V. Since the height of $\mathfrak{p}$ (and hence that of $\mathcal{P}$) is one, V is a DVR. Now let ϕ denote the composite map $R \to D(= R/Q) \to C \to V(= C_{\mathcal{P}})$. Clearly, $\phi^{-1}(\mathfrak{m}_V) = P$, $\phi(b) \neq 0$ and $V/\mathfrak{m}_V$ is algebraic over $k(P)$. □

We now prove the main result of this section.

Theorem 6.2. *Let R be a Noetherian domain containing a field of characteristic $p \geq 0$. Let $g \in R[X,Y,Z]$ $\left(= R^{[3]}\right)$ be of the form*

$bZ^n - a$ where $a, b \in R[X,Y]$, $b \neq 0$, and n is an integer ≥ 2 such that $p \nmid n$. Suppose that $R[X,Y,Z]/(g) = R^{[2]}$. Then $R[X,Y,Z] \otimes_R k(P) = (R[g,Z] \otimes_R k(P))^{[1]}$ and $R[X,Y] \otimes_R k(P) = (R[a] \otimes_R k(P))^{[1]}$ for all $P \in \operatorname{Spec}(R)$. Thus, if R contains $\mathbf{Q}$ or if R is seminormal, then $R[X,Y,Z] = R[g,Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$.

Proof. Fix $P \in \operatorname{Spec}(R)$. Let the images of b, a and g in $R[X,Y,Z] \otimes_R k(P)$ be $\overline{b}$, $\overline{a}$ and $\overline{g}$, respectively. Let k denote $k(P)$. We show that $k[X,Y] = k[\overline{a}]^{[1]}$ and $k[X,Y,Z] = k[\overline{g},Z]^{[1]}$.

If $\operatorname{ht} P = 0$, we are done by Theorem 4.5. So we assume that $\operatorname{ht} P = n \geq 1$. If $\overline{b} \neq 0$, then by Theorem 4.5, we are through. So we assume that $\overline{b} = 0$ (and hence $\overline{g} = \overline{a}$).

Using Lemma 6.1, we have a DVR (V,π) with a homomorphism $\phi : R \to V$ such that $\phi(b) \neq 0$, $\phi^{-1}(\pi) = P$ and $V/(\pi)$ is algebraic over $k(P)$. Note that $V[X,Y,Z]/(g) = V^{[2]}$, and hence, by Theorem 5.3, we have $V[X,Y,Z] = V[g,Z]^{[1]}$ and $V[X,Y] = V[a]^{[1]}$; in particular, $(V/(\pi)[X,Y,Z] = V/(\pi))[\overline{g},Z]^{[1]}$ and $(V/(\pi)[X,Y] = V/(\pi))[\overline{a}]^{[1]}$. Now, since $(k[X,Y]/(\overline{a}))[Z] = k^{[2]}$, by Theorem 2.5, we have $k[X,Y]/(\overline{a}) = k^{[1]}$. Since $V/(\pi)$ is algebraic over k and since $(V/(\pi)[X,Y] = V/(\pi))[\overline{a}]^{[1]}$, by [**9**, Proposition 1.16], we have $k[X,Y] = k[\overline{a}]^{[1]}$ and hence $k[X,Y,Z] = k[\overline{g},Z]^{[1]}$.

Thus,

$$R[X,Y,Z] \otimes_R k(P) = (R[g,Z] \otimes_R k(P))^{[1]}$$

and

$$R[X,Y] \otimes_R k(P) = (R[a] \otimes_R k(P))^{[1]}$$

for all $P \in \operatorname{Spec}(R)$.

Now, if R is seminormal or contains $\mathbf{Q}$, then $R[X,Y,Z] = R[g,Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$ by Theorem 2.8. □

Remark 6.3. (1) If R is seminormal or R contains $\mathbf{Q}$, then under the hypotheses of Theorem 6.2, one can show, by suitable reductions, that $R[X,Y,Z] = R[g,Z]^{[1]}$ and $R[X,Y] = R[a]^{[1]}$, even when R is non-Noetherian.

(2) If R is a UFD, then the proof of Theorem 5.3 shows that, under the hypotheses of Theorem 6.2, there exists an $X_0 \in R[X,Y]$ such that $b \in R[X_0]$ and $K[X,Y] = K[X_0, a]$. However, unlike in Theorem 5.3 (Theorem A), in the situation of Theorem 6.2 (even for Theorem B), there may not exist $X_0 \in R[X,Y]$ for which $b \in R[X_0]$ and $K[X,Y] = K[X_0, a]$. Consider the following example.

Example 6.4. Let k be any field, and let R be the normal affine k-domain $R = k[U,V,W]/(UV - W^2)$, where U,V,W are indeterminates over k. Let u,v,w denote, respectively, the images of U,V,W in R. Let K denote the field of fractions of R. Now set

$$\begin{aligned} X_1 &:= uX + wY, \\ a &:= Y, \\ b &:= {X_1}^3/u = u^2X^3 + 3uwX^2Y + 3uvXY^2 + vwY^3, \\ g &:= bZ^n - a, \text{ where } n \text{ is any positive integer, and} \\ X_2 &:= (X_1 + wg)/u = X + (uwX^3 + 3w^2X^2Y + 3vwXY^2 + v^2Y^3)Z^n. \end{aligned}$$

We first show that $R[X,Y,Z] = R[g,Z,X_2]$; in particular, $R[X,Y,Z]/(g) = R^{[2]}$. Let $B = R[g,Z,X_2]$ and $A = R[X,Y,Z]$. Then $B \subseteq A$ and

$$B[1/u] = R[1/u][g,Z,X_1] = R[1/u][Z,X_1,Y] = R[1/u][X,Y,Z] = A[1/u].$$

Note that $B = R^{[3]}$. Set

$$\begin{aligned} H &:= -g(1 + vwg^2Z^n) \quad \text{and} \\ G &:= (X_2 - v^2H^3Z^n)(1 - 3vwH^2Z^n). \end{aligned}$$

Then $H, G \in B$ and, under the canonical map $\phi : B \to A/uA$, we have $\phi(H) = Y$ and $\phi(G) = X$ in A/uA. It follows that the induced map

$$\overline{\phi} : B/uB \longrightarrow A/uA$$

is surjective and hence an isomorphism because both B/uB and A/uA are $(R/uR)^{[3]}$. In particular, $uA \cap B = uB$. Thus, $B = A$ (cf. [**6**, Lemma 2.1]).

Now suppose, if possible, that there exists an $X_0 \in R[X,Y]$ for which $b \in R[X_0]$ and $K[X,Y] = K[X_0, a] = K[X_0, Y]$. Then $K[X_0, Y] =$

$K[X_1, Y]$. Since $b \in (K[X_0] \cap K[X_1]) \setminus K$, it would follow that $K[X_0] = K[X_1]$. Hence, we would have $c, d \in K$ such that $X_0 = cX_1 + d = cuX + cwY + d$. But as $X_0 \in R[X, Y]$, we would have $cu, cw, d \in R$. Let $cu = r$. Replacing X_0 by $X_0 - d$, we have $X_0 = cX_1$ and $b = {X_1}^3/u = (u^2/r^3){X_0}^3$. Since $b \in R[X_0]$, we get $u^2 \in r^3R$. Since R is a graded ring with $R_0 = k$ and u is a homogeneous element in R of degree 1, it would follow that $rR = R$. But as $cw(= rw/u) \in R$, we would then have $w \in uR$, which does not hold. Thus there does not exist such an X_0.

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