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SOME PROPERTIES AND APPLICATIONS OF F-FINITE F-MODULES

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ABSTRACT. M. Hochster's work in [**7**] has shown that F-finite F-modules over regular local rings have finitely many F-submodules. In this paper we apply this theorem to prove that morphisms of F-finite F-modules have a particularly simple form and we also show that there exist finitely many submodules compatible with a given Frobenius near-splitting thus generalizing a similar result in [**1**] to the case where the base ring is not F-finite.

1. Introduction. The purpose of this paper is to describe several applications of finiteness properties of F-finite F-modules recently discovered by Hochster in [**7**] to the study of Frobenius maps on injective hulls, Frobenius near-splittings and to the nature of morphisms of F-finite F-modules.

Throughout this paper (R, m) shall denote a complete regular local ring of prime characteristic p. At the heart of everything in this paper is the Frobenius map $f : R \to R$ given by $f(r) = r^p$ for $r \in R$. We can use this Frobenius map to define a new R-module structure on R given by $r \cdot s = r^p s$; we denote this R-module $F_* R$. We can then use this to define the *Frobenius functor* from the category of R-modules to itself: given an R-module M we define $F_R(M)$ to be $F_* R \otimes_R M$ with R-module structure given by $r(s \otimes m) = rs \otimes m$ for $r, s \in R$ and $m \in M$. Henceforth we shall abbreviate F_R to F for the sake of readability.

Let $R[\Theta; f]$ be the skew polynomial ring which is the free R-module $\oplus_{i=0}^{\infty} R\Theta^i$ with multiplication $\Theta r = r^p \Theta$ for all $r \in R$. As in [**8**], $\mathcal{C}$ shall denote the category of $R[\Theta; f]$-modules which are Artinian as R-modules. For any two such modules M, N, we denote the morphisms between them in $\mathcal{C}$ with $\operatorname{Hom}_{R[\Theta;f]}(M, N)$; thus an element $g \in \operatorname{Hom}_{R[\Theta;f]}(M, N)$ is an R-linear map such that $g(\Theta a) = \Theta g(a)$ for

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all $a \in M$. The first main result of this paper (Theorem 3.3) shows that under some conditions on N, $\operatorname{Hom}_{R[\Theta;f]}(M,N)$ is a finite set.

An *F-module* (cf. the seminal paper [**10**] for an introduction to F-modules and their properties) over the ring R is an R-module $\mathcal{M}$ together with an R-module isomorphism $\theta_{\mathcal{M}} : \mathcal{M} \to F(\mathcal{M})$. This isomorphism $\theta_{\mathcal{M}}$ is the *structure morphism* of $\mathcal{M}$.

A *morphism of F-modules* $\mathcal{M} \to \mathcal{N}$ is an R-linear map g which makes the following diagram commute

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad g \quad} & \mathcal{N} \\ {\scriptstyle \theta_{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle \theta_{\mathcal{N}}} \\ F(\mathcal{M}) & \xrightarrow{F(g)} & F(\mathcal{N}) \end{array}$$

where $\theta_{\mathcal{M}}$ and $\theta_{\mathcal{N}}$ are the structure isomorphisms of $\mathcal{M}$ and $\mathcal{N}$, respectively. We denote $\operatorname{Hom}_{\mathcal{F}}(\mathcal{M},\mathcal{N})$ the R-module of all morphism of F-modules $\mathcal{M} \to \mathcal{N}$.

Given any finitely generated R-module M and R-linear map $\beta : M \to F(M)$ one can obtain an R-module

$$\mathcal{M} = \varinjlim \left(M \xrightarrow{\beta} F(M) \xrightarrow{F(\beta)} F^2(M) \xrightarrow{F^2(\beta)} \cdots \right).$$

Since

$$F(\mathcal{M}) = \varinjlim \left(F(M) \xrightarrow{F(\beta)} F^2(M) \xrightarrow{F^2(\beta)} F^3(M) \xrightarrow{F^3(\beta)} \cdots \right) = \mathcal{M},$$

we obtain an isomorphism $\mathcal{M} \cong F(\mathcal{M})$, and hence $\mathcal{M}$ is an F-module. Any F-module which can be constructed as a direct limit as $\mathcal{M}$ above is called an *F-finite* F-module with *generating morphism* β.

There is a close connection between $R[\Theta;f]$-modules and F-finite F-modules given by *Lyubeznik's functor* from $\mathcal{C}$ to the category of F-finite F-modules which is defined as follows (see [**10**, Section 4] for the details of the construction.) Given an $R[\Theta;f]$-module M one defines the R-linear map $\alpha : F(M) \to M$ by $\alpha(r \otimes m) = r\Theta m$; an application of Matlis duality then yields an R-linear map $\alpha^\vee : M^\vee \to F(M)^\vee \cong F(M^\vee)$ and one defines

$$\mathcal{H}(M) = \varinjlim \left(M^\vee \xrightarrow{\alpha^\vee} F(M^\vee) \xrightarrow{F(\alpha^\vee)} F^2(M^\vee) \xrightarrow{F^2(\alpha^\vee)} \cdots \right).$$

Since M is an Artinian R-module, $M^\vee$ is finitely generated and $\mathcal{H}(M)$ is an F-finite F-module with generating morphism $M^\vee \overset{\alpha^\vee}{\to} F(M^\vee)$. This construction is functorial and results in an exact contravariant functor from $\mathcal{C}$ to the category of F-finite F-modules.

Later in this paper we will need the following related constructions. Following [**8**] we shall denote $\mathcal{D}$ as the category of all R-linear maps $M \to F(M)$ where M is any finitely generated R-module, and where a morphism between $M \overset{a}{\to} F(M)$ and $N \overset{b}{\to} F(N)$ is a commutative diagram of R-linear maps

$$\begin{array}{ccc} M & \xrightarrow{\mu} & N \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ F(M) & \xrightarrow{F(\mu)} & F(N) \end{array}$$

Section 3 of [**8**] constructs a pair of functors $\Delta : \mathcal{C} \to \mathcal{D}$ and $\Psi : \mathcal{D} \to \mathcal{C}$ with the property that for all $L \in \mathcal{C}$, the $R[\Theta; f]$-module $\Psi \circ \Delta(L)$ is canonically isomorphic to L and for all $D = (B \overset{u}{\to} F(B)) \in \mathcal{D}$, $\Delta \circ \Psi(D)$ is canonically isomorphic to D. The functor Δ amounts to the "first step" in the construction of Lyubeznik's functor $\mathcal{H}$: for $L \in \mathcal{C}$ we define the R-linear map $\alpha : F(L) \to L$ to be the one given above, and we let $\Delta(L)$ be the map $\alpha^\vee : L^\vee \to F(L)^\vee \cong F(L^\vee)$ (cf. Section 3 in [**8**] for the details of the construction).

The main result in [**7**] is the surprising fact that for F-finite F-modules $\mathcal{M}$ and $\mathcal{N}$, $\operatorname{Hom}_{\mathcal{F}}(\mathcal{N}, \mathcal{M})$ is a finite set. In Section 3 of this paper we exploit this fact to prove the second main result in this paper (Theorem 3.4) to show the following. Let $\gamma : M \to F(M)$ and $\beta : N \to F(N)$ be generating morphisms for $\mathcal{M}$ and $\mathcal{N}$. Given an R-linear map g which makes the following diagram commute,

$$\begin{array}{ccc} N & \xrightarrow{\beta} & F(N) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F(g)} \\ M & \xrightarrow{\gamma} & F(M) \end{array}$$

one can extend that diagram to

$$\begin{array}{ccccccc} N & \xrightarrow{\beta} & F(N) & \xrightarrow{F(\beta)} & F^2(N) & \xrightarrow{F^2(\beta)} & \cdots \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F(g)} & & \downarrow{\scriptstyle F^2(g)} & & \\ M & \xrightarrow{\gamma} & F(M) & \xrightarrow{F(\gamma)} & F^2(M) & \xrightarrow{F^2(\gamma)} & \cdots \end{array}$$

and obtain a map between the direct limits of the horizontal sequences, i.e., an element in $\operatorname{Hom}_{\mathcal{F}}(\mathcal{N},\mathcal{M})$. We prove that all elements in $\operatorname{Hom}_{\mathcal{F}}(\mathcal{N},\mathcal{M})$ arise in this way (cf. Theorem 3.4); thus morphisms of F-finite F-modules have a particularly simple form. This answers a question implicit in [**10**, Remark 1.10(b)].

Finally, in Section 4 we consider the module $\operatorname{Hom}_R(F_* R^n, R^n)$ of *near-splittings* of $F_* R^n$. We establish a correspondence between these near-splittings and Frobenius actions on E^n which enables us to prove the third main result in this paper (Theorem 4.5) which asserts that, given a near-splitting ϕ corresponding to an injective Frobenius action, there are finitely many $F_* R$-submodules $V \subseteq F_* R^n$ such that $\phi(V) \subseteq V$. This generalizes a similar result in [**1**] to the case where R is not F-finite.

Our study of Frobenius near-splittings is based on the study of its dual notion, i.e., Frobenius maps on the injective hull $E = E_R(R/m)$ of the residue field of R. This injective hull is given explicitly as the module of inverse polynomials $\mathbb{K}[[x_1^-,\ldots,x_d^-]]$ where $x_1,\ldots,x_d$ are minimal generators of the maximal ideal of R (cf. [**3**, Section 12.4]). Thus E has a natural $R[T;f]$-module structure extending $T\lambda x_1^{-\alpha_1}\cdots x_1^{-\alpha_d} = \lambda^p x_1^{-p\alpha_1}\cdots x_d^{-p\alpha_d}$ for $\lambda \in \mathbb{K}$ and $\alpha_1,\ldots,\alpha_d > 0$. We can further extend this to a natural $R[T;f]$-module structure on E^n given by

$$T\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} Ta_1 \\ \vdots \\ Ta_n \end{pmatrix}.$$

Throughout this paper T will denote this natural Frobenius map, while Θ will be used for general Frobenius maps.

The results of Section 4 will follow from the fact that there is a dual correspondence between Frobenius near-splittings and sets of $R[\Theta;f]$-module structures on E^n.

2. Frobenius maps of Artinian modules and their stable submodules. Given an Artinian R-module M we can embed M in E^α for some $\alpha \geq 0$ and extend this inclusion to an exact sequence

$$0 \longrightarrow M \longrightarrow E^\alpha \xrightarrow{A^t} E^\beta \longrightarrow \cdots,$$

where $A^t \in \operatorname{Hom}_R(E_R^\alpha, E_R^\beta)$. In our setup Matlis duality gives $\operatorname{Hom}_R(E_R, E_R) \cong R$ and so $A^t \in \operatorname{Hom}_R(E_R^\alpha, E_R^\beta) \cong \operatorname{Hom}_R(R^\alpha, R^\beta)$ is a $\beta \times \alpha$ matrix with entries in R. Henceforth in this section we will describe certain properties of Artinian R-modules in terms of their representations as kernels of matrices with entries in R. We shall denote $\mathbf{M}_{\alpha,\beta}$ to be the set of $\alpha \times \beta$ matrices with entries in R, and for any such matrix A we will write $A^{[p]}$ to denote the matrix obtained by raising each of its entries to the pth power.

We now explore the duality between E^α with a given $R[\Theta; f]$-module structure and R-linear maps $R^\alpha \to R^\alpha$ for $\alpha \geq 1$ given by the functors Δ and Ψ defined in Section 1. Under this duality the $R[\Theta; f]$-module structure corresponding to the map $(R^\alpha \to R^\alpha) \in \mathcal{D}$ given by multiplication by $B \in \mathbf{M}_{\alpha,\alpha}$ is given by $\Theta = B^t T$ where T is the natural Frobenius map on E^α described in Section 1.

Proposition 2.1. *Let $M = \ker A^t \subseteq E^\alpha$ be an Artinian R-module where $A \in \mathbf{M}_{\alpha,\beta}$. Let $\mathbf{B} = \{B \in \mathbf{M}_{\alpha,\alpha} \mid \operatorname{Im} BA \subseteq \operatorname{Im} A^{[p]}\}$. For any $R[\Theta; f]$-module structure on M, $\Delta(M)$ can be identified with an element in $Hom_R(Coker\, A, Coker\, A^{[p]})$ and thus represented by multiplication by some $B \in \mathbf{B}$. Conversely, any such B defines an $R[\Theta; f]$-module structure on M which is given by the restriction to M of the Frobenius map $\phi : E^\alpha \to E^\alpha$ defined by $\phi(v) = B^t T(v)$ where T is the natural Frobenius map on E^α.*

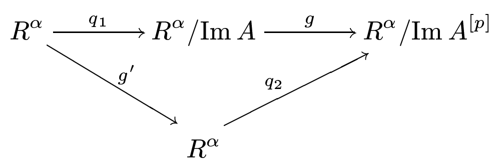
Proof. Matlis duality gives an exact sequence $R^\beta \xrightarrow{A} R^\alpha \to M^\vee \to 0$; hence,

$$\Delta(M) \in \operatorname{Hom}_R(M^\vee, F_R(M^\vee)) \cong \operatorname{Hom}_R(\operatorname{Coker} A, \operatorname{Coker} A^{[p]}).$$

Let $\Delta(M)$ be the map $g : \operatorname{Coker} A \to \operatorname{Coker} A^{[p]}$.

In view of Theorem 3.1 in [**8**] we only need to show that any such R-linear map is given by multiplication by an $B \in \mathbf{B}$, and that any such B defines an element in $\Delta(M)$.

Using the freeness of R^α, we find a map g' which makes the following diagram



commute, where q_1 and q_2 are quotient maps. The map g' is given by multiplication by some $\alpha \times \alpha$ matrix $B \in \mathbf{B}$. Conversely, any such matrix B defines a map g making the diagram above commute, and $\Psi(g)$ gives a $R[\Theta; f]$-module structure on M as described in the last part of the proposition. $\square$

Notation 2.2. We shall henceforth describe Artinian R-modules with a given $R[\Theta; f]$-module structure in terms of the two matrices in the statement of Proposition 2.1 and talk about Artinian R-modules $M = \operatorname{Ker} A^t \subseteq E^\alpha$ where $A \in \mathbf{M}_{\alpha,\beta}$ with $R[\Theta; f]$-module structure given by $B \in \mathbf{M}_{\alpha,\alpha}$.

5. Morphisms in $\mathcal{C}$. In this section we raise two questions. The first of these asks when for given $R[\Theta; f]$-modules M, N, the set $\operatorname{Hom}_{R[\Theta;f]}(M, N)$ is finite; later in this section we prove that this holds when N has no Θ-torsion. The following two examples illustrate why this set is not finite in general, and why it is finite in a special simple case.

Example 3.1. Let $\mathbb{K}$ be an infinite field of prime characteristic p, and let $R = \mathbb{K}[[x]]$. Let $M = \operatorname{ann}_E xR$, and fix an $R[\Theta; f]$-module structure on M given by $\Theta a = x^p T a$ where T is the standard Frobenius action on E. Note that $\Theta M = 0$ and that for all $\lambda \in \mathbb{K}$ the map $\mu_\lambda : M \to M$ given by multiplication by λ is in $\operatorname{Hom}_{R[\Theta;f]}(M, M)$, and hence this set is infinite.

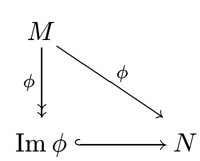
Example 3.2. Let $I, J \subseteq R$ be ideals, and fix $u \in (I^{[p]} : I)$ and $v \in (J^{[p]} : J)$. Endow $\operatorname{ann}_E I$ and $\operatorname{ann}_E J$ with $R[\Theta; f]$-module structures given by $\Theta a = uTa$ and $\Theta b = vTb$ for $a \in \operatorname{ann}_E I$ and $b \in \operatorname{ann}_E J$ where T is the standard Frobenius map on E.

If $g : \operatorname{ann}_E I \to \operatorname{ann}_E J$ is R-linear, an application of Matlis duality yields $g^\vee : R/J \to R/I$, and we deduce that g is given by multiplication by an element in $w \in (I : J)$. If in addition $g \in \operatorname{Hom}_{R[\Theta;f]}(\operatorname{ann}_E I, \operatorname{ann}_E J)$, we must have $wuTa = g(\Theta a) = \Theta g(a) = vTwa = vw^pTa$, for all $a \in \operatorname{ann}_E I$, hence $(vw^p - uw)T\operatorname{ann}_E I = 0$ and $vw^p - uw \in I^{[p]}$. The finiteness of $\operatorname{Hom}_{R[\Theta;f]}(\operatorname{ann}_E I, \operatorname{ann}_E J)$ translates in this setting to the finiteness of the set of solutions modulo $I^{[p]}$ for the variable w of the equation above, and it is not clear why this set should be finite. However, if we simplify to the case where $I = J = 0$, the set of solutions of $vw^p - uw = 0$ over the fraction field of R has at most p elements, and in this case we can deduce that $\operatorname{Hom}_{R[\Theta;f]}(E, E)$ also has at most p elements.

As in [**10**], for any $R[\Theta; f]$-module M we define the *submodule of nilpotent elements* to be $\operatorname{Nil}(M) = \{a \in M \mid \Theta^e a = 0 \text{ for some } e \geq 0\}$. We recall that when M is an Artinian R-module and there exists an $\eta \geq 0$ such that $\Theta^\eta \operatorname{Nil}(M) = 0$ (cf. [**6**, Proposition 1.11] and [**10**, Proposition 4.4]). We also define $M_{\mathrm{red}} = M/\operatorname{Nil}(M)$ and $M^* = \cap_{e \geq 0} R\Theta^e M$ where $R\Theta^e M$ denotes the R-module generated by $\{\Theta^e a \mid a \in M\}$. We also note that when M is an $R[\Theta; f]$-module which is Artinian as an R-module, there exists an $e \geq 0$ such that $M^* = R\Theta^e M$ and also $(M_{\mathrm{red}})^* = (M^*)_{\mathrm{red}}$ (cf. [**9**, Section 4]).

Theorem 3.3. *Let M, N be $R[\Theta; f]$-modules. Let $\phi \in \operatorname{Hom}_{R[\Theta;f]}(M, N)$. We have $\mathcal{H}(\operatorname{Im}\phi) = 0$ if and only if $\phi(M) \subseteq \operatorname{Nil}(N)$ and, consequently, if $\operatorname{Nil}(N) = 0$, the map $\mathcal{H} : \operatorname{Hom}_{R[\Theta;f]}(M, N) \to \operatorname{Hom}_{\mathcal{F}_R}(\mathcal{H}(N), \mathcal{H}(M))$ is an injection and $\operatorname{Hom}_{R[\Theta;f]}(M, N)$ is a finite set.*

Proof. We apply $\mathcal{H}$ to the commutative diagram



to obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{H}(N) & \twoheadrightarrow & \mathcal{H}(\operatorname{Im}\phi) \\ & \searrow^{\mathcal{H}(\phi)} & \downarrow \\ & & \mathcal{H}(M). \end{array}$$

Now $\mathcal{H}(\phi) = 0$ if and only if $\mathcal{H}(\operatorname{Im}\phi) = 0$, and by [**10**, Theorem 4.2] this is equivalent to $(\operatorname{Im}\phi)^*_{\text{red}} = 0$.

Choose $\eta \geq 0$ such that $\Theta^\eta \operatorname{Nil}(N) = 0$ and choose $e \geq 0$ such that $(\operatorname{Im}\phi)^* = R\Theta^e \operatorname{Im}\phi$.

Now
$$\begin{aligned} (\operatorname{Im}\phi)^*_{\text{red}} = 0 &\iff R\Theta^\eta R\Theta^e \phi(M) = 0 \\ &\iff R\Theta^{\eta+e}\phi(M) = 0 \\ &\iff \operatorname{Im}\phi \subseteq \operatorname{Nil}(N) \end{aligned}$$

The second statement now follows immediately. □

The second main result in this section, Theorem 3.4, shows that all morphisms of F-finite F-modules arise as images of maps of $R[\Theta; f]$-modules under Lyubeznik's functor $\mathcal{H}$.

Theorem 3.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be F-finite F-modules. For every $\phi \in \operatorname{Hom}_{\mathcal{F}_R}(\mathcal{N}, \mathcal{M})$ there exist generating morphisms $\gamma : M \to F(M) \in \mathcal{D}$ and $\beta : N \to F(N) \in \mathcal{D}$ for $\mathcal{M}$ and $\mathcal{N}$, respectively, and a morphism (in the category $\mathcal{D}$)*

$$\begin{array}{ccc} N & \xrightarrow{\beta} & F(N) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F(g)} \\ M & \xrightarrow{\gamma} & F(M) \end{array}$$

such that $\phi = \mathcal{H}(\Psi(g))$, i.e., such that ϕ is the map of direct limits

$$\begin{array}{ccccccc} N & \xrightarrow{\beta} & F(N) & \xrightarrow{F(\beta)} & F^2(N) & \xrightarrow{F^2(\beta)} & \cdots \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle F(g)} & & \downarrow{\scriptstyle F^2(g)} & & \\ M & \xrightarrow{\gamma} & F(M) & \xrightarrow{F(\gamma)} & F^2(M) & \xrightarrow{F^2(\gamma)} & \cdots \end{array}$$

Proof. Choose any generating morphisms

$$\mathcal{N} = \varinjlim \left(N \xrightarrow{\beta} F(N) \xrightarrow{F(\beta)} F^2(N) \xrightarrow{F^2(\beta)} \cdots \right)$$

and

$$\mathcal{M} = \varinjlim \left(M \xrightarrow{\gamma} F(M) \xrightarrow{F(\gamma)} F^2(M) \xrightarrow{F^2(\gamma)} \cdots \right)$$

and fix any $\phi \in \operatorname{Hom}_{\mathcal{F}_R}(\mathcal{N}, \mathcal{M})$.

For all $j \geq 0$ let ϕ_j be the restriction of ϕ to the image of $F^j(N)$ in $\mathcal{N}$.

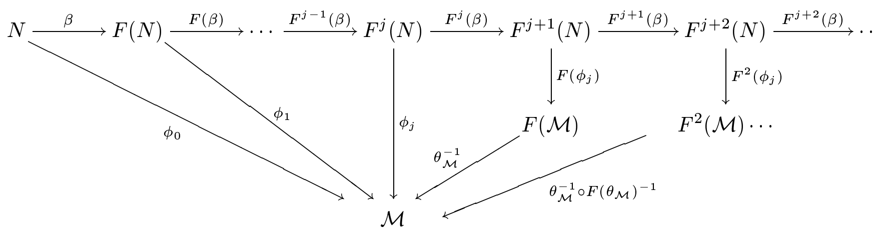
The fact that ϕ is a morphism of F-modules implies that for every $j \geq 0$ we have a commutative diagram

$$\begin{array}{ccc} F^j(N) & \xrightarrow{F^j(\beta)} & F^{j+1}(N) \\ \downarrow & & \downarrow \\ \mathcal{N} & \xrightarrow{\theta_{\mathcal{N}}} & F(\mathcal{N}) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle F(\phi)} \\ \mathcal{M} & \underset{\theta_{\mathcal{M}}}{\xrightarrow{\cong}} & F(\mathcal{M}) \end{array}$$

where $\theta_{\mathcal{M}}$ and $\theta_{\mathcal{N}}$ are the structure isomorphisms of $\mathcal{M}$ and $\mathcal{N}$, respectively, and where the compositions of the vertical maps are ϕ_j and $F(\phi_j)$. Repeated applications of the Frobenius functor yields a commutative diagram

$$\begin{array}{ccccc} F^j(N) & \xrightarrow{F^j(\beta)} & F^{j+1}(N) & \xrightarrow{F^{j+1}(\beta)} & \cdots \\ \downarrow{\scriptstyle \phi_j} & & \downarrow{\scriptstyle F(\phi_j)} & & \\ \mathcal{M} & \underset{\theta_{\mathcal{M}}}{\xrightarrow{\cong}} & F(\mathcal{M}) & \underset{F(\theta_{\mathcal{M}})}{\xrightarrow{\cong}} & \cdots \end{array}$$

and we can now extend this commutative diagram to the left to obtain



This commutative diagram defines an R-linear map $\psi_j : \mathcal{N} \to \mathcal{M}$. Furthermore, we show next that this ψ_j is a map of $\mathcal{F}$-modules, i.e., that for all $j \geq 0$, $F(\psi_j) \circ \theta_{\mathcal{N}} = \theta_{\mathcal{M}} \circ \psi_j$. Fix $j \geq 0$ and abbreviate $\psi = \psi_j$.

Pick any $a \in \mathcal{N}$ represented as an element of $F^e(N)$. If $e < j$, then the fact that ϕ is a morphism of F-modules implies that

$$\theta_{\mathcal{M}} \circ \psi(a) = \theta_{\mathcal{M}} \circ \phi(a) = F(\phi) \circ \theta_{\mathcal{N}}(a) = F(\psi) \circ \theta_{\mathcal{N}}(a).$$

Assume now that $e \geq j$; we have

$$\begin{aligned}\theta_{\mathcal{M}} \circ \psi(a) &= \theta_{\mathcal{M}} \circ \theta_{\mathcal{M}}^{-1} \circ F(\theta_{\mathcal{M}}^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\phi_j)(a)\\ &= F(\theta_{\mathcal{M}}^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\phi_j)(a)\end{aligned}$$

and

$$\begin{aligned}F(\psi) \circ \theta_{\mathcal{N}}(a) &= F\left(\theta_{\mathcal{M}}^{-1} \circ F(\theta_{\mathcal{M}}^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1})\right.\\ &\quad \left.\circ\, F^{e-j}(\phi_j)\right)\left(F^e(\beta)(a)\right)\\ &= F(\theta_{\mathcal{M}}^{-1})\\ &\quad \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e+1-j}(\phi_j)\left(F^e(\beta)(a)\right)\\ &= F(\theta_{\mathcal{M}}^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\theta_{\mathcal{M}}^{-1})\\ &\quad \circ F^{e-j}(\theta_{\mathcal{M}}) \circ F^{e-j}(\phi_j)(a)\\ &= F(\theta_{\mathcal{M}}^{-1}) \circ \cdots \circ F^{e-1-j}(\theta_{\mathcal{M}}^{-1}) \circ F^{e-j}(\phi_j)(a)\end{aligned}$$

where the penultimate inequality follows from the fact that ϕ is a morphism of F-modules.

Consider now the set $\{\psi_i\}_{i \geq 0}$; it is a finite set according to Theorem 5.1 in [**7**]; hence, we can find a sequence $0 \leq i_1 < i_2 < \cdots$

such that $\psi_{i_1} = \psi_{i_2} = \cdots$. By replacing $\mathcal{N}$ and $\mathcal{M}$ with $F^{i_1}(\mathcal{N})$ and $F^{i_1}(\mathcal{M})$ we may assume that $i_1 = 0$.

Pick $j \geq 0$ so that ϕ maps the image of N in $\mathcal{N}$ into $F^j(M)$. Since $\mathcal{M} \cong F^j(\mathcal{M})$ we may replace $\mathcal{M}$ with $F^j(\mathcal{M})$ and assume that $\phi(\operatorname{Im} N) \subseteq M$ and hence also that for all $e \geq 0$, $F^e(\phi)$ maps the image of $F^e(N)$ in $\mathcal{N}$ into $F^e(M)$.

Fix now any $e \geq 0$ and pick any $i_k > e$; the fact that $\psi_0 = \psi_{i_k}$ implies that for all $a \in F^e(N)$, $F^e(\phi_0)(a) = \psi_0(a) = \psi_{i_k}(a) = \phi(a)$ and since this holds for all $e \geq 0$ we deduce that ϕ is induced from the commutative diagram

$$\begin{array}{ccccccc} N & \xrightarrow{\beta} & F(N) & \xrightarrow{F(\beta)} & F^2(N) & \xrightarrow{F^2(\beta)} & \cdots \\ \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle F(\phi_0)} & & \downarrow{\scriptstyle F^2(\phi_0)} & & \\ M & \xrightarrow{\gamma} & F(M) & \xrightarrow{F(\gamma)} & F^2(M) & \xrightarrow{F^2(\gamma)} & \cdots \end{array}$$

An application of the functor Ψ to the leftmost square in the commutative diagram above yields a morphism of $R[\Theta; f]$-modules $g : M \to N$ and $\phi = \mathcal{H}(g)$. $\square$

4. Applications to Frobenius splittings. For any R-module M let F_*M denote the additive Abelian group M with R-module structure given by $r \cdot a = r^p a$ for all $r \in R$ and $a \in M$. In this section we study the module $\operatorname{Hom}_R(F_*R^n, R^n)$ of *near-splittings* of F_*R^n. Given such an element $\phi \in \operatorname{Hom}_R(F_*R^n, R^n)$ we will describe the submodules $V \subseteq F_*R^n$ for which $\phi(V) \subseteq V$. These submodules in the case $n = 1$, known as *ϕ-compatible ideals*, are of significant importance in algebraic geometry (cf. [**22**] for a study of applications of Frobenius splittings and their compatible submodules in algebraic geometry.) We will prove that under some circumstances these form a finite set and thus generalize a result in [**1**] obtained in the F-finite case.

We first exhibit the following easy implication of Matlis duality necessary for the results of this section.

Lemma 4.1. *For any* (*not necessarily finitely generated*) *R-module M,* $\operatorname{Hom}_R(M, R) \cong \operatorname{Hom}_R(R^\vee, M^\vee)$.

Proof. For all $a \in E$ let $h_a \in \operatorname{Hom}_R(R, E)$ denote the map sending 1 to a.

For any $\phi \in \operatorname{Hom}_R(M, R)$, $\phi^\vee \in \operatorname{Hom}_R(R^\vee, M^\vee)$ is defined as $(\phi^\vee(h_a))(m) = \phi(m)a$ for any $m \in M$ and $a \in E$. For any $\psi \in \operatorname{Hom}_R(R^\vee, M^\vee)$ we define $\widetilde{\psi} \in \operatorname{Hom}_R(M, R) \cong \operatorname{Hom}_R(M, E^\vee)$ as $(\widetilde{\psi}(m))(a) = (\psi(h_a))(m)$ for all $a \in E$ and $m \in M$. Note that the function $\psi \mapsto \widetilde{\psi}$ is R-linear.

Let $\psi \in \operatorname{Hom}_R(R^\vee, M^\vee)$ and fix an $m \in M$. Note that for all $a \in E$

$$\widetilde{\psi}^\vee(h_a)(m) = \widetilde{\psi}(m)a$$

when we view $\widetilde{\psi}$ as an element in $\operatorname{Hom}_R(M, R)$. After we identify $\operatorname{Hom}_R(M, E^\vee)$ with $\operatorname{Hom}_R(M, R)$ we can write

$$\widetilde{\psi}^\vee(h_a)(m) = \widetilde{\psi}(m)(a) = \psi(h_a)(m);$$

thus, $\widetilde{\psi}^\vee = \psi$.

It is now enough to show that for all $\phi \in \operatorname{Hom}_R(M, R)$, $\widetilde{\phi^\vee} = \phi$, and indeed for all $a \in E$ and $m \in M$

$$\left(\widetilde{\phi^\vee}(m)\right)(a) = \left(\phi^\vee(h_a)\right)(m) = \phi(m)a,$$

i.e., $(\widetilde{\phi^\vee}(m)) \in \operatorname{Hom}_R(E, E)$ is given by multiplication by $\phi(m)$ and so under the identification of $\operatorname{Hom}_R(E, E)$ with R, $\widetilde{\phi^\vee}$ is identified with ϕ. □

We can now prove a generalization of Lemma 1.6 in [**5**] in the form of the next two theorems.

Theorem 4.2. (a) *The F_*R-module $\operatorname{Hom}_R(F_*R, E)$ is injective of the form $\oplus_{\gamma\in\Gamma} F_*E \oplus H$ where Γ is non-empty, $H = \oplus_{\lambda\in\Lambda} F_*E(R/P_\lambda)$, Λ is a (possibly empty) set, P_λ is a non-maximal prime ideal of R for all $\lambda \in \Lambda$ and $E(R/P_\lambda)$ denotes the injective hull of R/P_λ.*

(b) *Write $\mathcal{B} = \operatorname{Hom}_{F_*R}(E, \oplus_{\gamma\in\Gamma} F_*E) \subseteq \prod_{\gamma\in\Gamma} \operatorname{Hom}_{F_*R}(E, F_*E)$. We have*

$$\operatorname{Hom}_R(F_*R, R) \cong \mathcal{B} \subseteq \prod_{\gamma\in\Gamma} \operatorname{Hom}_{F_*R}(E, F_*E) \cong \prod_{\gamma\in\Gamma} F_*RT$$

where T is the standard Frobenius map on E.

(c) *The set* Γ *is finite if and only if* $F_*\mathbb{K}$ *is a finite extension of* $\mathbb{K}$, *in which case* $\#\Gamma = 1$.

Proof. The functors $\operatorname{Hom}_R(-, E) = \operatorname{Hom}_R(- \otimes_{F_*R} F_*R, E)$ and $\operatorname{Hom}_{F_*R}(-, \operatorname{Hom}_R(F_*R, E))$ from the category of F_*R-modules to itself are isomorphic by the adjointness of Hom and $\otimes$, and since $\operatorname{Hom}_R(-, E)$ is an exact functor, so is $\operatorname{Hom}_{F_*R}(-, \operatorname{Hom}_R(F_*R, E))$; thus, $\operatorname{Hom}_R(F_*R, E)$ is an injective F_*R-module and hence of the form $G \oplus H$ where G is a direct sum of copies of F_*E and H is as in the statement of the Theorem. Write $G = \oplus_{\gamma\in\Gamma} F_*E$. To finish establishing (a) we need only to verify that $\Gamma \neq \varnothing$ and we do this below.

Pick any $h \in \operatorname{Hom}_R(E, \oplus_{\lambda\in\Lambda} F_*E(R/P_\lambda))$. For any $a \in E$, $h(a)$ can be written as a finite sum $b_{\lambda_1} + \cdots + b_{\lambda_s}$ where $\lambda_1, \dots, \lambda_s \in \Lambda$ and $b_{\lambda_1} \in F_*E(R/P_{\lambda_1}), \dots, b_{\lambda_s} \in F_*E(R/P_{\lambda_s})$. Use prime avoidance to pick a $z \in m \setminus \cup_{i=1}^s P_{\lambda_i}$; now z and its powers act invertibly on each of $F_*E(R/P_{\lambda_1}), \dots, F_*E(R/P_{\lambda_s})$ while a power of z kills a, and so we must have $h(a) = 0$. We deduce that $\operatorname{Hom}_R(E, \oplus_{\lambda\in\Lambda} F_*E(R/P_\lambda)) = 0$ and

$$\begin{aligned}
&\operatorname{Hom}_R(E, \operatorname{Hom}_R(F_*R, E)) \\
&\qquad \cong \operatorname{Hom}_R\left(E, G \oplus \bigoplus_{\lambda\in\Lambda} F_*E(R/P_\lambda)\right) \\
&\qquad \cong \operatorname{Hom}_R(E, G) \oplus \operatorname{Hom}_R\left(E, \bigoplus_{\lambda\in\Lambda} F_*E(R/P_\lambda)\right) \\
&\qquad \cong \operatorname{Hom}_R(E, G) \\
&\qquad \cong \operatorname{Hom}_R(E, \oplus_{\gamma\in\Gamma} F_*E) \\
&\qquad = \mathcal{B}.
\end{aligned}$$

Now $\operatorname{Hom}_R(E, F_*E)$ is the R-module of Frobenius maps on E which is isomorphic as an F_*R module to F_*RT and we conclude that $\operatorname{Hom}_R(E, \operatorname{Hom}_R(F_*R, E)) \subseteq \prod_{\gamma\in\Gamma} F_*RT$.

An application of the Matlis dual and Lemma 4.1 now gives

$$\operatorname{Hom}_R(F_*R, R) \cong \operatorname{Hom}_R(E, \operatorname{Hom}_R(F_*R, E))$$

and (b) follows.

Write $\mathbb{K} = R/m$ and note that $F_*\mathbb{K}$ is the field extension of $\mathbb{K}$ obtained by adding all pth roots of elements in $\mathbb{K}$. We next compute the cardinality of Γ as the $F_*\mathbb{K}$-dimension of $\operatorname{Hom}_{F_*\mathbb{K}}(F_*\mathbb{K}, G)$. A similar argument to the one above shows that

$$\operatorname{Hom}_{F_*\mathbb{K}}\left(F_*\mathbb{K}, \bigoplus_{\lambda\in\Lambda} F_*E(R/P_\lambda)\right) = 0;$$

hence $\operatorname{Hom}_{F_*\mathbb{K}}(F_*\mathbb{K}, G) = \operatorname{Hom}_{F_*\mathbb{K}}(F_*\mathbb{K}, \operatorname{Hom}_R(F_*R, E))$.

We may identify $\operatorname{Hom}_{F_*\mathbb{K}}(F_*\mathbb{K}, \operatorname{Hom}_R(F_*R, E))$ and $\operatorname{Hom}_{F_*R}(F_*\mathbb{K}, \operatorname{Hom}_R(F_*R, E))$. Another application of the adjointness of Hom and $\otimes$ gives

$$\begin{aligned}\operatorname{Hom}_{F_*R}\left(F_*\mathbb{K}, \operatorname{Hom}_R\left(F_*R, E\right)\right) &\cong \operatorname{Hom}_R\left(F_*\mathbb{K} \otimes_{F_*R} F_*R, E\right)\\ &\cong \operatorname{Hom}_R\left(F_*\mathbb{K}, E\right).\end{aligned}$$

Since $mF_*\mathbb{K} = 0$, we see that the image of any $\phi \in \operatorname{Hom}_R(F_*\mathbb{K}, E)$ is contained in $\operatorname{Ann}_E m \cong \mathbb{K}$ and we deduce that $\operatorname{Hom}_R(F_*\mathbb{K}, E) \cong \operatorname{Hom}_R(F_*\mathbb{K}, \mathbb{K})$. We can now conclude that the cardinality of Γ is the $F_*\mathbb{K}$-dimension of $\operatorname{Hom}_R(F_*\mathbb{K}, \mathbb{K})$. In particular Γ cannot be empty and (a) follows.

If $\mathcal{U}$ is a $\mathbb{K}$-basis for $F_*\mathbb{K}$ containing $1 \in F_*\mathbb{K}$,

$$\operatorname{Hom}_{\mathbb{K}}\left(F_*\mathbb{K}, \mathbb{K}\right) \cong \prod_{b\in\mathcal{U}} \operatorname{Hom}_{\mathbb{K}}\left(\mathbb{K}b, \mathbb{K}\right) \tag{1}$$

and when $\mathcal{U}$ is finite, this is a one-dimensional $F_*\mathbb{K}$-vector space spanned by the projection onto $\mathbb{K}1 \subset F_*\mathbb{K}$. If $\mathcal{U}$ is not finite, the dimension as $\mathbb{K}$-vector space of (1) is at least $2^{\#\mathcal{U}}$, hence $\operatorname{Hom}_{\mathbb{K}}(F_*\mathbb{K}, \mathbb{K})$ cannot be a finite-dimensional $F_*\mathbb{K}$-vector space. □

Our next result is to establish a connection between submodules of R^n compatible with a given $B \in \operatorname{Hom}_R(F_*R^n, R^n)$ and submodules of E^n fixed under a sequence of Frobenius actions determined by B. Note that the previous theorem allows us to view elements of $\operatorname{Hom}_R(F_*R^n, R^n) \cong \operatorname{Hom}_R(F_*R, R)^{n\times n} = \mathcal{B}^{n\times n}$ as elements in $\prod_{\gamma\in\Gamma} F_*R^{n\times n}T$, i.e., as sequences $(B_\gamma T)_{\gamma\in\Gamma}$ where each B_γ is an $n \times n$ matrix with entries in F_*R and T is the natural Frobenius action on E^n.

Theorem 4.3. *Let $G = \oplus_{\gamma\in\Gamma} F_* E$ and $\mathcal{B}$ be as in Theorem 4.2. Let $B \in \operatorname{Hom}_R(F_* R^n, R^n)$ be represented by $(B_\gamma T)_{\gamma\in\Gamma} \in \mathcal{B}^{n\times n}$. For all $\gamma \in \Gamma$ consider E^n as an $R[\Theta_\gamma; f]$-module with $\Theta_\gamma v = B_\gamma^t T v$ for all $v \in E^n$. Let V be an R-submodule of R^n and fix a matrix A whose columns generate V. If $B(F_* V) \subseteq V$, then $\operatorname{Ann}_{E^n} A^t$ is an $R[\Theta_\gamma; f]$ submodule of E^n for all $\gamma \in \Gamma$.*

Proof. Apply the Matlis dual to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_*V & \longrightarrow & F_*R^n & \longrightarrow & F_*R^n/F_*A & \longrightarrow & 0 \\ & & \downarrow B & & \downarrow B & & \downarrow \overline{B} & & \\ 0 & \longrightarrow & V & \longrightarrow & R^n & \longrightarrow & R^n/V & \longrightarrow & 0 \end{array}$$

where the rightmost vertical map is induced by the middle map to obtain

$$\begin{array}{ccccc} 0 & \longrightarrow & (R^n/V)^\vee & \longrightarrow & E^n \\ & & \downarrow \overline{B}^\vee & & \downarrow B^\vee \\ 0 & \longrightarrow & (F_*R^n/F_*V)^\vee & \longrightarrow & \operatorname{Hom}_R(F_*R^n, E) \end{array}$$

Note that the previous theorem shows that

$$\operatorname{Hom}_R(E^n, \operatorname{Hom}_R(F_*R^n, E)) \cong \operatorname{Hom}_R(E^n, \oplus_{\gamma\in\Gamma} F_*E^n).$$

Also note that under this isomorphism $B^\vee \in \operatorname{Hom}_R(E, \oplus_{\gamma\in\Gamma} F_*E)^{n\times n}$ is given by $(B_\gamma^t)_{\gamma\in\Gamma}$. and that the image of $B^\vee$ is contained in $\oplus_{\gamma\in\Gamma} F_*E^n$.

Using the presentation $F_*R^m \xrightarrow{F_*A} F_*R^n \to F_*R^n/F_*V \to 0$ we obtain the exact sequence

$$0 \longrightarrow (F_*R^n/F_*V)^\vee \longrightarrow \operatorname{Hom}_R(F_*R^n, E) \xrightarrow{F_*A^t} \operatorname{Hom}_R(F_*R^m, E);$$

thus,

$$(F_*R^n/F_*V)^\vee = \operatorname{Ann}_{\operatorname{Hom}(F_*R^n, E)} F_*A^t.$$

We now obtain the commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \operatorname{ann}_{E^n} A^t & \longrightarrow & E^n \\ & & \Big\downarrow {\scriptstyle (B^t_\gamma T)_{\gamma\in\Gamma}} & & \Big\downarrow {\scriptstyle (B^t_\gamma T)_{\gamma\in\Gamma}} \\ 0 & \longrightarrow & \oplus_{\gamma\in\Gamma} \operatorname{ann}_{F_* E^n} F_* A^t & \longrightarrow & \oplus_{\gamma\in\Gamma} F_* E^n \end{array}$$

and we deduce that $\operatorname{Ann}_{E^n} A^t$ is an $R[\Theta_\gamma; f]$-module for all $\gamma \in \Gamma$. ◻

Theorem 4.4. *Let M be an $R[\Theta; f]$-module with no nilpotents, and assume M is an Artinian R-module. Then M has finitely many $R[\Theta; f]$-submodules.* (*Compare with Corollary* 4.18 *in* [**1**].)

Proof. Write $\mathcal{M} = \mathcal{H}(M)$. In view of [**10**, Theorem 4.2], there is an injection between the set of inclusions of $R[\Theta; f]$-submodules $N \subseteq M$ and the set of surjections of F-finite F-modules $\mathcal{M} \to \mathcal{N}$; hence, it is enough to show that there are finitely many such surjections. By [**10**, Theorem 2.8] the kernels of these surjections are F-finite F-submodules of $\mathcal{M}$; hence, it is enough to show that $\mathcal{M}$ has finitely many submodules.

All objects in the category of F-finite F-modules have finite length (cf. [**10**, Theorem 3.2]) and the theorem now follows from [**7**, Corollary 5.2 (b)]. ◻

Corollary 4.5. *Let $B \in \operatorname{Hom}_R(F_* R^n, R)$ be represented by $(B^t_\gamma T)_{\gamma\in\Gamma} \in \mathcal{B}^{n\times n}$, and assume that $B^t_\gamma T : E^n \to E^n$ is injective for some $\gamma \in \Gamma$. Then there are finitely many B-compatible submodules of $F_* R^n$. In particular this holds when $n = 1$ and $(B_\gamma T)_{\gamma\in\Gamma} : E \to \oplus_{\gamma\in\Gamma} E$ is injective.*

Proof. Let V be an R-submodule of R^n and fix a matrix A whose columns generate V. Theorem 4.3 implies that if $F_* V \subseteq F_* R^n$ is B-compatible then for all $\gamma \in \Gamma$, $\operatorname{Ann}_{E^n} A^t \subseteq E^n$ is an $R[\Theta; f]$-submodule of E^n with the Frobenius action given by $B^t_\gamma T$. If there exists a $\gamma \in \Gamma$ such that $B^t_\gamma T$ is injective, then [**11**, Theorem 3.10] or [**4**, Theorem 3.6] imply that there must be finitely many $R[B^t_\gamma T; f]$-submodules of E^n and hence also finitely many B-compatible submodules of R^n.

Assume now that $n = 1$. For all $\gamma \in \Gamma$ write $Z_\gamma = \{v \in E \mid B_\gamma T v = 0\}$, and let $C_\gamma \subseteq R$ be the ideal for which $Z_\gamma = \operatorname{Ann}_E C_\gamma$. If

$C_\gamma \subseteq mR$ for all $\gamma \in \Gamma$, then $C = \sum_{\gamma\in\Gamma} C_\gamma \neq R$, and for any non-zero $v \in \operatorname{Ann}_E C \neq 0$, we have $B_\gamma T v = 0$ for all $\gamma \in \Gamma$. We conclude that there exists a $\gamma \in \Gamma$ such that, $C_\gamma = R$, i.e., that the Frobenius map $B_\gamma T$ on E is injective, and the last assertion of the corollary follows. □

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REFERENCES

1. M. Blickle and G. Böckle, *Cartier modules: Finiteness results*, preprint, oai:arXiv.org:0909.2531.

2. M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progr. Math. **231**, Birkhäuser Boston, Inc., Boston, MA, 2005.

3. M.P. Brodmann and R.Y. Sharp, *Local cohomology: An algebraic introduction with geometric applications*, Cambridge Stud. Adv. Math. **60**, Cambridge University Press, Cambridge, 1998.

4. F. Enescu and M. Hochster, *The Frobenius structure of local cohomology*, Algebra Number Theory **2** (2008), 721–754.

5. R. Fedder, *F-purity and rational singularity*, Trans. Amer. Math. Soc. **278** (1983), 461–480.

6. R. Hartshorne and R. Speiser, *Local cohomological dimension in characteristic p*, Annals Math. **105** (1977), 45–79.

7. M. Hochster, *Some finiteness properties of Lyubeznik's $\mathcal{F}$-modules*, Contemp. Math. **448**, American Mathematical Society, Providence, RI, 2007.

8. M. Katzman, *Parameter test ideals of Cohen Macaulay rings*, 0 Composit. Math. **144** (2008), 933–948.

9. ———, *Frobenius maps on injective hulls and their applications to tight closure*, J. L.M.S. **81** (2010), 589–607.

10. G. Lyubeznik, *F-modules: Applications to local cohomology and D-modules in characteristic $p > 0$*, J. reine Angew. Math. **491** (1997), 65–130.

11. R.Y. Sharp, *Graded annihilators of modules over the Frobenius skew polynomial ring, and tight closure*, Trans. Amer. Math. Soc. **359** (2007), 4237–4258.

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