JOURNAL OF COMMUTATIVE ALGEBRA
Volume 2, Number 4, Winter 2010

COSET DIAGRAMS IN THE STUDY OF FINITELY PRESENTED GROUPS WITH AN APPLICATION TO QUOTIENTS OF THE MODULAR GROUP

ANNA TORSTENSSON

ABSTRACT. We look at some ways in which coset diagrams have been used to study quotients, subgroups and structure of finitely presented groups. Then we apply one of those known methods to get a partial answer to what happens when adding one relator to the modular group. We find that the language of such words making the resulting group isomorphic to S_3 is regular and that the language of words making the group infinite contains a subset that is a context sensitive language.

1. Introduction. In this paper we will look at how coset diagrams have been used in the study of finitely presented groups. Such diagrams, sometimes under the name *Schreier diagrams*, is a relatively old concept and seems to originate from work by Otto Schreier and Kurt Reidemeister in the 1920s. A description in more modern terms can be found in [**7**] from 1966. The interest has risen in the last decades as the possibilities to use diagram techniques in combination with mathematical software has improved. In the first part of this paper we will describe a couple of different applications of coset diagrams to the study of finitely presented groups. The aim of this part is not to introduce new ideas, but rather to clarify how a number of more specific applications are based on the same diagram technique. In the second part we will confine ourselves to a particular type of finitely presented groups, namely one-relator quotients of the modular group. These have been studied before by Conder in [**3**] where he described all such groups with third relator of length at most 24. The method used was to either prove finiteness and find the structure of the group using coset enumeration or, in the cases where the computations did not finish indicating an infinite group, constructing infinite transitive coset diagrams to prove infinitude. This was done to fill in a gap in an investigation by Tucker

Received by the editors on September 15, 2009, and in revised form on March 31, 2010.

DOI:10.1216/JCA-2010-2-4-501

([**11**]) of groups having Cayley graphs which can be embedded in surfaces of small genus. In addition to providing a counterexample to a conjecture in Tucker's paper, Conder's work raises the question whether properties of the group G_ω can be effectively described in terms of the third relator ω regarded as a word. In this paper we will for instance show that the language of third relators such that G_ω is isomorphic to S_3 is a regular language. We also describe a subset of the language of words for which G_ω is infinite and prove that this subset is context sensitive but can be recognized by a linearly bounded automaton (LBA).

2. Cayley graphs and coset diagrams. Given a finitely presented group $G = \langle X \mid R \rangle$ with generating set $X = \{x_1, x_2, x_3 \dots , x_n\}$ and relators R, the Cayley graph $\Gamma(G, X)$ of G with respect to the generating set X is the colored directed graph with vertex set consisting of the elements of G and an edge of 'color' x_i between g and h if and only if $gx_i = h$.

The concept of Cayley graph can then be generalized by taking cosets modulo some subgroup H of G, instead of elements of G, as vertices. In other words the coset diagram $\Gamma(G, H, X)$ of G with respect to the generating set X and the subgroup H is the colored directed graph with vertex set consisting of the cosets of Hg_j, and an edge of 'color' x_i between Hg_j and Hg_k if and only if $Hg_jx_i = Hg_k$.

3. Using coset diagrams in the study of finitely presented groups. Coset diagrams have been used in the study of finitely presented groups in a number of different ways. One application that was first introduced by Graham Higman (not published by Higman, but described in [**2**]) and then developed further by Marston Conder in [**1**], aims to show that a certain finitely presented group has almost all symmetric or alternating groups as quotients. The idea is to find small coset diagrams of the group that can be combined together into coset diagrams of any desired (large enough) size. These diagrams also need an additional property (the existence of a group element whose action on the diagram regarded as a permutation has a cycle structure of a certain type) to make sure that the quotient is the full symmetric or alternating group. The method was used by Conder, Martin and Torstensson in [**3**] for maximal symmetry groups of hyperbolic 3-

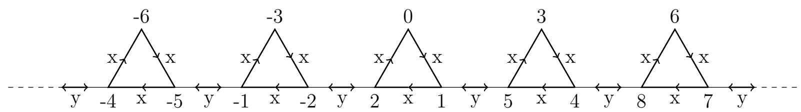

FIGURE 1. An infinite transitive coset diagram.

manifolds and by Conder for the $(2, 3, 7)$ triangle group in [**1**]. Also Stothers ([**9**]) developed a similar technique when studying subgroups of finite index in the $(2, 3, 7)$ triangle group where he related different properties of the subgroups to properties of the corresponding coset diagrams. In this context it is also worth mentioning that by further refining the methods used by Higman and Conder for the triangle group Brent Everitt has been able to show that not only the $(2, 3, 7)$ triangle group, but all Fuchsian groups, have all but finitely many alternating groups among its homomorphic images–a substantial generalization of earlier results ([**5**]).

Another, less sophisticated, but nevertheless useful application of coset diagrams is to prove the infinitude of a given finitely presented group by constructing an infinite connected coset diagram for the group. As mentioned in the introduction this method was used extensively by Conder in [**3**]. The rest of this paper will concern different ways of extending those results.

4. One-relator quotients of the modular group. We will study groups with presentations

$$G_\omega = \langle x, y \mid x^3 = y^2 = \omega(x, y) = 1 \rangle. \tag{1}$$

As noted in [**3**] we may alternatively choose $U = xy$ and $V = x^2 y$ as generators and express the last relator as a word in U and V.

5. Generalized use of some coset diagrams. In [**4**] the author constructs transitive permutation representations on the integers for some of the groups G_ω in order to prove that they are infinite. Here we will take a closer look at one of those constructions and show for which choices of a third relator it works. The language of such words will turn out to be context-sensitive but accepted by a linear-bounded automaton.

Furthermore we will find a regular language of words such that the construction results in some (finite or infinite) non-trivial permutation representation of the group. Especially one can conclude that for third relators in this language, which we will describe by an accepting finite state automata, the corresponding group is non-trivial. One can also make some conclusions about the structure of the group in the cases where the quotient is finite.

Let us start out with the simplest of the constructions given in [**4**] where the two permutations are given by

$$\phi(x) : k \longmapsto \begin{cases} k+1 & \text{if } k \equiv 0,\, 1 \bmod 3 \\ k-2 & \text{if } k \equiv 2 \bmod 3 \end{cases} \tag{2}$$

$$\phi(y) : k \longmapsto \begin{cases} k & \text{if } k \equiv 0 \bmod 3 \\ k+4 & \text{if } k \equiv 1 \bmod 3 \\ k-4 & \text{if } k \equiv 2 \bmod 3. \end{cases} \tag{3}$$

The corresponding coset diagram is depicted in Figure 1.

This gives a permutation representation of G_ω (by extending ϕ to G_ω) if and only if each of the three relators act in such a way that all points are fixed. The diagram is constructed in such a way that the two first relators always act as the identity so we get a representation if and only if $\phi(\omega)$ acts as the identity. For symmetry reasons it suffices to examine what happens to the three points in one of the triangles. Let us denote those points, corresponding to the integers 0, 1 and 2, respectively, by P_T, P_R and P_L. It is easy to verify that after each permutation induced by a group element we always have one point in each triangle position–top, right and left. Let $p(\omega)$ be the vector keeping track of in which triangle on the line the top, right and left points are after the permutation, where zero corresponds to the original triangle, $-k$ denotes k steps to the left and k denotes k steps to the right. Then

$$p(\omega U) = p(\omega) M_U + t_U \tag{4}$$

where

$$M_U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad t_U = (0, -1, 1),$$

as $U = xy$ acts by moving P_T to the left point of the triangle one step to the right, P_R to the right point in the triangle one step to the left and P_L to the top point of the original triangle. By the same line of argument we find that

$$p(\omega V) = p(\omega)M_V + t_V \tag{5}$$

where

$$M_V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad t_V = (0, -1, 1).$$

Let us now define the matrix M_ω for any word $\omega \in \{U, V\}^*$ recursively by the rule $M_{\omega\omega'} = M_\omega M_{\omega'}$.

Note that the condition $\phi(\omega) = \mathrm{Id}$ now can be broken down into two parts.

• A. The points P_T, P_R and P_L all end up in the original triangle or equivalently $p(\omega) = (0, 0, 0)$.

• B. The top (right/left) point is still top (right/left) after the permutation or equivalently $M_\omega = I$.

The second condition can be verified relatively easily if we note the following:

Proposition 1. *Each matrix M_ω equals exactly one of the six matrices $M_{\overline{\omega}}$, where $\overline{\omega} \in \{1, U, V, UV, VU, UVU\}$ is the normal form of ω under the following set of reductions: $U^2 \to 1$, $V^2 \to 1$, $VUV \to UVU$.*

Proof. It is easy to verify that the set of reductions constitute a confluent rewriting system using Knuth-Bendix completion procedure ([**8**]) (with shortlex ordering and $U \prec V$), and that the six words listed in the proposition are all the normal forms. Moreover $M_{U^2} = M_{V^2} = I$ and $M_{UVU} = M_{VUV}$ so $M_\omega = M_{\omega'}$ whenever ω' can be obtained from ω by reduction. Noting that all $M_{\overline{\omega}}$ are different this completes the proof of the proposition. □

Now we would like to describe how the position vector p changes when adding a letter at the end of ω. Unfortunately this depends on the entire word ω and not only on its normal form. As a simple example consider $p(\omega U) - p(\omega)$ for $\omega = 1$ and V^2, respectively. We have that

$$\begin{aligned} p(U) - p(1) &= (0,-1,1) - (0,0,0) = (0,-1,1) \\ &\neq p(V^2U) - p(V^2) \\ &= (2,-2,0) - (-1,-1,2) \\ &= (3,-1,-2) \end{aligned}$$

despite that fact that $\overline{V^2} = \overline{1} = 1$. However, if we instead consider the vectors $r(\omega) = p(\omega^R)$, where ω^R is the reversed word (obtained by reading ω backwards) we can describe the result of appending a letter.

Proposition 2. *Let $r(\omega)$ be the position vector of a word ω as described above. Then $r(\omega U) - r(\omega) = t_U M_{\overline{\omega}^R}$ and $r(\omega V) - r(\omega) = t_V M_{\overline{\omega}^R}$ holds for any $\omega \in \{U, V\}^*$.*

Proof. First of all let us note that for $\omega = \omega_1\omega_2\omega_3 \cdots \omega_k$,

$$p(\omega) = t_{\omega_k} + t_{\omega_{k-1}} M_{\omega_k} + t_{\omega_{k-2}} M_{\omega_{k-1}\omega_k} + \cdots + t_{\omega_1} M_{\omega_2\omega_3\cdots\omega_k}.$$

Using this, we obtain

$$\begin{aligned} r(\omega U) - r(\omega) &= p((\omega U)^R) - p(\omega^R) = p(U(\omega)^R) - p(\omega^R) \\ &= t_U M_{\omega^R} = t_U M_{\overline{\omega^R}} = t_U M_{\overline{\omega}^R}. \end{aligned}$$

Here the next to last equality holds by Proposition 1. The last equality uses that $\overline{\omega^R} = \overline{\omega}^R$ which in turn is a consequence of the fact that all words in the reduction rules are palindromes.

A similar computation for $r(\omega V) - r(\omega)$ now completes the proof of the proposition. □

The useful point of the above proposition is that what is added to the position vector $r(\omega)$ when appending a letter to ω only has a finite number of (more precisely six) different values. Due to this fact we can describe $r(\omega)$ by a finite graph where each node corresponds to a value of $\overline{\omega}^R$. Each node in the graph corresponds to a reduced word in

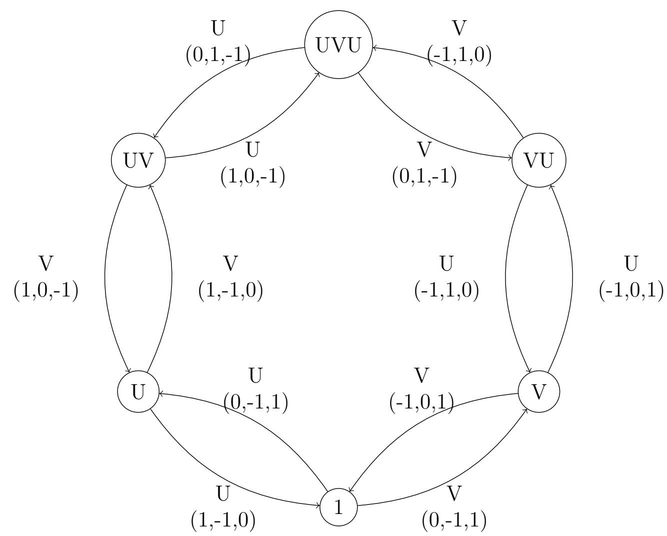

FIGURE 2. A graph to compute the position vector of a relator ω.

U and V and the position vector $p(\omega) = r(\omega^R)$ of the triangle points, corresponding to a certain third relator ω, is built up by adding the vectors that are labeling the nodes passed by when following the path of ω^R, starting at the node labeled 1. In this way condition A can be verified simply by computing $p(\omega)$ and checking if it is the zero vector. By Proposition 1 it is clear that Condition B corresponds to

$$\overline{\omega} = 1 \iff \overline{\omega}^R = 1 \overset{1}{\iff} \overline{\omega^R} = 1,$$

or in terms of the graph to ending up at the starting node.

Example. If we want to examine the group

$$G_\omega = \langle x, y \mid x^3 = y^2 = xyx^2yx^2yxyx^2yx^2y = 1 \rangle \tag{6}$$

we should consider the path of the reverse $\omega^R = V^2UV^2U$ of the third relator $\omega = UV^2UV^2$. This path is given by first traveling one step in anti clockwise direction and then three steps in clockwise direction and finally two more anti clockwise steps. It is clear that

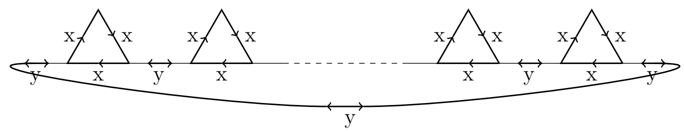

FIGURE 3. A finite variant of our coset diagram.

condition B is satisfied as we end up at the starting node. The position vector $p(\omega) = r(\omega^R) = (0,-1,1) + (-1,0,1) + (0,-1,1) + (1,-1,0) + (1,0,-1) + (1,-1,0) = (2,-4,2)$ is non-zero showing that for this particular relator condition A is not satisfied and our construction therefore does not give an infinite permutation representation. We can however see that we get a certain finite representation: All positions are altered by multiples of two so if we identify all points corresponding to numbers equivalent modulo six (geometrically this means rolling up our infinite string of triangles so that each orbit contains two triangles) we get a permutation of the numbers one to six. This homomorphic image of our group is the subgroup of S_6 generated by $a = (1,2,3)(4,5,6)$ and $b = (2,6)(3,5)$ and can easily be shown to equal S_4.

More generally, let H_k be the subgroup of the symmetric group S_{3k} given by

$$H_k = \langle (1,2,3)(4,5,6)\cdots(3k-2,3k-1,3k), (2,6)(5,9) \cdots(3k-4,3k)(3k-1,3)\rangle.$$

We would like to describe the relators ω for which H_k is a quotient of G_ω, but in order to do so we need the following uniqueness result.

Proposition 3. *If there exists a surjective homomorphism $\phi : G_\omega \to H_k$, where H_k is the subgroup of the symmetric group S_{3k} generated by the elements $a = (1,2,3)(4,5,6)\cdots(3k-2,3k-1,3k)$ and $b = (2,6)(5,9)\cdots(3k-4,3k)(3k-1,3)$, then (after renaming the points) we may assume that $\phi(x) = a$ and $\phi(y) = b$.*

Proof. The group H_k can be described by the coset diagram in Figure 3 with k triangles. Now $\phi(x)$ is an element of H_k and can be expressed as a word $\omega_1(a,b)$ in the generators a and b. The coset diagram of H_k contains three types of points: top points, left base points and right base points (according to the position they belong to in the triangle). Let us denote these point sets by T, L and R respectively. By the symmetry of the diagram it is clear that any element of H_k acts in the same way on all points in the same set. For instance if $\omega_1(a,b)$ moves one top point to the right base point two triangles to the left of the original triangle, then $\omega_1(a,b)$ act in this way on all other top points as well. As a consequence $\phi(x)$ and $\phi(y)$ act on the family of sets $\{T,L,R\}$. Let us denote these permutations by $\phi(x)^{SET}$ and $\phi(y)^{SET}$. We know that $\phi(x)$ and $\phi(y)$, being images of x and y, are of order three and two respectively. It follows that the only possible values of $\phi(x)^{SET}$ except the identity are the three-cycles (TLR) and (TRL). Similarly $\phi(y)^{SET}$ must equal (TL), (TR), (LR) or the identity element. In any case there is at least one fixed point of $\phi(y)^{SET}$. We conclude that $\phi(x)^{SET} \neq \mathrm{Id}$ as this would lead to a set being fixed by the whole of $H_k = \langle \phi(x), \phi(y) \rangle$, thus disconnecting this set from the rest of the coset diagram of H_k. This obviously contradicts the fact that the diagram in Figure 3 is connected. Now that we know that $\phi(x)^{SET}$ permutes the three point sets cyclically and $\phi(x)$ is of order three, $\phi(x)$ must permute all points as three-cycles. After renumbering of points we may assume that $\phi(x) = (1,2,3)(4,5,6)\cdots(3k-2,3k-1,3k) = a$.

It remains to prove that $\phi(y) = b$. Now, again by the connectedness of Figure 3, we find that $\phi(y)^{SET} \neq \mathrm{Id}$. By the aforementioned symmetry of the action of ϕ on elements it is now easy to see that $\phi(y)$ must connect all the triangles (= cycles of $\phi(x) = a$) to form the diagram in Figure 3. (Another cyclic renumbering within each triangle may be necessary, but this will not affect the action of x.) $\square$

From the above discussion it is clear that the graph in Figure 2 represents an automaton accepting L_2^R where L_2 is the language consisting of third relators ω for which G_ω either is infinite or has some quotient of the form

$$H_k = \langle (1,2,3)(4,5,6)\cdots(3k-2,3k-1,3k), (3,5)(6,8)\cdots(3k,3k+2) \rangle \leq S_{3k}.$$

For the first few H_k we have $H_1 = S_3$, $H_2 = S_4$, $H_3 = S_3 \times C_3$, $H_4 = S_4 \sim (C_2 \times C_2)$, $H_5 = S_3 \sim (C_5 \times C_5)$ and $H_6 = S_4 \times C_3$. Note that whenever $k \mid l$ there is some quotient of H_l isomorphic to H_k, which can be seen by identifying suitable triangles/integers in the set H_l acts on as a permutation group.

Moreover from the graph in Figure 2 it is not difficult to create, for a given k, a finite state automaton that accepts exactly those words ω for which condition B holds and the position vector contains only multiples of k. This is done by creating additional states so that the state also records the current value of the position vector modulo k. Thus we have the following

Corollary 4. *The language of third relators such that G_ω is isomorphic to S_3 is a regular language. The same holds if S_3 is replaced by any of the groups H_k described above.*

Let L_1 be the language of third relators ω such that the diagram in Figure 1 is a transitive coset diagram for G_ω. (Thus $L_1 \subseteq L$, the language of third relators for which G_ω is infinite.) Let us try to get an idea of the type of words that lie in L_1 by first restricting ourselves to words obtained by traveling around the graph in Figure 2 once in a clockwise direction. A straightforward computation using the mentioned graph shows the following

Proposition 5. *A necessary and sufficient condition for $U^{2\alpha+1}V^{2\beta+1}U^{2\gamma+1}V^{2\delta+1}U^{2\varepsilon+1}V^{2\phi+1}$ to belong to L_1 is that $\gamma+\delta \geq \alpha$,$\alpha+\beta \geq \delta$, $\phi = \gamma+\delta-\alpha$, $\varepsilon = \alpha+\beta-\delta$.*

This means that we can easily produce such relators by first assigning α, β, γ any non-negative integer values, then δ a number between $\alpha-\gamma$ and $\alpha+\beta$ and finally compute ϕ and ε from the conditional equations.

In [**3**] Conder noted that when he examined third relators of length at most twelve in U and V, the relators giving rise to infinite groups G_ω were all either of the form ω_1^r for some $r \geq 2$ or of the form $\omega_1(U,V)\omega_1(V,U)$ for some word ω_1. We can now see that this pattern holds only for short relators and that $UV^5UV^3U^3V^3$ is a counterexample of length 16.

Let us now return to the problem of getting to know the entire language L_1.

Proposition 6. *The language L_1 of third relators such that the diagram in Figure 1 is an infinite transitive coset diagram is closed under cyclic permutation of letters.*

Proof. This is immediate from the description of L_1 as relators making a certain diagram a coset diagram of G_ω, since permuting a relator cyclically leaves the group G_ω unchanged. □

Theorem 7. *The language L_1 of third relators such that the diagram in Figure 1 is an infinite transitive coset diagram is not context-free.*

Proof. We will use the pumping lemma for context-free languages stating that, for any such language L, there is an integer m such that any string s in L of length at least m can be written $s = abcde$ with bcd of length at most m, not both b and d empty such that all 'pumped' strings ab^kcd^ke also are in L.

From Proposition 6 it follows that we may assume that a is the empty string since $abcde \in L_1 \Leftrightarrow bcdea \in L_1$ and $ab^kcd^ke \in L_1 \Leftrightarrow b^kcd^kea \in L_1$. Thus, the pumping lemma says that if $bcdea$ is in L_1 and of length at least m, then b^kcd^kea is also in L_1 for all positive integers k. We will use this version in the following.

Considering our language of interest, L_1, let $s = U^{2m+1}VU^{2m+1} \times VU^{2m+1}V$, which belongs to L_1 by Proposition 5. Now the pumping string bcd, being of length at most m and a prefix of s is of the form $b = U^\alpha$, $c = U^\beta$ $d = U^\gamma$, and hence $ea = U^{2m+1-(\alpha+\beta+\gamma)}VU^{2m+1}V$ $U^{2m+1}V$. Letting $k = 3$ we obtain the pumped string $U^{2(m+\alpha+\gamma)+1} \times VU^{2m+1}VU^{2m+1}V$ which cannot belong to L_1 according to Proposition 5. (Unless both α and γ are zero, but this cannot be the case by the formulation of the pumping lemma.)

This shows that L_1 contains a word that cannot be pumped in the way described above, and hence shows that L_1 cannot be a context-free language. □

We have now seen that L_1 is not context-free. The context-free languages are exactly those that can be recognized by some pushdown automaton. (See [**10**], which we also use as a general reference for formal languages and automata theory.) Thus it is clear that we need a more powerful automaton to recognize our language. The next level of complexity in the Chomsky hierarchy of grammars consists of the context-sensitive grammars corresponding to languages recognized by linear-bounded automata. A linear-bounded automaton can be defined as a multitrack Turing Machine using only as much of the tape as the length of the input (the input string is assumed to be written on the first track of the tape).

Theorem 8. *There exists a linear-bounded automaton* (LBA) *accepting the language* L_1.

Proof. First we note that it suffices to prove that L_1 is accepted by a multitape Turing machine with each tape of length equal to the input string, by the construction in Section 9.5 in [**10**].

Let us now describe how to construct a Turing Machine with four tapes accepting the language L_1^R. The class of context-sensitive languages is closed under reversal so this will imply that L_1 also belongs to the same class.

We will construct our Turing Machine from the graph in Figure 2. In essence the machine will operate as follows: The first tape contains the input which will be scanned from left to right. The states will correspond to the states of the graph and the transition from one state to another will depend on the input symbol scanned as depicted in the graph. The three additional tapes will keep track of the value of the position vector (with the value zero represented by a blank tape). At each transition the value will be 'updated' according to the label in the graph. The accepting state is reached when being in the state labeled 1 and scanning four blanks.

Now the above construction has to be somewhat modified to meet the technical requirements of a Turing Machine. Since the alphabet used must be finite the value k in one of the position vector tapes must be represented by k consecutive ones, rather than by the symbol k. This can be obtained by adding some additional states at those transitions where something should be added (subtracted) to the position vector at a tape where it already has a positive (negative) value. Note that this does not use more tape than permitted as there are only as many updates of the position vector as there are letters in the input string. □

Corollary 9. *The language L_1 of third relators such that the diagram in Figure* 1 *is an infinite transitive coset diagram is of type* 1 *in the Chomsky hierarchy, that is, it is context-sensitive but not context-free.*

6. Ideas for future work. There are other interesting permutations of the integers that give homomorphic images for classes of G_ω, not considered in [**3**]. One of them is the following:

$$X(k) = \begin{cases} k+1 & \text{if } k \equiv 0,\ 1 \bmod 3 \\ k-2 & \text{if } k \equiv 2 \bmod 3 \end{cases} \tag{7}$$

$$Y(k) = \begin{cases} k+3 & \text{if } k \text{ even} \\ k-3 & \text{if } k \text{ odd.} \end{cases} \tag{8}$$

The construction of finite state automata accepting regular languages of third relators ω for which G_ω is isomorphic to a certain small group can be done also for other infinite coset diagrams, both those from [**4**] and others like the one described above. In this way one could build a small database of regular languages corresponding to small groups that G_ω is isomorphic to.

Also, for each choice of infinite coset diagram we can look at the language of relators for which the diagram works and in that way find different subsets of the language L of all relators that make the group infinite. It seems to be the case that all those languages are of type one in the Chomsky hierarchy but this remains to be verified in detail. Ideally one would like to find the complexity of L and construct the appropriate kind of automaton to accept this language. This cannot (in any straightforward way) be done by the methods presented here, but requires some additional ideas.

REFERENCES

1. M.D.E. Conder, *Generators for alternating and symmetric groups*, J. London Math. Soc. **22** (1980), 75–86.

2. ———, *Group actions on graphs, maps and surfaces with maximum symmetry*, in *Groups St. Andrews* 2001 *in Oxford.* Vol. I, London Math. Soc. Lect. Note Ser. **304**, Cambridge Univ. Press, Cambridge, 2003.

3. M.D.E. Conder, *Three-relator quotients of the modular group*, Quart. J. Math. Oxford **38** (1987), 427–447.

4. M.D.E. Conder, G. Martin and A. Torstensson, *Maximal symmetry groups of hyperbolic* 3*-manifolds*, New Zealand J. Math. **35** (2006), 37–62.

5. B.J. Everitt, *Alternating quotients of Fuchsian groups*, J. Algebra **223** (2000), 457–476.

6. G. Higman and Q. Mushtaq, *Coset diagrams and relations for* $\mathrm{PSL}(2, \mathbf{Z})$, Arab Gulf J. Sci. Res. **1** (1983), 159–164.

7. W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory*: *Presentations of groups in terms of generators and relations*, Interscience, New York, 1966.

8. C. Sims, *Computation with finitely presented groups* Cambridge University Press, Cambridge, 1994.

9. W.W. Stothers, *Subgroups of the* $(2,3,7)$ *triangle group*, Manuscr. Math. **20** (1977), 323–334.

10. T.A. Sudkamp, *Languages and machines–An introduction to the theory of computer science*, Addison-Wesley, Reading, Massachusetts, 1988.

11. T.W. Tucker, *A refined Hurwitz theorem for imbeddings of irredundant Cayley graphs*, J. Combin. Theory **36** (1984), 244–268.

Centre for Mathematical Sciences, Box 118, SE-221 00 Lund, Sweden
Email address: annat@maths.lth.se