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GORENSTEIN PROJECTIVE DIMENSION WITH RESPECT TO A SEMIDUALIZING MODULE

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Dedicated to the memory of Colleen Kilker

ABSTRACT. We introduce and investigate the notion of G_C-projective modules over (possibly non-Noetherian) commutative rings, where C is a semidualizing module. This extends Holm and Jørgensen's notion of C-Gorenstein projective modules to the non-Noetherian setting and generalizes projective and Gorenstein projective modules within this setting. We then study the resulting modules of finite G_C-projective dimension, showing in particular that they admit G_C-projective approximations, a generalization of the maximal Cohen-Macaulay approximations of Auslander and Buchweitz. Over a local ring, we provide necessary and sufficient conditions for a G_C-approximation to be minimal.

1. Introduction. Over a Noetherian ring R, Foxby [**9**], Golod [**10**] and Vasconcelos [**19**] independently initiated the study of semidualizing modules (under different names): a module C is semidualizing if $\operatorname{Hom}_R(C,C) \cong R$ and $\operatorname{Ext}_R^{\geqslant 1}(C,C) = 0$. Examples include the rank 1 free module and a dualizing (canonical) module, when one exists. Golod [**10**] used these to define G_C-dimension, a refinement of projective dimension, for finitely generated modules. The G_C-dimension of a finitely generated R-module M is the length of the shortest resolution of M by so-called totally C-reflexive modules; see Definition 4.1. Motivated by Enochs and Jenda's extensions in [**7**] of Auslander and Bridger's G-dimension [**2**], Holm and Jørgensen [**12**] have extended this notion to arbitrary modules over a Noetherian ring. The current paper provides a unified and generalized treatment of these concepts, in part by removing the Noetherian hypothesis. The tools

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developed in this paper have been particularly useful for investigating the similarities and differences between certain relative cohomology theories [**15, 16**] and the stability properties of operators on categories [**17**].

Section 2 is devoted to the study of the G_C-projective R-modules, which are built from projective and C-projective modules; see Definition 2.1. We show that every module that is either projective or C-projective is G_C-projective in Proposition 2.6. In particular, every R-module admits a G_C-projective resolution. Further properties of the class of G_C-projective modules are contained in the following result; see Theorem 2.8.

Theorem 1. *The class of G_C-projectives is projectively resolving and closed under direct summands. The class of finitely generated G_C-projective R-modules is closed under summands. The set of G_C-projective R-modules admitting a degreewise finite free resolution is finite projectively resolving.*

Section 2 ends with basic properties of the resulting G_C-projective dimension. In particular, we show that, for an R-module M of G_C-projective dimension $n > 0$, the nth kernel in any G_C-projective resolution is G_C-projective.

Within the class of G_C-projective resolutions, the proper ones exhibit particularly good lifting properties; see subsection 1.5. These are the subject of Section 3. Coupled with Proposition 3.4, the following result shows that every module of finite G_C-projective dimension admits a proper G_C-projective resolution; see Theorem 3.6.

Theorem 2. *If M is an R-module with finite G_C-projective dimension, then M admits a strict G_C-projective resolution, that is, a G_C-resolution of the form*

$$0 \longrightarrow C \otimes_R P_n \longrightarrow \cdots \longrightarrow C \otimes_R P_1 \longrightarrow G \longrightarrow M \longrightarrow 0$$

where G is G_C-projective and $P_1, \ldots, P_n$ are projective.

These strict G_C-projective resolutions give rise to G_C-projective approximations, which are similar to the maximal Cohen-Macaulay approximations of Auslander and Buchweitz in [**3**].

Section 4 is concerned with comparing the G_C-projective and totally C-reflexive properties; see Definition 4.1. The next result is Theorem 4.4, which extends a result of Avramov, Buchweitz, Martsinkovsky and Reiten [**5**, (4.2.6)].

Theorem 3. *If M and $\operatorname{Hom}_R(M,C)$ admit degree wise finite projective resolutions, then M is G_C-projective if and only if it is totally C-reflexive.*

The paper closes with several results on minimal proper G_C-projective resolutions of finitely generated modules over Noetherian local rings.

1. Preliminaries.

Throughout this work R is a commutative ring with unity, $\mathcal{X} = \mathcal{X}(R)$ is a class of unital R-modules, and $\mathcal{X}^f$ is the subclass of finitely generated R-modules in $\mathcal{X}$.

Homological dimensions built from resolutions are fundamental to this work. The prototypes are the projective and injective dimensions.

1.1. An *R-complex* is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots$$

such that $\partial^X_{n-1}\partial^X_n = 0$ for each integer n; the nth *homology module* of X is $\mathrm{H}_n(X) = \operatorname{Ker}(\partial^X_n)/\operatorname{Im}(\partial^X_{n+1})$. A morphism of complexes $\alpha\colon X \to Y$ induces homomorphisms $\mathrm{H}_n(\alpha)\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$, and α is a *quasiisomorphism* when each $\mathrm{H}_n(\alpha)$ is bijective.

The complex X is *bounded* if $X_n = 0$ for $|n| \gg 0$; it is *acyclic* if $X_{-n} = 0 = \mathrm{H}_n(X)$ for each $n > 0$. When X is acyclic, the natural morphism $X \to \mathrm{H}_0(X) = M$ is a quasiisomorphism, and X is an $\mathcal{X}$-*projective resolution* of M if each X_n is in $\mathcal{X}$; in this event, the exact sequence

$$X^+ = \cdots \xrightarrow{\partial^X_2} X_1 \xrightarrow{\partial^X_1} X_0 \longrightarrow M \longrightarrow 0$$

is the *augmented $\mathcal{X}$-projective resolution* of M associated to X. Dually, one defines $\mathcal{X}$-coresolutions and augmented $\mathcal{X}$-coresolutions. The $\mathcal{X}$-

projective dimension of M is defined as

$$\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-projective resolution of } M\}.$$

The nonzero modules in $\mathcal{X}$ are precisely the modules of $\mathcal{X}$-pd 0.

1.2. The class $\mathcal{X}$ is *projectively resolving* if

(a) $\mathcal{X}$ contains every projective R-module, and

(b) for every exact sequence of R-modules $0 \to M' \to M \to M'' \to 0$ with $M'' \in \mathcal{X}$, one has $M \in \mathcal{X}$ if and only if $M' \in \mathcal{X}$.

The class $\mathcal{X}$ is *finite projectively resolving* if

(a) $\mathcal{X}$ consists entirely of finitely generated R-modules,

(b) $\mathcal{X}$ contains every finitely generated projective R-module, and

(c) for every exact sequence of finitely generated R-modules $0 \to M' \to M \to M'' \to 0$ with $M'' \in \mathcal{X}$, one has $M \in \mathcal{X}$ if and only if $M' \in \mathcal{X}$.

1.3. Consider an exact sequence of R-modules

$$0 \to M' \to M \to M'' \to 0.$$

The class $\mathcal{X}$ is *closed under extensions* when M', $M'' \in \mathcal{X}$ implies $M \in \mathcal{X}$, *closed under kernels of epimorphisms* when M, $M'' \in \mathcal{X}$ implies $M' \in \mathcal{X}$ and *closed under cokernels of monomorphisms* when M', $M \in \mathcal{X}$ implies $M' \in \mathcal{X}$.

1.4. Let M be an R-module. If $X \in \mathcal{X}$ and $\phi: X \to M$ is a homomorphism, the pair (X, ϕ) is an *$\mathcal{X}$-precover* of M when, for every homomorphism $\psi: Y \to M$ where $Y \in \mathcal{X}$, there exists a homomorphism $f: Y \to X$ such that $\phi f = \psi$. Enochs and Jenda introduced this terminology, which can be found in [**8**].

1.5. An R-complex Z is *$\mathcal{X}$-proper* if the complex $\text{Hom}_R(Y, Z)$ is exact for each $Y \in \mathcal{X}$. If $\mathcal{X}$ contains R and Z is $\mathcal{X}$-proper, then Z is exact.

An $\mathcal{X}$-resolution X of M is *$\mathcal{X}$-proper* if the augmented resolution X^+ is $\mathcal{X}$-proper; by [**11**, (1.8)] $\mathcal{X}$-proper resolutions are unique up to homotopy. Accordingly, when M admits an $\mathcal{X}$-proper resolution X and N is an R-module, the *nth relative homology module* and the *nth relative cohomology module*

$$\operatorname{Tor}_n^{\mathcal{X}}(M,N) = \mathrm{H}_n(X \otimes_R N) \qquad \operatorname{Ext}_{\mathcal{X}}^n(M,N) = \mathrm{H}_{-n}\operatorname{Hom}_R(X,N)$$

are well-defined for each integer n.

1.6. A *degreewise finite projective* (*respectively, free*) *resolution* of an R-module M is a projective (respectively, free) resolution P of M such that each P_i is a finitely generated projective (respectively, free). Note that M admits a degreewise finite projective resolution if and only if it admits a degreewise finite free resolution. However, it is possible for a module to admit a bounded degreewise finite projective resolution but not admit a bounded degreewise finite free resolution. For example, if $R = k_1 \oplus k_2$, where k_1 and k_2 are fields, then $M = k_1 \oplus 0$ is a projective R module, but it does not admit a bounded free resolution.

The next result follows from well-known constructions, but the author is unable to locate an elementary reference.

Lemma 1.7. *The class of R-modules admitting a degreewise finite projective* (*respectively, free*) *resolution is closed under summands, extensions, kernels of epimorphisms, and cokernels of monomorphisms.*

1.8. An R-module C is *semidualizing* if

(a) C admits a degreewise finite projective resolution,

(b) The natural homothety map $R \to \operatorname{Hom}_R(C,C)$ is an isomorphism, and

(c) $\operatorname{Ext}_R^{\geqslant 1}(C,C) = 0$.

A free R-module of rank one is semidualizing. If R is Noetherian and admits a dualizing module D, then D is a semidualizing.

Note that this definition agrees with the established definition when R is Noetherian, in which case condition (a) is equivalent to C being finitely generated. Also, since $\operatorname{Hom}_R(C,C) \cong R$ any homomorphism $\phi: C^n \to C^m$ can be represented by an $m \times n$ matrix with entries in R.

Finally, note that the hypothesis that C admits a degreewise finite free resolution does not imply that R is Noetherian. As one example, take R to be a non-Noetherian ring and $C = R$. For an example with $C \neq R$, let $Q \to R$ be a flat local homomorphism of commutative rings, with Q Noetherian and R non-Noetherian. If C' is semidualizing over Q with degreewise finite projective resolution F, then $C = C' \otimes_Q R$ is semidualizing over the non-Noetherian ring R with degreewise finite projective resolution $F \otimes_Q R$.

1.9. Avramov and Martsinkovsky define a general notion of minimality for complexes in [**4**, Section 1]: A complex B is *minimal* if every homotopy equivalence $f: B \to B$ is an isomorphism. Furthermore, by [**4**, (1.7)] a complex B is minimal if and only if every morphism $f: B \to B$ homotopic to the identity map on B is an isomorphism.

1.10. Let M, N and F be R-modules. The *tensor evaluation* homomorphism

$$\omega_{MNF}: \operatorname{Hom}_R(M,N) \otimes_R F \longrightarrow \operatorname{Hom}_R(M, N \otimes_R F)$$

is defined by $\omega_{MNF}(\psi \otimes_R f)(m) = \psi(m) \otimes_R f$. It is straightforward to verify that this is an isomorphism when M is a finitely generated free (or projective) R-module.

Lemma 1.11. *Let F be a flat R-module.*

(a) *If M admits a degreewise finite projective resolution P, then for $i \geq 0$ there are isomorphisms $\operatorname{Ext}^i_R(M, C \otimes_R F) \cong \operatorname{Ext}^i_R(M,C) \otimes_R F$.*

(b) *If M admits a degreewise finite projective resolution and $\operatorname{Ext}^i_R(M, C) = 0$ for some $i \geq 0$, then $\operatorname{Ext}^i_R(M, C \otimes_R F) = 0$.*

(c) *If M admits a degreewise finite projective resolution, F is faithfully flat, and $\operatorname{Ext}^i_R(M, C \otimes_R F) = 0$ for some $i \geq 0$, then $\operatorname{Ext}^i_R(M,C) = 0$.*

Proof. (a) The maps ω_{P_iCF} are isomorphisms by 1.10; hence, the desired conclusion follows from the flatness of F and the resulting isomorphism of complexes

$$\operatorname{Hom}_R(P, C \otimes_R F) \cong \operatorname{Hom}_R(P, C) \otimes_R F.$$

(b) and (c). These follow directly from (a). □

1.12. An R-module is *C-projective* if it has the form $C \otimes_R P$ for some projective P. Set $\mathcal{P}_C = \mathcal{P}_C(R) = \{C \otimes_R P \mid P \text{ is projective}\}$. These modules are studied extensively (in the non-commutative setting) in [**12**]. We state for later use a Lemma that follows readily from [**12**, (3.6, 5.6, 6.8)].

Lemma 1.13. *Consider an exact sequence of R-modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

When M'' is a (finitely generated) C-projective, M' is a (finitely generated) C-projective if and only if M is a (finitely generated) C-projective. If all of the modules in (1) *are C-projective, then* (1) *splits.*

1.14. The *Bass class with respect to C*, denoted $\mathcal{B}_C$ or $\mathcal{B}_C(R)$, consists of all R-modules N satisfying

(a) $\operatorname{Ext}_R^{\geqslant 1}(C, N) = 0$,

(b) $\operatorname{Tor}^R_{\geqslant 1}(C, \operatorname{Hom}_R(C, N)) = 0$, and

(c) The evaluation map $\nu_{CN}\colon C \otimes_R \operatorname{Hom}_R(C, N) \to N$ is an isomorphism.

2. G_C-projective modules. In this section we define and develop properties of G_C-projective R-modules and the associated G_C-projective dimension. We begin with a definition which extends the notion of G_C-projective modules found in [**12**] (where they are referred to as C-Gorenstein projective modules) to the non-Noetherian setting.

Definition 2.1. A *complete PC-resolution* is an exact sequence of R-modules

$$(2) \qquad X = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \otimes_R Q^0 \longrightarrow C \otimes_R Q^1 \longrightarrow \cdots$$

where each P_i and Q^i is projective, and such that the complex $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact for each projective R-module Q.

An R-module M is *G_C-projective* if there exists a complete PC-resolution as in (2) with $M \cong \operatorname{coker}(P_1 \to P_0)$.

Note that when $C = R$, the definitions above correspond to the definitions of complete resolutions and Gorenstein projective modules. The definition immediately gives rise to the following, which generalizes [**11**, (2.3)].

Proposition 2.2. *A module M is G_C-projective if and only if $\operatorname{Ext}_R^{\geqslant 1}(M, C \otimes_R P) = 0$ and M admits a $\mathcal{P}_C$-coresolution Y with $\operatorname{Hom}_R(Y, C \otimes_R Q)$ exact for any projective Q.*

Observation 2.3. If M is a G_C-projective R-module, then M admits a complete PC-resolution of the form

$$(3) \qquad \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots$$

where each F_i and F^i is free. To construct such a sequence from a given complete PC-resolution, argue as in [**11**, (2.4)].

When X is a complex of the form (2), then the complex $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact for all projective R-modules Q if and only if the complex $\operatorname{Hom}_R(X, C \otimes_R F)$ is exact for all free R-modules F. One implication is immediate. For the other, note that if $Q \oplus Q'$ is free, then we have the following isomorphism of complexes $\operatorname{Hom}_R(X, C \otimes_R (Q \oplus Q')) \cong \operatorname{Hom}_R(X, C \otimes_R Q) \oplus \operatorname{Hom}_R(X, C \otimes_R Q')$.

The next three results provide ways to create G_C-projective modules.

Proposition 2.4. *If X_λ is a collection of complete PC-resolutions, then $\coprod_\lambda X_\lambda$ is a complete PC-resolution. Thus, the class of (finitely generated) G_C-projective R-modules is closed under (finite) direct sums.*

Proof. For any projective R-module Q there is an isomorphism,

$$\operatorname{Hom}_R\Big(\coprod_\lambda X_\lambda, C \otimes_R Q\Big) \cong \prod_\lambda \operatorname{Hom}_R(X_\lambda, C \otimes_R Q).$$

Thus, if the complex $\operatorname{Hom}_R(X_\lambda, C \otimes_R Q)$ is exact for all λ, then the complex $\operatorname{Hom}_R(\coprod_\lambda X_\lambda, C \otimes_R Q)$ is exact. It follows that a (finite) direct sum of (finitely generated) G_C-projective R-modules is a (finitely generated) G_C-projective$R$-module. □

Lemma 2.5. *Let P and Q be projective R-modules, and let X be a complex of R-modules. If the complex $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact, then the complex $\operatorname{Hom}_R(P \otimes_R X, C \otimes_R Q)$ is exact. Thus, if X is a complete PC-resolution of an R-module M, then $P \otimes_R X$ is a complete PC-resolution of $P \otimes_R M$. The converses hold when P is faithfully projective.*

Proof. Assume the complex $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact. Since $\operatorname{Hom}_R(P, -)$ is an exact functor, the isomorphism of complexes given by Hom-tensor adjointness

$$\operatorname{Hom}_R(P \otimes_R X, C \otimes_R Q) \cong \operatorname{Hom}_R(P, \operatorname{Hom}_R(X, C \otimes_R Q))$$

implies that $\operatorname{Hom}_R(P \otimes_R X, C \otimes_R Q)$ is exact. It is now straightforward to see that if X is a complete PC-resolution of an R-module M, then $P \otimes_R X$ is a complete PC-resolution of $P \otimes_R M$.

If P is faithfully projective, then the complex $\operatorname{Hom}_R(P, \operatorname{Hom}_R(X, C \otimes_R Q))$ is exact if and only if the complex $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact. □

Proposition 2.6. *If P is R-projective, then P and $C \otimes_R P$ are G_C-projective. Thus, every R-module admits a G_C-projective resolution.*

Proof. Using Lemma 2.5, it suffices to construct complete PC-resolutions of C and R. By definition, C admits an augmented degreewise finite free resolution

$$X = \cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow C \longrightarrow 0,$$

and this is a complete PC-resolution of C. Indeed, the complex X is exact by definition and $C \cong \operatorname{Coker}(R^{\beta_1} \to R^{\beta_0})$. Furthermore, the complex $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact for all projective R-modules Q by Lemma 1.11 (b), because $\operatorname{Ext}_R^{\geqslant 1}(C, C) = 0$. Thus, C is G_C-projective.

We now show that

$$\operatorname{Hom}_R(X,C) = 0 \longrightarrow R \longrightarrow C^{\beta_0} \longrightarrow C^{\beta_1} \longrightarrow \cdots$$

is a complete PC-resolution of R. First, left exactness of $\operatorname{Hom}_R(-,C)$ and the equality $\operatorname{Ext}_R^{\geqslant 1}(C,C)=0$ imply $\operatorname{Hom}_R(X,C)$ is exact. Moreover, since $\operatorname{Hom}_R(X,C)$ consists of finitely presented modules, for any projective R-module Q, tensor evaluation provides the first isomorphism of complexes

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Hom}_R(X,C), C\otimes_R Q) &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(X,C),C)\otimes_R Q\\ &\cong X\otimes_R Q.\end{aligned}$$

The second isomorphism follows from the fact that $\operatorname{Hom}_R(C,C)\cong R$. These complexes are exact since the complex X is exact and Q is flat.

Finally, since the class of G_C-projective R-modules contains the class of projective R-modules, every R-module admits a G_C-projective resolution. □

When $C=R$, the following proposition is contained in [**11**, (2.3)]. The proof is similar to that of [**4**, (2.2)].

Proposition 2.7. *If X is a complete PC-resolution and L is an R-module admitting a bounded $\mathcal{P}_C$-projective resolution, then the complex* $\operatorname{Hom}(X,L)$ *is exact. Thus, if M is G_C-projective, then* $\operatorname{Ext}_R^{\geqslant 1}(M,L)=0$.

The following result is Theorem 1 from the introduction.

Theorem 2.8. *The class of G_C-projectives is projectively resolving and closed under direct summands. The class of finite G_C-projective R-modules is closed under summands. The class of G_C-projective R-modules admitting a degreewise finite projective resolution is finite projectively resolving.*

Proof. Consider an exact sequence

$$0\longrightarrow M' \xrightarrow{\ \iota\ } M \xrightarrow{\ \rho\ } M'' \longrightarrow 0$$

of R-modules. First, assume that M' and M'' are G_C-projective with complete PC-resolutions X' and X'', respectively. Use the Horseshoe lemmas in [**11**, (1.7)] and [**14**, (6.20)], together with the fact that the classes of projective and C-projective R-modules are closed under extensions to construct a complex

$$X = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \otimes_R Q^0 \longrightarrow C \otimes_R Q^1 \longrightarrow \cdots$$

with P_i and Q^i projective and a degreeswise split exact sequence of complexes

$$0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

such that $\operatorname{Coker}(P_1 \to P_0) \cong M$. To show that M is G_C-projective, it suffices to show that $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact for all projective R-modules Q. The sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_R(X'', C \otimes_R Q) \longrightarrow \operatorname{Hom}_R(X, C \otimes_R Q) \\ \longrightarrow \operatorname{Hom}_R(X', C \otimes_R Q) \longrightarrow 0 \end{aligned}$$

is an exact sequence of complexes. Since the outer two complexes are exact, the associated long exact sequence in homology shows that the middle one is also exact.

Next, assume that M and M'' are G_C-projective with complete PC-resolutions X and X'', respectively. Comparison lemmas for resolutions, see e.g., [**11**, (1.8)] and by [**14**, (6.9)], provide a morphism of chain complexes $\phi\colon X \to X''$ inducing ρ on the degree 0 cokernels. By adding complexes of the form $0 \to P_i'' \xrightarrow{\text{id}} P_i'' \to 0$ and $0 \to C \otimes_R (Q^i)'' \xrightarrow{\text{id}} C \otimes_R (Q^i)'' \to 0$ to X, one can assume ϕ is surjective. Since both the class of projective and C-projective modules are closed under kernels of epimorphisms, see Theorem 1.13, the complex $X' = \ker(\phi)$ has the form

$$X' = \cdots \longrightarrow P_1' \longrightarrow P_0' \longrightarrow C \otimes_R (Q^0)' \longrightarrow C \otimes_R (Q^1)' \longrightarrow \cdots$$

with P_i' and $(Q^i)'$ projective. The exact sequence $0 \to X' \to X \to X'' \to 0$ is degreewise split by Lemma 1.13, so an argument similar to that of the previous paragraph implies that X' is a complete PC-resolution and M' is G_C-projective.

Since the class of G_C-projective R-modules is projectively resolving by the previous paragraphs and closed under arbitrary direct sums by Proposition 2.4, it follows from Eilenberg's swindle [**11**, (1.4)]holm:ghd that they are also closed under direct summands.

When the exact sequence (4) consists of modules admitting a degree-wise finite projective resolution, one can check that the above constructions can be carried out using finite modules. Finally, if G is a finitely generated G_C-projective, then any summand is also G_C-projective. Since summands of finitely generated modules are finitely generated, this implies that the class of finitely generated G_C-projective modules is closed under summands. □

When $C = R$, the next proposition follows readily from the symmetry of the definition of the Gorenstein projectives. However, in the case of G_C-projectives, the situation is more subtle. Nonetheless, significant symmetry exists.

Proposition 2.9. *Every cokernel in a complete PC-resolution is G_C-projective.*

Proof. Consider a complete PC-resolution

$$X = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \otimes_R Q^0 \longrightarrow C \otimes_R Q^1 \longrightarrow \cdots$$

and set $M = \operatorname{Coker}(P_1 \to P_0)$ and $K = \operatorname{Coker}(P_2 \to P_1)$. Since M and P_0 are G_C-projective, the exact sequence

$$0 \longrightarrow K \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

shows that K is G_C-projective; see Theorem 2.8. Inductively, one can show that $\operatorname{Coker}(P_{i+1} \to P_i)$ is G_C-projective for every positive integer i.

Set $N_{-1} = M$, $N_0 = \operatorname{Coker}(P_0 \to C \otimes_R Q^0)$, and $N_i = \operatorname{Coker}(C \otimes_R Q^{i-1} \to C \otimes_R Q^i)$ for $i \geq 1$. Using Proposition 2.2, we will be done once we verify that $\operatorname{Ext}_R^{\geqslant 1}(N_i, C \otimes_R Q) = 0$ for all projective R-modules Q. For each $i > -1$, consider the exact sequence

$$Y_i = 0 \longrightarrow N_i \longrightarrow C \otimes_R Q^{i+1} \longrightarrow N_{i+1} \longrightarrow 0.$$

By induction, one has $\operatorname{Ext}_R^{\geqslant 1}(N_i, C \otimes_R Q) = 0$. Proposition 2.6 implies that $C \otimes_R Q^{i+1}$ is G_C-projective for each $i \geq 0$, and hence $\operatorname{Ext}_R^{\geqslant 1}(C \otimes_R Q^{i+1}, C \otimes_R Q) = 0$. The long exact sequence in $\operatorname{Ext}_R(-, C \otimes_R Q)$ associated to Y_i provides $\operatorname{Ext}_R^{\geqslant 2}(N_{i+1}, C \otimes_R Q) = 0$. Furthermore, since $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact, so is the complex $\operatorname{Hom}_R(Y_i, C \otimes_R Q)$. Therefore, since $\operatorname{Ext}_R^1(C \otimes_R Q^{i+1}, C \otimes_R Q) = 0$, one has $\operatorname{Ext}_R^1(N_{i+1}, C \otimes_R Q) = 0$. □

The class of G_C-projective R-modules can be used to define the G_C-projective dimension, denoted $\mathrm{G_C}\text{-}\mathrm{pd}_R(-)$; see1.1. The following five results are proved similarly to [**11**, (2.18), 2.19), (2.20), (2.21), (2.24)]. We collect them here for ease of reference.

Proposition 2.10. *Let $0 \to K \to G \to M \to 0$ be an exact sequence of R-modules where G is G_C-projective. If M is G_C-projective, then so is K. Otherwise, one has $\mathrm{G_C}\text{-}\mathrm{pd}_R(K) = \mathrm{G_C}\text{-}\mathrm{pd}_R(M) - 1$.*

Proposition 2.11. *If $(M_\lambda)_{\lambda \in \Lambda}$ is a collection of R-modules, then*

$$\mathrm{G_C}\text{-}\mathrm{pd}_R\left(\coprod_\lambda M_\lambda\right) = \sup\{\mathrm{G_C}\text{-}\mathrm{pd}_R(M_\lambda) \mid \lambda \in \Lambda\}.$$

Proposition 2.12. *Let M be an R-module such that $\mathrm{G_C}\text{-}\mathrm{pd}_R(M)$ is finite, and let n be an integer. The following are equivalent.*

(i) $\mathrm{G_C}\text{-}\mathrm{pd}_R(M) \leq n$.

(ii) $\operatorname{Ext}_R^i(M, L) = 0$ *for all $i > n$ and all R-modules L with* $\mathcal{P}_\mathrm{C}\text{-}\mathrm{pd}(L) < \infty$.

(iii) $\operatorname{Ext}_R^i(M, C \otimes_R P) = 0$ *for all $i > n$ and all projective R-modules P.*

(iv) *In every exact sequence $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ where the G_i are G_C-projective, one has thatK_n is also G_C-projective.*

Proposition 2.13. *Let M be an R-module with $\mathrm{G_C}\text{-}\mathrm{pd}_R(M) < \infty$. If M admits a degreewise finite projective resolution, then there is an equality $\mathrm{G_C}\text{-}\mathrm{pd}_R(M) = \sup\{i \in \mathbf{Z} \mid \operatorname{Ext}_R^i(M, C) \neq 0\}$.*

Proposition 2.14. *If two modules in an exact sequence have finite G_C-projective dimension, then so does the third.*

When $C = R$, there are numerous proofs (see e.g., [**4**, (3.4)] or [**11**, (2.27)]) of the following: if M is an R-module of finite projective dimension, then there is an equality $\mathrm{pd}_R(M) = \text{G-pd}_R(M)$. Since G_C-dimension can be viewed as a refinement of projective dimension, it makes sense to ask the following:

Question 2.15. If M is an R-module of finite projective dimension, must $\mathrm{pd}_R(M) = \mathrm{G_C}\text{-pd}_R(M)$?

Over a Noetherian, local ring, the affirmative answer in the case of finitely generated modules follows immediately from the AB-formulas for projective dimension and G_C-dimension. Over a non-local Noetherian ring, an affirmative answer follows from work in [**11, 12**]. However, as of the writing of this paper, the author does not know the answer to this question in general.

However, arguably the more natural comparison is between $\mathcal{P}_C$-dimension and G_C-dimension. We have the following.

Proposition 2.16. *If M is an R-module of finite $\mathcal{P}_C$-projective dimension, then $\mathcal{P}_\mathrm{C}\text{-pd}_R(M) = \mathrm{G_C}\text{-pd}_R(M)$.*

Proof. Using Proposition 2.12, it suffices to show that if M is G_C-projective with finite $\mathcal{P}_C$-projective dimension, then M is C-projective. To this end, consider an exact sequence of the form

$$0 \longrightarrow K \longrightarrow C \otimes_R P \longrightarrow M \longrightarrow 0$$

where P is projective and $\mathrm{G_C}\text{-pd}_R(K) < \infty$. By Proposition 2.12, $\mathrm{Ext}\,^1_R(M, K) = 0$ so the above sequence splits, forcing M to be a summand of $C \otimes_R P$. Since the class of C-projectives is closed under summands by 1.13, this implies that M is C-projective, as desired. □

3. **G_C-projective resolutions and approximations.** In this section we prove the existence of strict and proper G_C-projective resolutions and of G_C-projective approximations. These will give rise to well-defined relative (co)homology functors, see Remark 3.7, which are further studied in [**15, 16**]. We begin with the requisite definitions.

Definition 3.1. Let M be an R-module of finite G_C-projective dimension. A *strict G_C-projective resolution* of M is a bounded G_C-projective resolution G such that for $i \geq 1$, there exists a projective R-module P_i such that $G_i \cong C \otimes_R P_i$. This gives rise to an associated *G_C-projective approximation* of M; that is, an exact sequence of R-modules

$$0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0$$

in which $\mathcal{P}_{\mathrm{C}}\text{-}\dim_R(K)$ is finite and G is G_C-projective.

We provide two examples. The first corresponds to the situation when C is dualizing, the second to when $C = R$.

Example 3.2. When R is a local Cohen-Macaulay ring with dualizing module D, Auslander and Buchweitz [**3**] show that every finitely generated module M admits a maximal Cohen-Macaulay approximation, that is, an exact sequence of the form

$$0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0$$

where K has finite injective dimension and G is maximal Cohen-Macaulay. This gives rise to a resolution of the form

$$0 \longrightarrow D^{\alpha_n} \longrightarrow \cdots \longrightarrow D^{\alpha_0} \longrightarrow G \longrightarrow M \longrightarrow 0$$

where G is a maximal Cohen-Macaulay module.

Example 3.3. When R is Noetherian and M is an R-module of finite G-dimension, Avramov and Martsinkovsky [**4**, (3.8)] and Holm [**11**, (2.10)] provide several constructions of G-approximations, that is, exact sequences of the form

$$0 \longrightarrow K \longrightarrow G \longrightarrow M \longrightarrow 0$$

where K has finite projective dimension and G is totally reflexive (see 4.1). These give rise to strict G-approximations, namely, exact sequences of the form

$$0 \longrightarrow R^{\alpha_n} \longrightarrow \cdots \longrightarrow R^{\alpha_0} \longrightarrow G \longrightarrow M \longrightarrow 0$$

where G is totally reflexive.

The existence of strict G_C-projective resolutions implies the existence of proper G_C-projective resolutions.

Proposition 3.4. *Augmented strict G_C-projective resolutions are G_C-proper.*

Proof. Let H be a G_C-projective R-module and

$$(6) \qquad 0 \longrightarrow C \otimes_R P_n \longrightarrow \cdots \longrightarrow C \otimes_R P_1 \longrightarrow G \longrightarrow M \longrightarrow 0$$

an augmented strict G_C-projective resolution. Since $\operatorname{Ext}^1_R(H, C \otimes_R P_n) = 0$ by Proposition 2.12, applying $\operatorname{Hom}_R(H, -)$ to the exact sequence $0 \to C \otimes_R P_n \to C \otimes_R P_{n-1} \to K_{n-2} \to 0$ provides an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(H, C \otimes_R P_n) \longrightarrow \operatorname{Hom}_R(H, C \otimes_R P_{n-1}) \longrightarrow \operatorname{Hom}_R(H, K_{n-2}) \longrightarrow 0.$$

Continuing to break the exact sequence (6) into short exact sequences and applying Proposition 2.12 shows that (6) is G_C-proper. □

The existence of a strict G_C-projective resolution for a module M of finite G_C-projective dimension which is in the Bass class of R with respect to C (see 1.14) was shown in [**12**, (5.9)]. We offer an alternative construction, motivated by [**3**], that has the added advantage of not requiring any Bass class assumption. When R is Noetherian and M is finitely generated, this is [**1**, (2.13)]. We begin by proving a lemma.

Lemma 3.5. *Let $\phi\colon G \to V$ be a homomorphism between G_C-projective R-modules. If $0 \to G \xrightarrow{\psi} U \to N \to 0$ is an exact sequence of*

R-modules such that N is G_C-projective, then the pushout module H of the maps ϕ and ψ is G_C-projective.

Proof. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{\psi} & U & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & V & \longrightarrow & H & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

Since N and V are G_C-projective, Proposition 2.8 implies H is G_C-projective. □

The next result contains Theorem 2 from the introduction.

Theorem 3.6. *If M is an R-module with finite G_C-projective dimension, then M admits a strict G_C-projective resolution and hence a G_C-projective approximation.*

Proof. Assume $\mathrm{G_C}\text{-pd}_R(M) = n$. By Proposition 2.12, truncating an augmented free resolution of M yields an augmented G_C-projective resolution of M

$$0 \longrightarrow G_n \xrightarrow{\phi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

A complete PC-resolution of G_n gives rise to an exact sequence

$$0 \longrightarrow G_n \xrightarrow{\psi} C \otimes_R P_n \longrightarrow N \longrightarrow 0$$

where P_n is projective and N is G_C-projective. Lemma 3.5 provides a commutative diagram (note that the orientation is not the same as in the previous lemma)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_n & \xrightarrow{\phi_n} & F_{n-1} & \longrightarrow & F_{n-2} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \psi} & & \downarrow & & & & \\ 0 & \longrightarrow & C \otimes_R P_n & \xrightarrow{\phi'} & G_{n-1} & & & & \end{array}$$

with exact rows in which G_{n-1} is G_C-projective. As G_{n-1} is a pushout module, the maps ϕ_n and ϕ' have isomorphic cokernels, resulting in a G_C-resolution

$$0 \longrightarrow C \otimes_R P_n \xrightarrow{\phi'} G_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0.$$

Continuing this process yields a strict G_C-projective resolution of M. □

Remark 3.7. As noted in the introduction, Proposition 3.4 and Theorem 3.6 imply the following: every module M of finite G_C-projective dimension admits a proper G_C-projective resolution. Hence, the relative (co)homology functors $\operatorname{Ext}^n_{G_C}(M,-)$ and $\operatorname{Tor}^n_{G_C}(M,-)$ are well-defined for each integer n; see 1.5.

We close the section with a complement to Proposition 2.11, which is proved as in [**12**, (2.11)].

Corollary 3.8. *Let* $0 \to G' \to G \to M \to 0$ *be an exact sequence of* R*-modules. Assume* G *and* G' *are* G_C*-projective and that* $\operatorname{Ext}^1_R(M, C \otimes_R Q) = 0$ *for all projective* R*-modules* Q*. Then* M *is* G_C*-projective.*

4. Connections with totally C-reflexive modules. In this section, we reconnect with Golod's G_C-dimension.

Definition 4.1. Let M be an R-module, and assume that M and $\operatorname{Hom}_R(M,C)$ admit a degreewise finite projective resolution. The module M is *totally* C*-reflexive* if the following conditions hold

(a) The natural biduality map $M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C)$ is an isomorphism,

(b) $\operatorname{Ext}^{\geqslant 1}_R(M,C) = 0$, and

(c) $\operatorname{Ext}^{\geqslant 1}_R(\operatorname{Hom}_R(M,C),C) = 0$.

Observation 4.2. Finitely generated free modules are totally C-reflexive, as is the R-module C^n for any positive integer n. If M is totally C-reflexive, then it is straightforward to check that any summand M' of M is also totally C-reflexive (using Lemma 1.7 to see that M' admits a degreewise finite free resolution). Thus, finitely generated projective R-modules are also totally C-reflexive, and so every finitely generated R-module admits a resolution by totally C-reflexive modules.

When R is Noetherian, the homological dimension which arises by resolving a given module by totally C-reflexive modules is known as the G_C*-dimension* of a module, which was first introduced by Golod; see [**10**]. In the case $C = R$, this is Auslander and Bridger's G*-dimension* [**2**].

Next we provide a useful characterization of totally C-reflexive modules, which generalizes [**5**, (4.1.4)].

Lemma 4.3. *An R-module M is totally C-reflexive if and only if there is an exact sequence of the form*

$$X = \cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow C^{\alpha_0} \longrightarrow C^{\alpha_1} \longrightarrow \cdots \tag{7}$$

with $M \cong \mathrm{Coker}\,(R^{\beta_1} \to R^{\beta_0})$ and such that $\mathrm{Hom}_R(X, C)$ *is exact.*

Proof. Set $(-)^\dagger = \mathrm{Hom}_R(-, C)$. Assume first that M is totally C-reflexive. By definition, there exist augmented degreewise finite free resolutions

$$F = \cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0$$
$$G = \cdots \longrightarrow R^{\alpha_1} \longrightarrow R^{\alpha_0} \longrightarrow M^\dagger \longrightarrow 0.$$

The complexes $F^\dagger$ and $G^\dagger$ are exact, as $\mathrm{Ext}_R^{\geqslant 1}(M, C) = 0 = \mathrm{Ext}_R^{\geqslant 1}(M^\dagger, C)$. The isomorphism $M \cong M^{\dagger\dagger}$ shows that $G^\dagger$ has the form

$$G^\dagger \cong 0 \longrightarrow M \longrightarrow C^{\alpha_0} \longrightarrow C^{\alpha_1} \longrightarrow \cdots .$$

Splicing together the complexes F and $G^\dagger$ provides an exact sequence X of the form (7) with $M \cong \mathrm{Coker}\,(R^{\beta_1} \to R^{\beta_0})$. The fact that $F^\dagger$ and G are exact implies that $X^\dagger$ is exact.

Conversely, assume that M admits a resolution X of the form (7) such that

$$X^\dagger = \cdots \longrightarrow R^{\alpha_1} \longrightarrow R^{\alpha_0} \longrightarrow C^{\beta_0} \longrightarrow C^{\beta_1} \longrightarrow \cdots \tag{8}$$

is exact. Consider the following “soft truncations” of X

$$\begin{aligned} F &= \cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow M \longrightarrow 0 \\ H &= 0 \longrightarrow M \longrightarrow C^{\alpha_0} \longrightarrow C^{\alpha_1} \longrightarrow \cdots . \end{aligned}$$

The complex $X^\dagger$ is exact and therefore so are $F^\dagger$ and $H^\dagger$.

Since F is an augmented free resolution of M, this implies that $\operatorname{Ext}_R^{\geqslant 1}(M, C) = 0$. The biduality maps and exactness of $H^\dagger$ provide a commutative diagram

$$\begin{array}{ccccccccc} H = 0 & \longrightarrow & M & \longrightarrow & C^{\alpha_0} & \longrightarrow & C^{\alpha_1} & \longrightarrow & \cdots \\ & & \big\downarrow \delta_M^C & & \big\downarrow \delta_{C^{\alpha_0}}^C & & \big\downarrow \delta_{C_1^{\alpha}}^C & & \\ H^{\dagger\dagger} = 0 & \longrightarrow & \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C) & \longrightarrow & C^{\alpha_0} & \longrightarrow & C^{\alpha_1} & \longrightarrow & \cdots . \end{array}$$

The top row is exact by definition, while a routine diagram chase and the fact that $\operatorname{Hom}_R(-, C)$ is left exact shows that the bottom row is exact. Since $\delta_{C^{\alpha_1}}^C$ and $\delta_{C^{\alpha_0}}^C$ are isomorphisms, the snake lemma implies that the map δ_M^C is an isomorphism. Finally, the exact sequence $H^\dagger$ is an augmented degreewise finite free resolution of $M^\dagger$. Thus, exactness of $H^{\dagger\dagger}$ implies that $\operatorname{Ext}_R^{\geqslant 1}(M^\dagger, C) = 0$ and thus M is totally C-reflexive. $\square$

The next result is Theorem 3 from the introduction.

Theorem 4.4. *If M and $\operatorname{Hom}_R(M, C)$ admit degreewise finite projective resolutions, then M is G_C-projective if and only if it is totally C-reflexive.*

Proof. Set $(-)^\dagger = \operatorname{Hom}_R(-, C)$ and let F and G be degreewise finite free resolutions of M and $\operatorname{Hom}_R(M, C)$, respectively.

Assume first that M is totally C-reflexive. By Lemma 4.3, there is an exact sequence

$$X = \cdots \longrightarrow R^{\beta_1} \longrightarrow R^{\beta_0} \longrightarrow C^{\alpha_0} \longrightarrow C^{\alpha_1} \longrightarrow \cdots$$

with $M \cong \operatorname{Coker}(R^{\beta_1} \to R^{\beta_0})$ and such that $\operatorname{Hom}_R(X, C)$ is exact. An argument similar to the one used in the proof of Lemma 1.11 implies that the complex $\operatorname{Hom}_R(X, C \otimes_R P)$ is exact, and so X is a complete PC-resolution of M.

Conversely, assume that M is G_C-projective, and let

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow C \otimes_R F^0 \longrightarrow C \otimes_R F^1 \longrightarrow \cdots$$

be a complete PC-resolution of M in which each F^i is a free R-module; see Observation 2.3. We show M is totally C-reflexive by constructing a complex X as in Lemma 4.3. To this end, it suffices to construct an augmented $\mathcal{P}_C^f$-coresolution

$$Y = 0 \longrightarrow M \longrightarrow C \otimes_R R^{\alpha_0} \longrightarrow C \otimes_R R^{\alpha_1} \longrightarrow \cdots$$

where each α_i is a non-negative integer and $Y^\dagger$ is exact. Indeed, Proposition 2.12 implies that $\operatorname{Ext}_R^{\geqslant 1}(M, C \otimes_R P) = 0$ for any projective R-module P; in particular $\operatorname{Ext}_R^{\geqslant 1}(M, C) = 0$. It follows that $(F^+)^\dagger$ is exact. Splicing together the complexes F and Y provides the desired complex X.

We now build the complex Y piece by piece. Consider the exact sequence

$$0 \longrightarrow M \longrightarrow C \otimes_R F^0 \longrightarrow G \longrightarrow 0$$

arising from the given complete PC-resolution of M. By Proposition 2.9 we know that G is G_C-projective. Since $C \otimes_R F^0$ is a direct sum of copies of C we know that the image of the finitely generated module M is contained in a finite direct sum of copies C. That is, the image of M is contained in a finitely generated submodule $C \otimes_R R^{\alpha_0}$ of $C \otimes_R F^0$. Thus, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & C \otimes_R R^{\alpha_0} & \longrightarrow & H & \longrightarrow & 0 \\ & & \big\downarrow{=} & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & C \otimes_R F^0 & \longrightarrow & G & \longrightarrow & 0. \end{array} \tag{9}$$

Let P be a projective R-module, and set $\mathcal{F} = \operatorname{Hom}_R(-, C \otimes_R P)$. Since $C \otimes_R R^{\alpha_0}$ and G are G_C-projective, we have $\operatorname{Ext}_R^1(G, C \otimes_R P) =$

$0 = \operatorname{Ext}^1_R(C \otimes_R R^{\alpha_0}, C \otimes_R P)$. Hence, applying $\mathcal{F}$ to (9) yields a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}(G) & \longrightarrow & \mathcal{F}(C \otimes_R F^0) & \longrightarrow & \mathcal{F}(M) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow = & & & & \\
0 & \longrightarrow & \mathcal{F}(H) & \longrightarrow & \mathcal{F}(C \otimes_R R^{\alpha_0}) & \longrightarrow & \mathcal{F}(M) & \longrightarrow & \operatorname{Ext}^1_R(H, C \otimes_R P) & \longrightarrow & 0.
\end{array}$$

A routine diagram chase shows that $\operatorname{Ext}^1_R(H, C \otimes_R P) = 0$. Proposition 2.12 and Proposition 2.14 then imply that H is G_C-projective. Since M and R^{α_0} admit degreewise finite projective resolutions, so does H. Applying $\operatorname{Hom}_R(-, C)$ to the exact sequence

$$0 \longrightarrow M \longrightarrow C \otimes_R R^{\alpha_0} \longrightarrow H \longrightarrow 0$$

gives rise to an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(H, C) \longrightarrow R^{\alpha_0} \longrightarrow \operatorname{Hom}_R(M, C) \longrightarrow 0.$$

Here we used the facts that $\operatorname{Hom}_R(C \otimes_R R^{\alpha_0}, C) \cong R^{\alpha_0}$ and $\operatorname{Ext}^1_R(H, C) = 0$ because H is G_C-projective. Since $\operatorname{Hom}_R(M, C)$ and R^{α_0} admit degreewise finite projective resolutions, so does $\operatorname{Hom}_R(H, C)$; see Lemma 1.7. Thus, we can proceed inductively to construct the complex Y with the given properties. □

Corollary 4.5. *If M and $\operatorname{Hom}_R(M, C)$ admit degreewise finite projective resolutions, then M has finite G_C-projective dimension if and only if it has finite G_C-dimension. Moreover, these values coincide.*

Combining this with the AB-formula for G_C-dimension, see [**6**, (3.14)], and Proposition 2.16, we have an AB-formula for modules of finite $\mathcal{P}_C$-dimension.

Corollary 4.6. *Let R be a local, Noetherian ring. If M is a finitely generated R-module of finite $\mathcal{P}_C$-dimension, then*

$$\mathcal{P}_{\mathrm{C}}\text{-pd}_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M).$$

The next result compares with Theorem 3.6.

Corollary 4.7. *If M and* $\operatorname{Hom}_R(M,C)$ *admit degreewise finite projective resolutions and* $\mathrm{G_C}\text{-}\mathrm{pd}_R(M)$ *is finite, then M admits a strict G_C^f-resolution.*

We conclude the paper with results on minimal proper G_C-projective resolutions; see 1.9 for the definition of a minimal complex. Note that Proposition 4.10 (b) shows, in particular, that such resolutions are strict. We begin with two lemmas, the first of which follows as in [**4**, (8.1)].

Lemma 4.8. *Over a local ring R, a complex H consisting of modules in $\mathcal{P}_C^f$ is minimal if and only if $\partial(H) \subseteq \mathfrak{m}H$.*

Lemma 4.9. *Let R be local, Noetherian and M a finitely generated R-module which admits a bounded $\mathcal{P}_C^f$-resolution. Then M admits a minimal $\mathcal{P}_C^f$-resolution.*

Proof. An augmented bounded $\mathcal{P}_C^f$-resolution of M

$$X^+ = \ 0 \longrightarrow C^{\alpha_n} \longrightarrow \dots \longrightarrow C^{\alpha_1} \longrightarrow C^{\alpha_0} \longrightarrow M \longrightarrow 0$$

is also an augmented strict G_C-projective resolution of M and so Proposition 3.4 implies that it is proper. Applying the functor $\operatorname{Hom}_R(C,-)$ to X and using the fact that $\operatorname{Hom}_R(C,C) \cong R$ yields an exact sequence

$$\begin{aligned}\operatorname{Hom}_R(C,X^+) = 0 \longrightarrow R^{\alpha_n} \longrightarrow \dots \longrightarrow R^{\alpha_1}& \\ \longrightarrow R^{\alpha_0} \longrightarrow \operatorname{Hom}_R(C,M) \longrightarrow 0,&\end{aligned}$$

which is an augmented finite free resolution of $\operatorname{Hom}_R(C,M)$. There is an isomorphism of complexes $\operatorname{Hom}_R(C,X) \cong F \oplus G$ where F is an augmented minimal free resolution of $\operatorname{Hom}_R(C,M)$ and G is a contractible complex of free modules. Recall that G is *contractible* if the identity map on G is homotopic to the zero map.

Since M has finite $\mathcal{P}_C$-dimension, [**18**, (2.9)] implies that $M \in \mathcal{B}_C(R)$ (see 1.14 for the definition). This provides the first isomorphism below

$$\begin{aligned} X^+ &\cong C \otimes_R \operatorname{Hom}_R(C, X^+) \\ &\cong (C \otimes_R F) \oplus (C \otimes_R G) \end{aligned}$$

while the second follows from the isomorphism $\operatorname{Hom}_R(C, X^+) \cong F \oplus G$ and the fact that finite direct sums commute with tensor products. It is now straightforward to verify that the complex $C \otimes_R F$ is contractible and that the complex $C \otimes_R F$ is a minimal $\mathcal{P}_C$-resolution of M, as desired. □

The following structure result is the key to demonstrating the differences between the relative cohomology theories $\operatorname{Ext}_{\mathcal{P}_C}$, $\operatorname{Ext}_{\mathcal{G}_C}$, and Ext_R in [**15**].

Proposition 4.10. *Assume that R is local, and let M be a finitely generated R-module of finite G_C-projective dimension. If M and $\operatorname{Hom}_R(M, C)$ admit degreewise finite projective resolutions, then the following hold.*

(a) *The module M admits a minimal proper G_C-projective resolution.*

(b) *A given G_C-projective resolution H of M is minimal if and only if the following conditions hold.*

(1) *$H_n \cong C^{\alpha_n}$ for all $n \geq 1$,*

(2) *$\partial_n^H(H_n) \subseteq \mathfrak{m}H_{n-1}$ for all $n \geq 2$, and*

(3) *$\partial_1^H(H_1)$ contains no nonzero C-summand of H_0.*

Proof. We begin by showing that a G_C-projective resolution H satisfying conditions (1)–(3) is minimal. First, observe that H_0 is finitely generated because M and H_1 are so. Let $\gamma: H \to H$ be a morphism that is homotopic to id_H. Using 1.9, we need to show that γ_n is an isomorphism for each integer n.

For $n \geq 0$, let $\theta_n: H_n \to H_{n+1}$ be maps such that $\gamma_n - \mathrm{id}_{H_n} = \theta_{n-1}\partial_n^H + \partial_{n+1}^H\theta_n$, which exist since γ is homotopic to id_H. For $n \geq 2$, condition (2) implies $\partial_n^H \otimes_R k = 0$, and so $\gamma_n \otimes_R k - \mathrm{id}_{H_n} \otimes_R k = 0$. Nakayama's lemma implies γ_n is a surjective endomorphism, and hence bijective.

Now let $n = 1$. We verify the containment $\operatorname{Im}(\theta_0\partial_1^H) \subseteq \mathfrak{m}C^{\alpha_1}$ and then an argument similar to that in the previous paragraph shows that γ_1 is an isomorphism. Suppose $\operatorname{Im}(\theta_0\partial_1^H) \nsubseteq \mathfrak{m}C^{\alpha_1}$. This means the matrix representation of $\theta_0\partial_1^H$ contains a unit; see 1.8. Thus, there exist maps $\rho\colon C^{\alpha_1} \to C$ and $\iota\colon C \to C^{\alpha_1}$ such that $\rho\theta_0\partial_1^H\iota = \mathrm{id}_C$. This provides a splitting $\partial_1^H\iota$ of $\rho\theta_0\colon H_0 \to C$, and so $H_0 \cong C \oplus \ker(\rho\theta_0)$. Finally, the summand $C\oplus 0$ is isomorphic to $\operatorname{Im}(\partial_1^H\iota)$ which is contained in $\operatorname{Im}(\partial_1^H) \subseteq H_0$, contradicting assumption (3).

The fact that γ_0 is an isomorphism now follows as in [**4**, (8.5)].

Next we show that M admits a resolution satisfying (1)–(3). With the first part of this proof, this will establish part (a). First, note that by Corollary 4.7, there exists a G_C-projective approximation of M

$$Y = 0 \longrightarrow K \longrightarrow H \longrightarrow M \longrightarrow 0$$

where H is totally C-reflexive and K admits a bounded $\mathcal{P}_C^f$-resolution.

If possible, write $H \cong C \oplus H'$ and assume K contains the nonzero C-summand $C \oplus 0$ of H, say $K = C \oplus K'$ for some K'. One checks readily that the compatibility of the two splittings gives rise to a split exact sequence of complexes, written vertically

$$\begin{array}{cccccccccc}
0 & & & 0 & & 0 & & 0 & & \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \\
Y' & 0 & \longrightarrow & K' & \longrightarrow & H' & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \\
Y & 0 & \longrightarrow & K & \longrightarrow & H & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & & \\
X & 0 & \longrightarrow & C & \longrightarrow & C & \longrightarrow & 0 & & \\
\downarrow & & & \downarrow & & \downarrow & & & & \\
0 & & & 0 & & 0. & & & &
\end{array}$$

Since Y' is a G_C^f-projective approximation of M, one can repeat this process. Finitely many iterations yield a G_C-approximation $Y_0 = 0 \to$

$K_0 \to H_0 \to M \to 0$ where K_0 does not contain a nonzero C-summand of H_0. Lemma 4.9 implies that K admits a minimal $\mathcal{P}_C^f$-resolution Z. Splicing together Y_0 and Z at K provides a resolution of M satisfying (1)–(3).

Finally, let G be a resolution of M satisfying conditions (1)–(3), and let H be a minimal proper G_C-projective resolution of M. It follows from [**11**, (1.8)] that G and H are homotopy equivalent. Since G and H are minimal, it follows from Definition 1.9 that they are isomorphic, and so H has the prescribed form. □

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