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PROPERTIES OF FACTORIZATIONS WITH SUCCESSIVE LENGTHS IN ONE-DIMENSIONAL LOCAL DOMAINS

WOLFGANG HASSLER

ABSTRACT. Let D be an atomic domain. Then every non-unit $a \in D\backslash\{0\}$ decomposes (in general in a highly non-unique way) into a product

$$a = u_1 \cdot \ldots \cdot u_n \tag{1}$$

of irreducible elements (atoms) u_i of D. The integer n is called the length of (1) and $\mathsf{L}(a) = \{n \in \mathbb{N} \mid a \text{ decomposes into } n \text{ irreducible elements of } D\}$ is called the set of lengths of a. Two integers $k < l$ are called successive lengths of a if $\mathsf{L}(a) \cap \{m \in \mathbb{N} \mid k \leq m \leq l\} = \{k, l\}$.

Suppose that D is a one-dimensional local domain with finite normalization. Then it is well known that $\Delta(D) = \{l - k \mid 0 \neq a \in D\backslash D^{\times}, k < l \text{ are successive lengths of } a\}$ is finite. Let $0 \neq a \in D$ be a non-unit and denote by $\mathsf{Z}_n(a)$ the set of factorizations of a with length n. In the present paper we investigate the structure of $\mathsf{Z}_n(a)$ and the relations between $\mathsf{Z}_k(a)$ and $\mathsf{Z}_l(a)$ if k and l are successive lengths of a. We prove that $\mathsf{Z}_k(a)$ and $\mathsf{Z}_l(a)$ are "similar" in a very strong sense except if k and l are close to the "boundaries" of $\mathsf{L}(a)$. We show by examples that in the latter case $\mathsf{Z}_k(a)$ and $\mathsf{Z}_l(a)$ may have a completely different structure. Finally, we apply our results to local quadratic orders of algebraic number fields.

1. Introduction.

Let D be an integral domain. A nonzero non-unit $u \in D$ is called irreducible (or an atom) if u does not decompose into a product of two non-units of D. D is called atomic if every nonzero non-unit $a \in D$ has a factorization

$$a = u_1 \cdot \ldots \cdot u_n \tag{2}$$

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into irreducible elements u_i of D. Important examples for atomic domains are Noetherian domains and Krull domains. The integer n is called the length of the factorization (2). In general, elements of atomic domains decompose into atoms in a highly non-unique way. Let D be an atomic domain. For $0 \neq a \in D \backslash D^{\times}$ (where $D^{\times}$ denotes the group of units of D) we call

$$\begin{aligned} &\mathsf{L}(a) = \mathsf{L}_D(a) \\ &= \{n \in \mathbb{N} \mid a \text{ has a factorization into } n \text{ irreducible elements of } D\} \end{aligned}$$

the set of lengths of a. Sets of lengths play an important role in the theory of non-unique factorizations. The reader is referred to [**6**] and [**17**] for survey articles on this topic. Two integers $k, l \in \mathbb{N}$ with $k < l$ are called successive lengths of $a \in D$ if $\{m \in \mathsf{L}(a) \mid k \leq m \leq l\} = \{k, l\}$. The invariant

$$\Delta(D) = \{l - k \mid 0 \neq a \in D \backslash D^{\times},\ k < l \text{ are successive lengths of } a\} \subset \mathbb{N}$$

is called the set of differences of D. It provides a measure for the size of the gaps occurring in sets of lengths of elements of D.

If D is a one-dimensional local domain, then it is known that the sets of lengths of D have a special structure: they are, up to a bounded initial and final segment, arithmetical progressions with some period d which only depends on D. For more details see Definition 2.4 and Theorem 2.5. In particular, $\Delta(D)$ is a finite set. Furthermore, D has finite catenary degree, i.e. there exists a bound $B \in \mathbb{N}$ (which only depends on D) such that for every element $0 \neq a \in D \backslash D^{\times}$ and for all factorizations z and z' of a there exists a finite sequence of factorizations

$$z = z_0, \ldots, z_s = z' \tag{3}$$

of a such that z_{i-1} and z_i differ by at most B irreducible elements for all $1 \leq i \leq s$. In the case when D is analytically unramified (i.e. if the integral closure of D is a finitely generated D-module) this was proved in [**13**], Proposition 7.3. For the analytically ramified case see [**19**].

Let D be a one-dimensional local domain whose normalization is a finitely generated D-module and let $0 \neq a \in D \backslash D^{\times}$. For $m \in \mathsf{L}(a)$ we denote by $\mathsf{Z}_m(a)$ the set of factorizations of a with length m. In the present paper we are interested in the structure of the sets $\mathsf{Z}_m(a)$. We study the question whether $\mathsf{Z}_k(a)$ and $Z_l(a)$ are "similar" with respect to the natural metric (see (4) in section 2) if k and l are successive lengths of a. Furthermore, we study whether for arbitrary $n \in \mathsf{L}(a)$ and factorizations $z, z' \in \mathsf{Z}_n(a)$ there exists a chain $z = z_0, z_1, \ldots, z_s = z'$ of elements $z_i \in \mathsf{Z}_n(a)$ such that z_{i-1} and z_i have bounded distance for all $1 \leq i \leq s$ (where the bound should only depend on D). Note that in the above mentioned case of the (ordinary) catenary degree no conditions on the lengths of the z_i in (3) are imposed. One motivation to study these questions is to investigate the structure of chains of factorizations. For example, is it always possible to choose the chain in (3) in such a way that the lengths of the z_i form a monotone sequence of integers?

It turns out that in general the answer to our questions is negative (except if D is a Cohen-Kaplansky domain, i.e. if $D \backslash \{0\}$ is a finitely generated monoid, see [**3**] and [**8**]). Already simple examples show that factorizations with given successive lengths may have a completely different structure (see Examples 6.3 and 6.5). However, we prove that we get a positive answer to our questions if k, l and n are not contained in a bounded neighborhood of $\min \mathsf{L}(a)$ and $\max \mathsf{L}(a)$. To be more precise, there exist some constant $C \in \mathbb{N}$ and some bound $B \in \mathbb{N}$ (which both only depend on D) such that for all $a \in D$ and for all $n \in \mathsf{L}(a)$ with $\min \mathsf{L}(a) + C \leq n \leq \max \mathsf{L}(a) - C$ the strong successive distance $\delta_n(a)$ at length n (cf. Definition 3.4) and the catenary degree $c_n(a)$ at length n (cf. Definition 3.3) are bounded by B. Thus the observed "irregular" behavior of factorizations with successive lengths is due to boundary phenomena.

The methods we use to prove this result (see Theorem 4.1) are combinatorial in nature. We couch our considerations in the language of monoids which turned out to be very convenient in the theory of non-unique factorizations. The theorem is obtained by studying a suitable multiplicative model for the domain D (cf. Definition 4.2). It should be mentioned that our proof strongly makes use of Geroldinger's Structure Theorem for sets of lengths of D (cf. Theorem 2.5).

As a corollary of our result (Corollary 4.16) we give a new proof of Theorem 4.8 3) in [**8**]. Our investigations moreover show that this Theorem also holds if D has infinite residue class field. As a second application of our result we prove that every localization of an order of some quadratic number field at a singular place has finite monotone catenary degree, i.e. in the case of these rings we indeed can choose the chain in (3) in such a way that the lengths of the z_i form a monotone sequence (see Theorem 5.3). However, we must leave it open whether the global version of this theorem is also true.

The organization of the paper is as follows: in section 2 we fix our notation and recall some facts from factorization theory which are needed in the sequel. In section 3 we define the invariants which lie in the center of our interest. Section 4 is devoted to the proof of the Main Theorem (Theorem 4.1). In section 5 we show the finiteness of the monotone catenary degree for local quadratic orders. In section 6 we construct examples for rings whose strong successive distance and monotone catenary degree are infinite.

2. Preliminaries and results from factorization theory. We denote by $\mathbb{N}$ (resp. $\mathbb{N}_0$) the set of positive (resp. non-negative) integers. For sets A and B we write $A \subset B$ if A is a subset of B and equality may hold. We write $A \subsetneq B$ if $A \subset B$ and $A \neq B$. For $a, b \in \mathbb{Z}$ we set $[a, b] = \{x \in \mathbb{Z} \mid \min\{a, b\} \le x \le \max\{a, b\}\}$. If $(M, \le)$ is a partially ordered set and $x \in M$, we put $M_{\le x} = \{y \in M \mid y \le x\}$ and $M_{\ge x} = \{y \in M \mid y \ge x\}$.

We call a commutative ring R local (resp. semi-local) if it has only one (resp. finitely many) maximal ideals and if it is Noetherian.

By a monoid we always mean a (usually multiplicatively written) commutative semigroup H with identity element for which the cancellation law holds, i.e. $ab = ac$ implies $b = c$ for all $a, b, c \in H$. The main examples for monoids we have in mind are the multiplicative semigroups of nonzero divisors of commutative rings. For monoids the notions “irreducible element” and “prime element” are defined completely analogously as in case of domains. Let H be a monoid. Then $H^\times$ denotes the group of invertible elements of H, $\mathsf{A}(H)$ denotes the set of irreducible elements (atoms) of H and $\mathsf{P}(H)$ denotes the set of prime elements of H. If $E \subset H$ is a subset, then $[E]$ denotes the submonoid of H which is generated by E. H is called atomic (resp. factorial) if $H^\times \cup \mathsf{A}(H)$ (resp. $H^\times \cup \mathsf{P}(H)$) generates H. We define the reduced monoid of H by $H_{\mathrm{red}} = H/H^\times$ and H itself is called reduced if $H^\times = \{1\}$. Since H satisfies the cancellation law, we can form the quotient group of H. We denote it by $\mathcal{Q}(H)$.

Let P be a set. We write $\mathcal{F}(P)$ for the free monoid generated by P. Then every $x \in \mathcal{F}(P)$ can be uniquely written as a product $x = \prod_{p \in P} p^{n_p}$, where $n_p \in \mathbb{N}_0$ and $n_p = 0$ for almost all $p \in P$. We call $|x| = |x|_{\mathcal{F}(P)} = \sum_{p \in P} n_p$ the length of x. We have a canonical metric $\mathsf{d} = \mathsf{d}_{\mathcal{F}(P)} : \mathcal{F}(P) \times \mathcal{F}(P) \to \mathbb{N}_0$ given by

$$\mathsf{d}(x, y) = \max\Big\{\Big|\frac{x}{\gcd(x, y)}\Big|, \Big|\frac{y}{\gcd(x, y)}\Big|\Big\}, \tag{4}$$

cf. for instance [**11**], section 2.

Let H be an atomic monoid. The monoid $\mathsf{Z}(H) = \mathcal{F}(\mathsf{A}(H_{\mathrm{red}}))$ is called the factorization monoid of H. For $x, y \in \mathsf{Z}(H)$ we call $\mathsf{d}(x, y) = \mathsf{d}_H(x, y) = \mathsf{d}_{\mathsf{Z}(H)}(x, y)$ the distance of x and y and we call $|x| = |x|_H = |x|_{\mathsf{Z}(H)}$ the length of x. We denote by $\pi = \pi_H : \mathsf{Z}(H) \to H_{\mathrm{red}}$ the canonical homomorphism. For $a \in H$ we denote by $\mathsf{Z}(a) = \mathsf{Z}_H(a) = \pi^{-1}(aH^\times)$ the set of factorizations of a. If $k \in \mathbb{N}_0$, then $\mathsf{Z}_k(a) = \{z \in \mathsf{Z}(a) \mid |z| = k\}$ denotes the set of factorizations of a with length k. We call

$$\mathsf{L}(a) = \mathsf{L}_H(a) = \{|z| \mid z \in \mathsf{Z}(a)\}$$

the set of lengths of a. H is called a BF-monoid if $\mathsf{L}(a)$ is a finite set for all $a \in H$. Important examples for BF-monoids are the multiplicative monoids of Noetherian domains, see [**2**], Proposition 2.2.

Let H be an atomic monoid. The quantity

$$\rho(H) = \sup\Big\{\frac{\sup \mathsf{L}(a)}{\min \mathsf{L}(a)} \;\Big|\; a \in H \setminus H^\times\Big\} \in \mathbb{R}_{\geq 1} \cup \{\infty\}$$

is called the elasticity of H. The reader is referred to [**4**] or [**1**] for survey articles on this important invariant. An atomic monoid H is called half-factorial if its elasticity is equal to one.

Definition 2.1.

(1) Let $T \subset \mathbb{Z}$. Two elements $k, l \in T$ are called successive elements of T if $k \neq l$ and $T \cap [k, l] = \{k, l\}$.

(2) We call

$$\Delta(T) = \{|k - l| \mid k \text{ and } l \text{ are successive elements of } T\} \subset \mathbb{N}$$

the set of differences of T. (Observe that $\Delta(T) = \varnothing$ if and only if $|T| \le 1$.)

(3) Let H be an atomic monoid. We call

$$\Delta(H) = \bigcup_{a \in H} \Delta(\mathsf{L}(a)) \subset \mathbb{N}$$

the set of differences of H.

The set of differences measures the size of "gaps" which occur in the sets of lengths of an atomic monoid.

Next we recall the notion of a finitely primary monoid. Finitely primary monoids were introduced by F. Halter-Koch in [**16**] as multiplicative models for one-dimensional local domains with finite normalization (cf. Proposition 6.1, where we prove a slightly more general result). Since that time they proved many times to be a very useful tool for the investigation of multiplicative properties of these rings. However, we need to refine this notion later (Definition 4.2).

Definition 2.2. A monoid H is called finitely primary of rank $s \in \mathbb{N}$ and exponent $\alpha \in \mathbb{N}$ if it is a submonoid of a factorial monoid F with s pairwise non-associated prime elements $p_1, \dots, p_s$,

$$H \subset F = F^\times \times [p_1, \dots, p_s]$$

such that the following conditions are satisfied:

(1) $(p_1 \cdot \ldots \cdot p_s)^\alpha F \subset H$.

(2) If $\varepsilon p_1^{\alpha_1} \cdot \ldots \cdot p_s^{\alpha_s} \in H$, where $\varepsilon \in F^\times$, then either $\alpha_1 = \cdots = \alpha_s = 0$ and $\varepsilon \in H^\times$ or $\alpha_1 \ge 1, \dots, \alpha_s \ge 1$.

If H is a finitely primary monoid, then the factorial monoid F in Definition 2.2 is isomorphic to the complete integral closure $\widehat{H}$ of H, see [**10**].

Definition 2.3. Let H be a finitely primary monoid with rank s and complete integral closure $\widehat{H} = \widehat{H}^\times \times [p_1, \dots, p_s]$. We denote by $\mathsf{v}_i : \mathcal{Q}(H) \to \mathbb{Z}$ the p_i-valuation and we define the group homomorphism $\mathsf{v} : \mathcal{Q}(H) \to \mathbb{Z}^s$ by setting $\mathsf{v}(a) = (\mathsf{v}_1(a), \dots, \mathsf{v}_s(a))$ for all $a \in \mathcal{Q}(H)$.

Note that a finitely primary monoid H is strongly primary, i.e. for every $a \in H$ there exists some $N \in \mathbb{N}$ such that $a \mid_H b$ for every $b \in H$ with $\max \mathsf{L}(b) \geq N$. We denote the smallest such N by $\mathcal{M}(a)$.

Definition 2.4. A non-empty finite set $L \subset \mathbb{Z}$ is called an almost arithmetical progression with bound $M \in \mathbb{N}$ and period $d \in \mathbb{N}$ if there exists a decomposition $L = L_1 \cup L^* \cup L_2$ such that $L^* \neq \varnothing$, $L_1 \subset [-M, -1] + \min L^*$, $L_2 \subset \max L^* + [1, M]$ and $L^* = [\min L^*, \max L^*] \cap (\min L^* + d\mathbb{Z})$.

Theorem 2.5. *(Structure Theorem for sets of lengths) Let H be a finitely primary monoid. Then there exists some $M \in \mathbb{N}$ such that $\mathsf{L}(a)$ is an almost arithmetical progression with bound M and period $d = \min \Delta(H)$ for every $a \in H$. In particular, H has a finite set of differences $\Delta(H)$.*

Proof. See Theorem 5.1 and Corollary 5.2 of [**12**].

Remark 2.6. The Structure Theorem for sets of lengths indeed holds for any one-dimensional local domain D. If D is analytically unramified (i.e. if D has finite normalization), then $D \backslash \{0\}$ is finitely primary and the assertion follows from Theorem 2.5. For the analytically ramified case see [**19**], Theorem 3.5. Furthermore, the Structure Theorem for sets of lengths holds (in a slightly more general form) for domains and monoids which play important roles in Algebraic Number Theory and Algebraic Geometry, see [**14, 15, 20**].

Next we recall the notion of local tameness of factorizations. This notion plays a crucial role in recent papers, cf. [**14, 15**].

Definition 2.7. Let H be an atomic monoid and $a \in H$.

(1) The tame degree $\mathsf{t}(H, a)$ of a is the minimum of all $N \in \mathbb{N}_0 \cup \{\infty\}$ with the following property: if $b \in H$ with $a \mid_H b$, $z \in \mathsf{Z}(b)$ and $x \in \mathsf{Z}(a)$, then there exists a factorization $z' \in \mathsf{Z}(b)$ with $x \mid_{\mathsf{Z}(H)} z'$ and $\mathsf{d}(z, z') \leq N$.

(2) H is called locally tame if $\mathsf{t}(H, a) < \infty$ for every $a \in H$.

By [**13**] Lemma 5.3, every finitely primary monoid is locally tame.

3. The monotone catenary degree and the strong successive distance. In this section we define the invariants which lie in the center of our interest.

Definition 3.1. Let (X, d) be a metric space and $f : X \to \mathbb{R}$ a map. Let $A \subset X$, $r \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ and $x, x' \in A$. An f-monotone r-chain from x to x' in A is a finite sequence $x_0, x_1, \ldots, x_k$ in A such that $x = x_0$, $x' = x_k$, $d(x_{i-1}, x_i) \leq r$ for all $i \in [1, k]$ and such that $f(x_0), \ldots, f(x_k)$ forms a monotone sequence of real numbers. We call

$$\mathsf{c}_f(A) = \inf\{r \in \mathbb{R}_{\geq 0} \cup \{\infty\} \mid \text{for all } x, x' \in A \text{ there exists an } f\text{-monotone } r\text{-chain from } x \text{ to } x' \text{ in } A\}$$

the f-monotone catenary degree of A. (Observe that $\mathsf{c}_f(\varnothing) = 0$.) The f-monotone catenary of A with $f = 0$ is called the catenary degree of A. It is denoted by $\mathsf{c}(A)$.

Definition 3.2. Let (X, d) be a metric space and let $A, B \subset X$ be nonempty subsets.

(1) We set $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$.

(2) $\mathrm{Dist}(A, B) = \sup\{d(\{a\}, B), d(A, \{b\}) \mid a \in A, b \in B\}$ is called the strong distance of A and B.

Let H be an atomic monoid. In the following we always regard the factorization monoid $\mathsf{Z}(H)$ of H as a metric space via the natural distance function (4). If not otherwise stated, a monotone chain in $\mathsf{Z}(H)$ is always an f-monotone chain, where f is the length function. Let $a \in H$. Two integers $k, l \in \mathbb{N}$ are called successive lengths of a if k, l are successive elements of $\mathsf{L}(a)$ (cf. Definition 2.1).

Definition 3.3. Let H be an atomic monoid and let $a \in H$. Let $f = |\,.\,|_H : \mathsf{Z}(H) \to \mathbb{N}_0$ be the length function.

(1) The catenary degree $\mathsf{c}(\mathsf{Z}(a))$ of $\mathsf{Z}(a)$ is called the (ordinary) catenary degree of a and is denoted by $\mathsf{c}(a)$. We call $\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H\}$ the (ordinary) catenary degree of H.

(2) $\mathsf{c}_f(\mathsf{Z}(a))$ is called the monotone catenary degree of a. We denote it by $\mathsf{c}_{\mathrm{mon}}(a)$. The quantity $\mathsf{c}_{\mathrm{mon}}(H) = \sup\{\mathsf{c}_{\mathrm{mon}}(a) \mid a \in H\}$ is called the monotone catenary degree of H.

(3) Let $k \in \mathbb{N}_0$. The quantity $\mathsf{c}(\mathsf{Z}_k(a)) = \mathsf{c}_f(\mathsf{Z}_k(a))$ is called the catenary degree of a at length k. We denote it by $\mathsf{c}_k(a)$.

Definition 3.4. Let H be an atomic monoid and let $a \in H$, $k \in \mathbb{N}_0$. If there exists some $l > k$ such that k, l are successive lengths of a (i.e. if $k \in \mathsf{L}(a)$ and $k \neq \sup \mathsf{L}(a)$), then we set

$$\delta_k(a) = \mathrm{Dist}(\mathsf{Z}_k(a), \mathsf{Z}_l(a)) \in \mathbb{N}.$$

Otherwise we set $\delta_k(a) = 0$. We call $\delta_k(a)$ the strong successive distance of a at length k.

A few remarks are in order. The ordinary catenary degree was introduced in [**9**]. It is known that every finitely primary monoid has finite catenary degree, see [**13**], Proposition 7.3. Indeed, every one-dimensional local domain has finite catenary degree, see [**19**]. The notions of the monotone catenary degree and the strong successive distance first appeared in [**8**]. A. Foroutan proved in [**8**], Theorem 3.9 that for every monoid H for which H_{red} is a finitely generated monoid, the quantities $\mathsf{c}_{\mathrm{mon}}(H)$ and $\sup\{\delta_k(a) \mid a \in H, k \in \mathsf{L}(a)\}$ are finite. Let H be an atomic monoid. Note that $\mathsf{c}_k(a) \leq \mathsf{c}_{\mathrm{mon}}(H)$ for all $a \in H$, $k \in \mathbb{N}_0$. On the other hand, if $\sup\{\delta_k(a) \mid a \in H, k \in \mathsf{L}(a)\}$ and $\sup\{\mathsf{c}_k(a) \mid a \in H, k \in \mathsf{L}(a)\}$ are both finite, then $\mathsf{c}_{\mathrm{mon}}(H)$ is finite, too.

Our main interest in this paper is focused on the quantities $\mathsf{c}_k(a)$ and $\delta_k(a)$ if H is the multiplicative monoid of a one-dimensional local domain.

4. Proof of the Main Theorem. In this section we prove our main result (Theorem 4.1). To reach this end, we first introduce a suitable multiplicative model for a one-dimensional local domain with finite normalization (cf. Definition 4.2 and Theorem 4.3). Then we prove the analogous theorem for this model (see Theorem 4.14). At the end of the section we examine the structure of chains of factorizations by means of our results (see Corollary 4.16).

We first state the Main Theorem. Its proof is an immediate consequence of Theorem 4.3 and Theorem 4.14.

Theorem 4.1. (Main Theorem). *Let D be a one-dimensional local domain whose integral closure is a finitely generated D-module. Put $H = D\setminus\{0\}$. Then there exists some constant $C \in \mathbb{N}_0$ with the following properties:*

(1) $\sup\{\mathsf{c}_k(a) \mid a \in H, k \in \mathbb{N}\,,\, \min \mathsf{L}(a)+C \le k \le \max \mathsf{L}(a)-C\} < \infty$.

(2) $\sup\{\delta_k(a) \mid a \in H, k \in \mathbb{N}\,,\, \min \mathsf{L}(a)+C \le k \le \max \mathsf{L}(a)-C\} < \infty$.

It was shown in [**8**], Theorem 3.9 that the Theorem holds for $C = 0$ if D is either a discrete valuation ring or a Cohen-Kaplansky domain (i.e. D has finite residue class field and is analytically irreducible, cf. [**3**]). In general, this is not true, see Examples 6.3 and 6.5.

Next we give the definition of our multiplicative models.

Definition 4.2. Let H be a finitely primary monoid with complete integral closure $\widehat{H}$ and rank $s \in \mathbb{N}$.

(1) We call H a ring-like finitely primary monoid if the following conditions are fulfilled:

(a) There exist some exponent $\alpha \in \mathbb{N}$ of H and some system $\{p_1, \dots, p_s\}$ of representatives of prime elements of $\widehat{H}$ with the following property: for all $i \in [1, s]$ and for all $a \in \widehat{H}$ with $\mathsf{V}_i(a) \ge \alpha$ we have $p_i a \in H$ if and only if $a \in H$.

(b) Either $\widehat{H}^\times/H^\times$ is finite or $\mathsf{V}(H\setminus H^\times) \subset \mathbb{N}^s$ possesses a smallest element with respect to the partial order.

(2) We call H strongly ring-like if it is ring-like and both conditions in (b) are satisfied.

Theorem 4.3. *Let $(D, \mathfrak{m})$ be a one-dimensional, local domain such that the integral closure $\overline{D}$ of D is a finitely generated D-module. Then $D\setminus\{0\}$ is a ring-like finitely primary monoid.*

Supplement: Let s denote the number of maximal ideals of $\overline{D}$. If $s \leq |D/\mathfrak{m}|$, then the set of values of $D\backslash(D^\times \cup \{0\})$ has a smallest element. In particular, if $s \leq |D/\mathfrak{m}| < \infty$, then $D\backslash\{0\}$ is strongly ring-like.

Proof. In [**18**], Theorem 2.7 it was shown that the finitely primary monoid $D\backslash\{0\}$ satisfies condition (1)(a) of Definition 4.2. In order to verify (1)(b) assume without loss of generality that D is not a discrete valuation ring. Then $D/\mathfrak{m}$ is finite if and only if $\overline{D}^\times/D^\times$ is finite, see for instance [**21**], Theorem 2.1. Hence it is enough to prove the supplement. By [**7**], Proposition 1.1 or by Proposition 6.1 (3), the semigroup of values of the completion $\widehat{D}$ of D coincides with the semigroup of values of D. Since the number of maximal ideals of $\overline{D}$ equals the number of minimal primes of $\widehat{D}$, the supplement follows from [**7**], Proposition 1.2. $\square$

Our next goal is to prove Theorem 4.14. This requires a large amount of preparatory work. Throughout the rest of the section we keep the following notation.

General Notation. *If H is a finitely primary monoid, then $s \in \mathbb{N}$ denotes its rank and $\alpha \in \mathbb{N}$ is an exponent of H. We denote by $\widehat{H} = \widehat{H}^\times \times [p_1, \ldots, p_s]$ the complete integral closure of H. If H is ring-like, we assume that α and the prime elements $p_1, \ldots, p_s$ of $\widehat{H}$ are chosen in such a way that condition* (1)(a) *of Definition* 4.2 *is satisfied.*

We set $\mathbb{A}(H) = \{q \in \mathsf{A}(H) \mid \mathsf{V}_i(q) \leq 2\alpha$ for all $i \in [1, s]\}$. Note that if $\widehat{H}^\times/H^\times$ is finite, then $\mathbb{A}(H)$ is a finite set.

Let $L \subset \mathbb{Z}$ be a finite set and $M \in \mathbb{N}_0$. We set $L\langle M\rangle = \{x \in L \mid \min L + M \leq x \leq \max L - M\}$.

If $X = \prod_{i\in I} X_i$ is a product of non-empty sets and $\boldsymbol{x} \in X$, then we denote by $\boldsymbol{x}_i \in X_i$ the i-th component of $\boldsymbol{x}$.

The following Proposition plays a key role in our investigations:

Proposition 4.4. *Let H be a ring-like finitely primary monoid. Let $u \in H$ and $i \in [1, s]$. If $\mathsf{V}_i(u) \geq 2\alpha$, then u is irreducible if and only if $p_i u$ is irreducible.*

Proof. Let $u \in H$ and $i \in [1, s]$ with $\mathsf{V}_i(u) \geq 2\alpha$. Clearly, $p_i u \in H$ since H is ring-like. If $u = bc$ is a decomposition into non-units, then we can assume without restriction that $\mathsf{V}_i(b) \geq \alpha$. But then $p_i b \in H$, whence we have a decomposition $p_i u = (p_i b)c$. The "only" part is proved similarly. □

The (P_γ) property we now define is the pivotal point in our proof of Theorem 4.14.

Definition 4.5. Let H be a finitely primary monoid and $\gamma \in \mathbb{N}_0$. We say that H has property (P_γ) if for all $M \in \mathbb{N}_0$ there exists some constant $C \in \mathbb{N}$ with the following property: if $a \in H$ and $z \in \mathsf{Z}(a)$ with $|z| \in \mathsf{L}(a)\langle M + \gamma\rangle$, then there exists some $x \in \mathsf{Z}(H)$ such that $x \mid_{\mathsf{Z}(H)} z$, $|x| \leq C$ and $|x| \in \mathsf{L}(\pi(x))\langle M\rangle$. The smallest C with this property is denoted by $\mathcal{C}(\gamma, M)$.

The following two lemmas are invoked in the proof of Proposition 4.8.

Lemma 4.6. *Let $s \in \mathbb{N}$, $\boldsymbol{N} = (N_1, \ldots, N_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in \mathbb{N}_0^s$ with $\sum_{i=1}^n \boldsymbol{x}_i \geq \boldsymbol{N}$. Set $N = \sum_{i=1}^s N_i \in \mathbb{N}_0$. Then there exists some subset $J \subset [1, n]$ such that $|J| \leq N$ and $\sum_{j \in J} \boldsymbol{x}_j \geq \boldsymbol{N}$.*

Proof. We prove the Lemma by induction on N. The case $N = 0$ is clear. Let $\boldsymbol{N}$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ be as in the assumptions and suppose that $N > 0$. Then there exists some $i \in [1, s]$ such that $N_i > 0$. Without restriction let $i = 1$. Set $\boldsymbol{N'} = (N_1 - 1, N_2, \ldots, N_s)$. Then the induction hypothesis implies that there exists some subset $J' \subset [1, n]$ such that $|J'| \leq N - 1$ and $\boldsymbol{S'} \geq \boldsymbol{N'}$, where $\boldsymbol{S'} = \sum_{j \in J'} \boldsymbol{x}_j$. If $\boldsymbol{S'} \geq \boldsymbol{N}$, then we set $J = J'$ and we have nothing else to show. Hence assume $\mathbf{S'} \not\geq \boldsymbol{N}$. Then the first component of $\boldsymbol{S'}$ is equal to $N_1 - 1$. But from $\sum_{i=1}^n \boldsymbol{x}_i \geq \boldsymbol{N}$ we infer that there exists some $j \in [1, n] \backslash J'$ such that the first component of $\boldsymbol{x}_j$ is non-vanishing. Hence the assertion follows if we set $J = J' \cup \{j\}$. □

Lemma 4.7 *Let H be a finitely primary monoid with rank one. Set $\mu_{\max} = \max\{\mathsf{V}(q) \mid q \in \mathsf{A}(H)\} \in \mathbb{N}$.*

(1) *For every $b \in H$ we have the estimate*

$$\min \mathsf{L}(b) \le \frac{\mathsf{V}(b)}{\mu_{\max}} + 3\alpha.$$

(2) *Let $M \in \mathbb{N}_0$, $a \in H$ and $z \in \mathsf{Z}(a)$ such that $|z| \ge \min \mathsf{L}(a) + M + 3\alpha$. Then there exists some $x'' \in \mathsf{Z}(H)$ with $x'' \mid_{\mathsf{Z}(H)} z$ and $\min \mathsf{L}(\pi(x'')) + M \le |x''| \le \mu_{\max}(3\alpha + M)$.*

Proof. (1) Since $\mu_{\max}$ is the valuation of an atom $q_{\max}$ of H, we have $\mu_{\max} < 2\alpha$. Assume without restriction that $\mathsf{V}(b) \ge \alpha$ and let $t \in \mathbb{N}_0$ and $r \in [0, \mu_{\max} - 1]$ with $\mathsf{V}(b) - \alpha = \mu_{\max} t + r$. Then $\mathsf{V}(bq_{\max}^{-t}) = \alpha + r$, whence $bq_{\max}^{-t} \in H$. Let $\xi \in \mathsf{Z}(bq_{\max}^{-t})$ be arbitrary and set $x = \xi q_{\max}^{t} \in \mathsf{Z}(b)$. Then $|x| = t + |\xi| \le t + \alpha + r \le \mathsf{V}(b)\mu_{\max}^{-1} + 3\alpha$.

(2) Let M, a and z be as in the assumptions. Suppose that $z = z_1 \cdot \ldots \cdot z_k$ with $z_i \in \mathsf{A}(H)$ and consider the estimate $\mu_{\max}|z| - \mathsf{V}(a) \ge \mu_{\max}(3\alpha + M + \min \mathsf{L}(a)) - \mathsf{V}(a) = \mu_{\max}(3\alpha + M) + \mu_{\max} \min \mathsf{L}(a) - \mathsf{V}(a) \ge \mu_{\max}(3\alpha + M)$. Then

$$\sum_{i=1}^{k} h_i \ge (3\alpha + M)\mu_{\max},$$

where $h_i = \mu_{\max} - \mathsf{V}(z_i) \in \mathbb{N}_0$ for all $i \in [1, k]$. Hence there exists some $I \subset [1, k]$ such that $|I| \le (3\alpha + M)\mu_{\max}$ and $\sum_{i \in I} h_i \ge (3\alpha + M)\mu_{\max}$. Set $x'' = \prod_{i \in I} z_i$. Then $\mathsf{V}(\pi(x'')) = -\sum_{i \in I} h_i + |I|\mu_{\max} \le (|I| - 3\alpha - M)\mu_{\max}$. Therefore $\min \mathsf{L}(\pi(x'')) \le (|I| - 3\alpha - M) + 3\alpha = |x''| - M$ by (1). □

Proposition 4.8. *Let H be a ring-like finitely primary monoid. Then H has property (P_γ) for some $\gamma \in \mathbb{N}$.*

Proof. Without loss of generality we assume that H is reduced. The proof is divided into two different parts. We first assume that $\widehat{H}^\times$ is finite. Then we treat the case when the semigroup of values of H has a smallest element. Note that if H has rank one, then $\mathsf{V}(H \setminus \{1\})$ always has a smallest element.

Let $\widehat{H}^\times$ be finite and assume that the rank of H is bigger than one. Set $\gamma = \alpha$ and $\mathbb{A} = \mathbb{A}(H)$. To begin with, we define maps $\Theta : H \to H$ and $R : H \to \mathbb{N}_0^s$ as follows: let $a \in H$ and $i \in [1, s]$. If $\mathsf{v}_i(a) > 2\alpha$, we set $k_i = \mathsf{v}_i(a) - 2\alpha$. If $\mathsf{v}_i(a) \le 2\alpha$, then we set $k_i = 0$. We define

$$\Theta(a) = a \prod_{i=1}^{s} p_i^{-k_i} \quad \text{and} \quad R(a) = (k_1, \ldots, k_s).$$

Then we indeed have $\Theta(H) \subset H$. Furthermore, $\Theta(\mathsf{A}(H)) \subset \mathbb{A}$ by Proposition 4.4.

The maps Θ and R induce homomorphisms on the factorization monoid of H:

$$\overline{\Theta} : \begin{cases} \mathsf{Z}(H) & \longrightarrow & \mathcal{F}(\mathbb{A}) \cong \mathbb{N}_0^{\mathbb{A}} \\ z_1 \cdot \ldots \cdot z_n & \mapsto & \Theta(z_1) \cdot \ldots \cdot \Theta(z_n) \end{cases} \quad \text{and}$$

$$\overline{R} : \begin{cases} \mathsf{Z}(H) & \longrightarrow & \mathbb{N}_0^s \\ z_1 \cdot \ldots \cdot z_n & \mapsto & R(z_1) + \cdots + R(z_n). \end{cases}$$

Next we define the homomorphism $\Psi : \mathsf{Z}(H) \to \mathbb{N}_0^{\mathbb{A}} \times \mathbb{N}_0^s$ by setting $\Psi(z) = (\overline{\Theta}(z), \overline{R}(z))$. Finally, we set

$$\Pi : \begin{cases} \mathbb{N}_0^{\mathbb{A}} \times \mathbb{N}_0^s & \longrightarrow & \widehat{H} \\ (\boldsymbol{m}, \boldsymbol{n}) & \mapsto & \prod_{q \in \mathbb{A}} q^{\boldsymbol{m}_q} \prod_{i=1}^s p_i^{\boldsymbol{n}_i}. \end{cases}$$

Let $M \in \mathbb{N}_0$. Set

$$\mathcal{M} = \Big\{ (\boldsymbol{m}, \boldsymbol{n}, \widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{n}}) \in (\mathbb{N}_0^{\mathbb{A}} \times \mathbb{N}_0^s)^2 \mid \Pi(\boldsymbol{m}, \boldsymbol{n}) = \Pi(\widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{n}}) \quad \text{and} \quad \sum_{q \in \mathbb{A}} (\widetilde{\boldsymbol{m}}_q - \boldsymbol{m}_q) \ge \alpha + M \Big\}.$$

By Dickson's Theorem (see [**25**], Satz 12), the set $\operatorname{Min}(\mathcal{M})$ of minimal points of $\mathcal{M}$ is finite (note that $\mathbb{A}$ is finite!). Furthermore, for every $\boldsymbol{u} \in \mathcal{M}$ there exists some $\boldsymbol{v} \in \operatorname{Min}(\mathcal{M})$ with $\boldsymbol{v} \le \boldsymbol{u}$. Let $\operatorname{Min}(\mathcal{M}) = \{(\boldsymbol{m_1}, \boldsymbol{n_1}, \widetilde{\boldsymbol{m}}_{\boldsymbol{1}}, \widetilde{\boldsymbol{n}}_{\boldsymbol{1}}), \ldots, (\boldsymbol{m_t}, \boldsymbol{n_t}, \widetilde{\boldsymbol{m}}_{\boldsymbol{t}}, \widetilde{\boldsymbol{n}}_{\boldsymbol{t}})\}$. Since we assumed that the rank of H is greater than one, $L = \sup\{\min \mathsf{L}(b) \mid b \in H\}$ is finite, see [**17**], Proposition 4.1. Set

$$C = \max\{|\boldsymbol{m}_i| + |\boldsymbol{n}_i| \mid i \in [1, t]\} + M + L,$$

where $|\boldsymbol{m}|$ denotes $\sum_{i=1}^{r} m_i$ for some vector $\boldsymbol{m} = (m_1, \dots, m_r) \in \mathbb{N}_0^r$.

Let $a \in H$ and $z \in \mathsf{Z}(a)$ with $|z| \in \mathsf{L}(a)\langle \alpha + M\rangle$. We write $z = z_1 \cdot \ldots \cdot z_k$ with $z_i \in \mathsf{A}(H)$ for all $i \in [1, k]$ and we can assume without loss of generality that $k = |z| \geq C$ (otherwise we set $x = z$ and we are done). Let $w \in \mathsf{Z}(a)$ be a factorization with $|w| = \max \mathsf{L}(a)$. Then $(\Psi(z), \Psi(w)) \in \mathcal{M}$. Let $\boldsymbol{p} = (\boldsymbol{m}, \boldsymbol{n}, \widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{n}}) \in \mathrm{Min}(\mathcal{M})$ be an element with $\boldsymbol{p} \leq (\Psi(z), \Psi(w))$. Set $m = |\boldsymbol{m}|$, $n = |\boldsymbol{n}|$, $\widetilde{m} = |\widetilde{\boldsymbol{m}}|$ and $\widetilde{n} = |\widetilde{\boldsymbol{n}}|$. By reordering the z_i (if necessary) we can assume that $\overline{\Theta}(z_1 \cdot \ldots \cdot z_m) = \boldsymbol{m}$. Since $\overline{R}(z) \geq \boldsymbol{n}$, there exists some $y' \in \mathsf{Z}(H)$ such that $y' \,|_{\mathsf{Z}(H)}\, z_{m+1} \cdot \ldots \cdot z_k$, $|y'| \leq n$ and $\overline{R}(z_1 \cdot \ldots \cdot z_m y') \geq \boldsymbol{n}$. Then $|z| - |z_1 \cdot \ldots \cdot z_m y'| \geq C - m - n \geq M + L$. Let y'' be an arbitrary divisor of $z(z_1 \cdot \ldots \cdot z_m y')^{-1}$ with $|y''| = M + L$ and set $y = y'y''$. Set

$$x = z_1 \cdot \ldots \cdot z_m y \in \mathsf{Z}(H).$$

Then $x \,|_{\mathsf{Z}(H)}\, z$ and $M + L = |y''| \leq |x| = m + |y| = m + |y'| + M + L \leq C$. Since $\min \mathsf{L}(\pi(x)) \leq L$, we get $\min \mathsf{L}(\pi(x)) + M \leq |x|$.

Our next aim is to show that $h = \pi(x)\Pi(\widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{n}})^{-1}\pi(\overline{\Theta}(y))^{-1} \in \mathcal{Q}(\widehat{H})$ is already contained in $\widehat{H}$. We have $\Psi(x) = (\overline{\Theta}(x), \overline{R}(x)) = (\boldsymbol{m}, \boldsymbol{n}) + (\overline{\Theta}(y), \boldsymbol{\eta})$, where $\boldsymbol{\eta} = \overline{R}(x) - \boldsymbol{n} \geq \boldsymbol{0}$. Thus $\pi(x) = \Pi(\boldsymbol{m}, \boldsymbol{n})\Pi(\overline{\Theta}(y), \boldsymbol{\eta}) = \Pi(\widetilde{\boldsymbol{m}}, \widetilde{\boldsymbol{n}})\Pi(\overline{\Theta}(y), \boldsymbol{\eta})$, whence we immediately see that $h \in \widehat{H}$.

To simplify notation we rewrite $\overline{\Theta}(y) \prod_{q \in \mathbb{A}} q^{\widetilde{\boldsymbol{m}}_q}$ as a product $u_1 \cdot \ldots \cdot u_\mu$ with $u_i \in \mathbb{A}$ and $\mu = \widetilde{m} + |y|$. Since $\widetilde{m} - m \geq \alpha + M$, we have $\mu \geq \widetilde{m} \geq \alpha + M \geq \alpha$. Thus we can form the element

$$\xi = u_1 \cdot \ldots \cdot u_\alpha h \prod_{i=1}^{s} p_i^{\widetilde{\boldsymbol{n}}_i}$$

and it is contained in H. Let $\xi_1 \cdot \ldots \cdot \xi_r \in \mathsf{Z}(\xi)$ be an arbitrary factorization of ξ into atoms $\xi_i \in \mathsf{A}(H)$ and set $v = \xi_1 \cdot \ldots \cdot \xi_r u_{\alpha+1} \cdot \ldots \cdot u_\mu \in \mathsf{Z}(H)$. Then we see easily that $\pi(v) = \pi(x) \in H$. Furthermore, we get the estimate

$$\begin{aligned} |v| - |x| &= (\widetilde{m} + |y| - \alpha + r) - (m + |y|) \\ &= \widetilde{m} - m - \alpha + r \geq (\alpha + M) - \alpha + r = M + r \geq M. \end{aligned}$$

This proves the Proposition if $\widehat{H}^\times$ is finite.

Now we come to the second part of the proof. We assume that H has arbitrary rank and that the set of values $\{(\mathsf{V}_1(a),\dots,\mathsf{V}_s(a))\in\mathbb{N}^s \mid a\in H\backslash\{1\}\}$ possesses a smallest element $\boldsymbol{\mu}=(\mu_1,\dots,\mu_s)$.

Let $M\in\mathbb{N}_0$, $\gamma\in\mathbb{N}_0$, $a\in H$ and $z\in\mathsf{Z}(a)$ with $|z|\in\mathsf{L}(a)\langle M+\gamma\rangle$. We see easily that $\mathrm{V}_i(a)\geq\mu_i\max\mathsf{L}(a)$ holds for all $i\in[1,s]$. Hence we have $\mathsf{V}_i(a)-\mu_i|z|\geq\mu_i\max\mathsf{L}(a)-\mu_i|z|=\mu_i(\max\mathsf{L}(a)-|z|)\geq\mu_i(\gamma+M)$ for all $i\in[1,s]$.

Write $z=z_1\cdot\ldots\cdot z_k$ with $z_i\in\mathsf{A}(H)$ and set $\boldsymbol{g}_i=\mathrm{V}(z_i)-\boldsymbol{\mu}\in\mathbb{N}_0^s$. Then

$$\sum_{j=1}^{k}\boldsymbol{g}_j=\sum_{j=1}^{k}\mathsf{V}(z_j)-k\boldsymbol{\mu}=\mathsf{V}(a)-|z|\boldsymbol{\mu}\geq(\gamma+M)\boldsymbol{\mu}.$$

Hence we see by Lemma 4.6 that there exists some subset $J\subset[1,k]$ with $|J|\leq(\gamma+M)\sum_{i=1}^{s}\mu_i$ and $\sum_{j\in J}\boldsymbol{g}_j\geq(\gamma+M)\boldsymbol{\mu}$. Define

$$x'=\prod_{j\in J}z_j\in\mathsf{Z}(H).$$

Then $x'\mid_{\mathsf{Z}(H)}z$ and $|x'|=|J|\leq(\gamma+M)\sum_{i=1}^{s}\mu_i$. Set $c'=\pi(x')\in H$ and consider the estimate

$$\mathsf{V}(c')-(M+|J|)\boldsymbol{\mu}=\sum_{j\in J}\boldsymbol{g}_j-M\boldsymbol{\mu}\geq\gamma\boldsymbol{\mu}. \tag{5}$$

Let $q_{\min}\in\mathsf{A}(H)$ be an atom with $\mathsf{V}(q_{\min})=\boldsymbol{\mu}$. If we assume that $\gamma\geq\alpha$, we get $c'q_{\min}^{-(M+|J|)}\in H$ from (5) (recall that α is an exponent of H). Therefore, $\max\mathsf{L}(c')\geq M+|J|=M+|x'|$.

In order to finish the proof we distinguish two cases:

Case 1. H has rank one. Set $\gamma=3\alpha$ and let $\mu_{\min}=\mu_1$ (resp. $\mu_{\max}$) denote the minimum (resp. maximum) of the finite set $\{\mathsf{V}(q)\mid q\in\mathsf{A}(H)\}\subset\mathbb{N}$. By Lemma 4.7(2) there exists some $x''\in\mathsf{Z}(H)$ with $x''\mid_{\mathsf{Z}(H)}z$ and $\min\mathsf{L}(\pi(x''))+M\leq|x''|\leq\mu_{\max}(3\alpha+M)$. Set $x=\gcd(x'x'',z)$ and $C=(\mu_{\min}+\mu_{\max})(3\alpha+M)$. Then $x\mid_{\mathsf{Z}(H)}z$, $|x|\leq C$ and $\min\mathsf{L}(\pi(x))+M\leq|x|\leq\max\mathsf{L}(\pi(x))-M$. The last inequality follows from the respective properties of x' and x''.

Case 2: H has rank greater than one. Set $\gamma=\alpha$ and define $C=(\alpha+M)\sum_{i=1}^{s}\mu_i+M+L$, where $L=\max\{\min\mathsf{L}(a)\mid a\in H\}$. Without restriction we may assume that $|z| = k \ge C$. Then $|zx'^{-1}| \ge M + L$. Let $x'' \in \mathsf{Z}(H)$ with $x'' \mid_{\mathsf{Z}(H)} zx'^{-1}$ and $|x''| = M + L$. Set $x = x'x''$. Then $|x| = |x'| + |x''| \le C$, $x \mid_{\mathsf{Z}(H)} z$ and $\min \mathsf{L}(\pi(x)) + M \le L + M \le |x| \le \max \mathsf{L}(\pi(x)) - M$. □

Lemma 4.9. *Let H be an atomic BF-monoid and $a, b \in H$. Set $l = \min \mathsf{L}(a) + \min \mathsf{L}(b)$ and $L = \max \mathsf{L}(a) + \max \mathsf{L}(b)$. Then $l - \mathsf{t}(H, a) \le \min \mathsf{L}(ab) \le l$ and $L \le \max \mathsf{L}(ab) \le L + \mathsf{t}(H, a)$.*

Proof. The inequalities $\min \mathsf{L}(ab) \le l$ and $L \le \max \mathsf{L}(ab)$ are obvious. We first show $l - \mathsf{t}(H, a) \le \min \mathsf{L}(ab)$. Let $x \in \mathsf{Z}(a)$ and $z \in \mathsf{Z}(ab)$ whose lengths are minimal. Then there exists some $z' \in \mathsf{Z}(ab)$ such that $x \mid_{\mathsf{Z}(H)} z'$ and $\mathsf{d}(z, z') \le \mathsf{t}(H, a)$. Thus $|z'| - |z| \le \mathsf{t}(H, a)$ and $|z'| \ge l$. Hence $|z| = \min \mathsf{L}(ab) = |z| - |z'| + |z'| \ge -\mathsf{t}(H, a) + |z'| \ge -\mathsf{t}(H, a) + l$. Likewise, one proves the remaining inequality. □

Lemma 4.10 *Let H be a finitely primary monoid.*

(1) *There exists some constant $L \in \mathbb{N}_0$ such that for all $a \in H$ and $m \in \mathsf{L}(a)\langle L\rangle$ we have $m + \min \Delta(H) \in \mathsf{L}(a)$ and $m - \min \Delta(H) \in \mathsf{L}(a)$.*

(2) *There exists some constant $L \in \mathbb{N}_0$ with the following property: if $q \in \mathbb{A}(H)$, $b \in H$ and $m \in \mathsf{L}(b)\langle L\rangle$, then $q \mid_H b$ and $m - 1 \in \mathsf{L}(bq^{-1})$.*

Proof. (1) is an immediate consequence of Theorem 2.5. In order to show (2)., let T denote the maximum of the set $\{\mathsf{t}(H, q) \mid q \in \mathbb{A}(H)\}$ (this set is indeed finite, see [**13**], Lemma 5.3) and set $\mathcal{M} = \max\{\mathcal{M}(q) \mid q \in \mathbb{A}(H)\} \in \mathbb{N}$ (for the definition of $\mathcal{M}(q)$ see the paragraph after Definition 2.3). The Structure Theorem for sets of lengths of H implies that there exists some $M \in \mathbb{N}_0$ such that every set of lengths of H is an almost arithmetical progression bounded by M. Set $L = \max\{M + T, \mathcal{M} + 1\}$. Let $b \in H$, $m \in \mathsf{L}(b)\langle L\rangle$ and $q \in \mathbb{A}(H)$. Then $\max \mathsf{L}(b) \ge \mathcal{M} + 1$, whence $c = bq^{-1} \in H \setminus H^\times$. From Theorem 2.5 we know that there is a decomposition $\mathsf{L}(c) = L_1 \cup L^* \cup L_2$ such that $L^* \ne \varnothing$, $L_1 \subset [-M, -1] + \min L^*$, $L_2 \subset \max L^* + [1, M]$ and $L^* = [\min L^*, \max L^*] \cap (\min L^* + d\mathbb{Z})$, where $d = \min \Delta(H)$. Since $\mathrm{L}(c) + 1 \subset \mathrm{L}(b)$ we in particular have $L^* + 1 \subset \mathrm{L}(b)$. But $\Delta(L^*) \subset \{d\}$, where d is the minimal possible difference of consecutive elements in sets of lengths of H. Therefore the inequality

$$\min L^* + 1 \le m \le \max L^* + 1 \tag{6}$$

already implies that $m \in L^* + 1$. We first show the left inequality in (6). From the decomposition $\mathsf{L}(c) = L_1 \cup L^* \cup L_2$ we get $\min \mathsf{L}(c) \ge \min L^* - M$ and Lemma 4.9 yields $\min \mathsf{L}(c) + 1 - T \le \min \mathsf{L}(b)$. These two inequalities imply $\min L^* + 1 \le \min \mathsf{L}(b) + L \le m$. The second inequality in (6) is proved the same way. $\square$

The following Proposition is the key ingredient for the proof of Theorem 4.14(1). Its proof is now easy.

Proposition 4.11. *Let H be a ring-like finitely primary monoid. Then there exists some constant $K \in \mathbb{N}_0$ with the following property: for all $q \in \mathbb{A}(H)$ and for all $a \in H$, $z \in \mathsf{Z}(a)$ with $|z| \in \mathsf{L}(a)\langle K \rangle$ there exists some $z' \in \mathsf{Z}(a)$ such that $q \,|_{\mathsf{Z}(H)}\, z'$, $|z| = |z'|$ and $\mathsf{d}(z, z') \le K$.*

Proof. Let $\gamma \in \mathbb{N}_0$ such that H has (P_γ) and let $L \in \mathbb{N}_0$ such that condition (2) of Lemma 4.10 is satisfied. Put $K = \max\{L + \gamma, \mathcal{C}(\gamma, L)\}$ and let a, z and q be as in the assumptions. Then there exists some $x \in \mathsf{Z}(H)$ such that $x \,|_{\mathsf{Z}(H)}\, z$, $|x| \le K$ and $|x| \in \mathsf{L}(\pi(x))\langle L \rangle$. Let $y \in \mathrm{Z}(\pi(x)q^{-1})$ with $|y| = |x| - 1$. Then $z' = qyx^{-1}z$ has the required properties. $\square$

Lemma 4.12. *Let H be a ring-like finitely primary monoid. Let $a \in H$ and $z \in \mathsf{Z}(a)$ with $|z| \ge 2^s$. Then there exists some $z' \in \mathsf{Z}(a)$ such that $|z| = |z'|$, $\mathsf{d}(z, z') \le 2$ and such that there exists some $q \in \mathbb{A}(H)$ with $q \,|_{\mathsf{Z}(H)}\, z'$.*

Proof. Define $\phi : H \to \{0,1\}^s$ by setting $\phi(h)_i = 1$ if and only if $\mathsf{V}_i(h) > 2\alpha$ for $h \in H$, $i \in [1, s]$. Suppose without restriction that $q \notin \mathbb{A}(H)$ for all atoms q dividing z. Since $|z| \ge 2^s$ there exist $v, w \in \mathsf{A}(H)$ such that $vw \,|_{\mathsf{Z}(H)}\, z$ and $\phi(v) = \phi(w)$. By Proposition 4.4 we then can rearrange the prime elements dividing v and w to get atoms v' and w' with $\pi_H(v'w') = \pi_H(vw)$ and $\{v', w'\} \cap \mathbb{A}(H) \ne \varnothing$. $\square$

Lemma 4.13. *Let H be a non half-factorial finitely primary monoid. Then there exist $\kappa \in \mathbb{Q}_{>0}$ and $\lambda \in \mathbb{Q}_{\geq 0}$ such that the inequality*

$$\max \mathsf{L}(a) - \min \mathrm{L}(a) \geq \kappa \max \mathsf{L}(a) - \lambda$$

holds for all $a \in H \backslash H^{\times}$.

Proof. If the rank of H is greater than one, then the set $\{\min \mathsf{L}(a) \mid a \in H\}$ is bounded, see [**17**], Proposition 4.1. Hence assume that H has rank one.

Let $\mu_{\min}$ (resp. $\mu_{\max}$) denote the minimum (resp. maximum) of the set $\{\mathsf{V}(q) \mid q \in \mathsf{A}(H)\}$. Since $\rho(H) > 1$, we have $\mu_{\max} > \mu_{\min}$. We claim that $\kappa = (1 - \mu_{\min}\mu_{\max}^{-1})$ and $\lambda = 4\alpha + 1$ satisfy the inequality of the Lemma. Let $a \in H \backslash H^{\times}$. If $\mathsf{V}(a) < \alpha$, then the assertion is certainly true. Hence assume $\mathsf{V}(a) \geq \alpha$ and let $t \in \mathbb{N}_0$ and $r \in [0, \mu_{\min} - 1]$ with $\mathsf{V}(a) - \alpha = t\mu_{\min} + r$. If $q_{\min} \in \mathsf{A}(H)$ is an atom with $\mathsf{V}(q_{\min}) = \mu_{\min}$, then $\mathsf{V}(aq_{\min}^{-t}) \geq \alpha$. Hence we see that $\max \mathsf{L}(a) \geq t$, i.e. we have the inequality $\max \mathsf{L}(a) \geq \mu_{\min}^{-1}(\mathrm{V}(a) - \alpha - r) \geq \mu_{\min}^{-1}(\mathsf{V}(a) - \alpha) - 1$. This yields

$$\mathsf{V}(a) \leq \mu_{\min}(\max \mathsf{L}(a) + 1) + \alpha. \tag{7}$$

Using (1) of Lemma 4.7 we obtain $\max \mathsf{L}(a) - \min \mathsf{L}(a) \geq \max \mathsf{L}(a) - \mathrm{V}(a)\mu_{\max}^{-1} - 3\alpha$. An easy calculation using (7) now yields the result. □

We are now ready to prove Theorem 4.14.

Theorem 4.14 *Let H be a ring-like finitely primary monoid. Then there exists some constant $C \in \mathbb{N}_0$ with the following properties:*

(1) $\sup\{\mathsf{c}_k(a) \mid a \in H, k \in \mathbb{N}, \min \mathsf{L}(a) + C \leq k \leq \max \mathsf{L}(a) - C\} < \infty$.

(2) $\sup\{\delta_k(a) \mid a \in H, k \in \mathbb{N}, \min \mathsf{L}(a) + C \leq k \leq \max \mathsf{L}(a) - C\} < \infty$.

Proof. (1) Without restriction we can assume that H is reduced and not half-factorial. Let $K \in \mathbb{N}_{\geq 2^s}$ be a natural number for which the assertion of Proposition 4.11 holds. Let $a \in H$, $k \in \mathsf{L}(a)\langle K \rangle$ and $z, \widetilde{z} \in \mathsf{Z}_k(a)$. Let T denote the maximum of the set $\{\mathsf{t}(H, q) \mid q \in \mathbb{A}(H)\} \subset \mathbb{N}_0$

(for the finiteness of this set see [**13**], Lemma 5.3) and let $\kappa > 0$ and $\lambda \geq 0$ be constants for which the assertion of Lemma 4.13 holds. Set $C = \max\{(2K + 2T + \lambda)\kappa^{-1}, K + T + 2^s\}$. We prove by induction on k that there exists some C-chain in $\mathsf{Z}_k(a)$ which concatenates z and $\widetilde{z}$.

Case 1: $\mathsf{L}(a)\langle K + T\rangle = \varnothing$. Then $\max \mathsf{L}(a) - \min \mathsf{L}(a) \leq 2(K + T)$. Hence $\max \mathsf{L}(a) \leq C$ by Lemma 4.13 and there is nothing more to show.

Case 2: $\mathsf{L}(a)\langle K + T\rangle \neq \varnothing$. Then either

$$\min \mathsf{L}(a) + K + T \leq k \quad \text{or} \quad k \leq \max \mathsf{L}(a) - (K + T). \tag{8}$$

Suppose first that the first inequality is satisfied. Then the assumption $k \in \mathsf{L}(a)\langle K\rangle$ yields

$$\min \mathsf{L}(a) + K + T \leq k \leq \max \mathsf{L}(a) - K. \tag{9}$$

Let $w \in \mathsf{Z}(a)$ be a factorization with $|w| = \max \mathsf{L}(a)$. Then $|w| \geq K \geq 2^s$ and by Lemma 4.12 we can assume without restriction that there exists some $q \in \mathbb{A}(H)$ with $q \mid_{\mathsf{Z}(H)} w$. If we apply Proposition 4.11 to z and $\widetilde{z}$ respectively, we obtain factorizations z' and $\widetilde{z}'$ of a such that $|z'| = k$, $|\widetilde{z}'| = k$, $q \mid_{\mathsf{Z}(H)} z'$, $q \mid_{\mathsf{Z}(H)} \widetilde{z}'$, $\mathsf{d}(z, z') \leq K$ and $\mathsf{d}(\widetilde{z}, \widetilde{z}') \leq K$. Set $b = aq^{-1}$, $x = z'q^{-1}$ and $\widetilde{x} = \widetilde{z}'q^{-1}$. Since $q \mid w$ we have $\max \mathsf{L}(b) = \max \mathsf{L}(a) - 1$. We also have $|x| = |\widetilde{x}| = k - 1$. From Lemma 4.9 we get $\min \mathsf{L}(b) + 1 - T \leq \min \mathsf{L}(a)$. Therefore we see from (9) that $\min \mathsf{L}(b) + K \leq (k - 1)$. Thus the induction hypothesis applies to b, x and $\widetilde{x}$ and there exists a C-chain $x = x_0, x_1, \ldots, x_l = \widetilde{x}$ with $x_i \in \mathsf{Z}_{k-1}(b)$. But then $z, qx_0, qx_1, \ldots, qx_l, \widetilde{z}$ is a C-chain in $\mathsf{Z}_k(a)$.

Suppose now that the first inequality in (8) is not satisfied. If $\min \mathsf{L}(a) < 2^s$, then $k < K + T + 2^s \leq C$ and there is nothing more to show. Hence we can assume that the second inequality in (8) holds and that $\min \mathsf{L}(a) \geq 2^s$. Then we pick some factorization $v \in \mathsf{Z}(a)$ with $|v| = \min \mathsf{L}(a)$ and may assume that $q \mid_{\mathsf{Z}(H)} v$ for some $q \in \mathbb{A}(H)$ (again by Lemma 4.12). Now we can argue completely the same way as before.

(2) Let $\gamma \in \mathbb{N}_0$ such that H has (P_γ) and let $L \in \mathbb{N}_0$ be as in Lemma 4.10(1). Let $b \in H$, $m \in \mathsf{L}(b)\langle L + \gamma\rangle$ and $z \in \mathsf{Z}_m(b)$. Then the (P_γ) property of H yields some $x \in \mathsf{Z}(H)$ such that $x \mid_{\mathsf{Z}(H)} z$, $|x| \leq \mathcal{C}(\gamma, L)$ and $|x| \in \mathsf{L}(c)\langle L\rangle$, where $c = \pi(x)$. Since L satisfies (1) of Lemma 4.10 we infer that $|x| \pm \min \Delta(H) \in \mathsf{L}(c)$. Let $y^+, y^- \in \mathsf{Z}(c)$ be

factorizations with $|y^+| = |x| + \min\Delta(H)$ and $|y^-| = |x| - \min\Delta(H)$. Set $z^\pm = y^\pm z x^{-1} \in \mathsf{Z}(b)$. Then $|z^-|, |z|$ (resp. $|z|, |z^+|$) are successive lengths of b. Furthermore, we have the estimate

$$\mathrm{d}(z^\pm, z) \le |x| + \min\Delta(H) \le \mathcal{C}(\gamma, L) + \min\Delta(H).$$

Now set $C = L + \gamma + \min\Delta(H)$ and let $a \in H$ and $k \in \mathsf{L}(a)\langle C\rangle$. Set $l = k + \min\Delta(H)$. Then the above consideration applies to $b = a$ and $m = k$ or $m = l$. Therefore we have

$$\delta_k(a) = \mathrm{Dist}(\mathrm{Z}_k(a), \mathsf{Z}_l(a)) \le \mathcal{C}(\gamma, L) + \min\Delta(H). \qquad \square$$

The following technical Lemma is needed for the proof of Corollary 4.16.

Lemma 4.15. *Let H be a finitely primary monoid which is not half-factorial. Let $C \in \mathbb{N}$. Then there exists some $L = L(C) \in \mathbb{N}$ such that for all $b \in H$ with $\max\mathsf{L}(b) \ge L$ and for all $z \in \mathsf{Z}(b)$ with $|z| \in \{k \in \mathsf{L}(b) \mid k < \min\mathsf{L}(b) + C$ or $k > \max\mathsf{L}(b) - C\}$ there exists some $y \in \mathsf{Z}(b)$ such that $\min\mathsf{L}(b) + C \le |y| \le \max\mathsf{L}(b) - C$ and $\mathsf{d}(z, y) \le L$.*

Proof. Since H is not half-factorial there exists some $a \in H$ possessing factorizations w_0, w_1, w_2 with $|w_i| - |w_{i-1}| \ge C$ for $i \in [1, 2]$. Set $L = \max\{\mathcal{M}(a), \mathsf{t}(H, a) + \max\mathsf{L}(a)\}$ (for the definition of $\mathcal{M}(a)$ see the paragraph after Definition 2.3) and let $b \in H$ with $\max\mathsf{L}(b) \ge L$. Then a is a divisor of b. Let $z \in \mathsf{Z}(b)$. Assume first that $|z| < \min\mathsf{L}(b) + C$. By the definition of the tame degree there exists some $\widetilde{z} \in \mathsf{Z}(b)$ such that $w_0 |_{\mathsf{Z}(H)} \widetilde{z}$ and $\mathsf{d}(z, \widetilde{z}) \le \mathsf{t}(H, a)$. Set $y = \widetilde{z} w_0^{-1} w_1 \in \mathsf{Z}(b)$. Then $\mathsf{d}(z, y) \le \mathsf{d}(z, \widetilde{z}) + \mathsf{d}(\widetilde{z}, y) \le \mathsf{t}(H, a) + \max\mathsf{L}(a) \le L$ and $|y| = |\widetilde{z}| + |w_1| - |w_0| \ge |\widetilde{z}| + C \ge \min\mathsf{L}(b) + C$. On the other hand we see that $\max\mathsf{L}(b) - |y| \ge |y w_1^{-1} w_2| - |y| \ge C$. The case when $|z| > \max\mathsf{L}(b) - C$ is treated in a similar way. $\square$

Corollary 4.16 [cf. [**8**], Theorem 4.8] *Let H be a ring-like finitely primary monoid. Then there exists some constant $K \in \mathbb{N}_0$ having the following properties:*

(1) *For every* $a \in H$ *and each two factorizations* $z_1, z_2 \in \mathsf{Z}(a)$ *there exist factorizations* $y_1, y_2 \in \mathsf{Z}(a)$ *such that* $\mathsf{d}(z_i, y_i) \le K$ *for* $i \in [1,2]$ *and such that there exists a monotone* K*-chain from* y_1 *to* y_2 *in* $\mathsf{Z}(a)$.

(2) *For every* $a \in H$ *and each two factorizations* $z_1, z_2 \in \mathsf{Z}(a)$ *with* $\big||z_1| - |z_2|\big| \ge K$ *there exists a monotone* K*-chain in* $\mathsf{Z}(a)$ *which concatenates* z_1 *and* z_2.

Proof. Assume first that H is half-factorial. Then (2) is trivially satisfied for $K \ge 1$. Since H is half-factorial, $\mathsf{c}_k(a)$ coincides with the ordinary catenary degree of a if k is the length of some (and hence every) factorization of a. But H has finite ordinary catenary degree. Thus we are done if we set $y_i = z_i$ and $K = \mathsf{c}(H)$.

Suppose now that H is not half-factorial and let C be as in Theorem 4.14. Let $L = L(C)$ be a constant which fulfills the assertion of Lemma 4.15. Set $K = \max\{2L, C, B\}$, where $B = \max\{\delta_k(a), \mathsf{c}_k(a) \mid a \in H\,, k \in \mathbb{N}\,, \min \mathsf{L}(a) + C \le k \le \max \mathsf{L}(a) - C\}$. Let $a \in H$, $z_1, z_2 \in \mathsf{Z}(a)$. If $\max \mathsf{L}(a) \le K$ there is nothing to show. Hence assume $\max \mathsf{L}(a) > K$. Now we define factorizations y_i as follows: if $|z_i| \in \{k \in \mathsf{L}(a) \mid k < \min \mathsf{L}(a) + C \text{ or } k > \max \mathsf{L}(a) - C\}$ we let $y_i \in \mathsf{Z}(a)$ with $\min \mathsf{L}(a) + C \le |y_i| \le \max \mathsf{L}(a) - C$ and $\mathsf{d}(y_i, z_i) \le L$ by Lemma 4.15. Otherwise we set $y_i = z_i$. Then Theorem 4.14 immediately implies that y_1 and y_2 can be concatenated by a monotone B-chain. To show (2) assume that $|z_2| - |z_1| \ge K$. This implies that we cannot have $|z_1| > \max \mathsf{L}(a) - C$ or $|z_2| < \min \mathsf{L}(a) + C$. Therefore we get $|y_1| \ge |z_1|$ and $|y_2| \le |z_2|$ by the construction of the y_i. We moreover have $|y_2| - |y_1| = |z_2| - |z_1| + (|y_2| - |z_2|) + (|z_1| - |y_1|) \ge K - 2L \ge 0$. Now Theorem 4.14 implies (2). □

5. The monotone catenary degree of local quadratic orders. In this section we show the finiteness of the monotone catenary degree for one-dimensional local domains $(R, \mathfrak{m})$ having the following properties:

- The integral closure $\overline{R}$ of R is a finitely generated R-module.
- $\overline{R}$ has two maximal ideals.
- The residue class field $R/\mathfrak{m}$ is finite.

Clearly, the multiplicative monoid of a ring R with the above properties is strongly ring-like by Theorem 4.3. Hence the assertion is a direct consequence of Theorem 5.3. Note that (in view of Example 6.3) the analogous assertion if $\overline{R}$ has more than two maximal ideals is not true.

Rings of the above type appear naturally as localizations of orders $\mathcal{O}$ of quadratic number fields at singular places. Together with Theorem 3.9 in [**8**] this shows that every such order $\mathcal{O}$ has locally finite monotone catenary degree. However, we do not know whether this result globalizes, i.e. whether $\mathcal{O}$ itself has finite monotone catenary degree.

Assume that the following notation holds throughout the rest of the section: H is a reduced strongly ring-like finitely primary monoid with rank two and complete integral closure $\widehat{H} = \widehat{H}^\times \times [p_1, p_2]$. The prime elements p_1, p_2 and an exponent α of H are chosen in such a way that they satisfy condition (1)(a) of Definition 4.2. We denote by $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbb{N}^2$ the smallest element of $\mathsf{V}(H\setminus\{1\})$ and we set $\mathbb{A}(H) = \{q \in \mathsf{A}(H) \mid \mathsf{V}_i(q) \le 2\alpha \text{ for } i \in [1,2]\}$.

In order to show the theorem we need two preliminary lemmas.

Lemma 5.1. *The inequality* $\min\{\mathsf{V}_i(a) - \mu_i \max \mathsf{L}(a) \mid i \in [1,2]\} < \alpha$ *holds for every* $a \in H$.

Proof. Let $a \in H$ and suppose that $\mathrm{V}_i(a) - \mu_i \max \mathsf{L}(a) \ge \alpha$ for all $i \in [1,2]$. Let $q_{\min} \in H$ with $\mathsf{V}(q_{\min}) = \boldsymbol{\mu}$. Then we have $a q_{\min}^{-L} \in H\setminus\{1\}$, where $L = \max \mathsf{L}(a)$. Clearly, $q_{\min}$ is irreducible. Thus a possesses a factorization which is strictly longer than L, a contradiction. $\square$

Lemma 5.2. *There exist constants* $N, E \in \mathbb{N}$ *with the following property: if* $q_1, \ldots, q_N \in \mathsf{A}(H)$, $u \in \mathbb{A}(H)$ *with* $\mathsf{V}_1(q_1) = \cdots = \mathsf{V}_1(q_N) = \mathsf{V}_1(u)$ *and* $q \in \mathsf{A}(H)$ *with* $\mathsf{V}_2(q) \ge E$, *then there exists some* $w \in \mathsf{Z}(\pi(q q_1 \cdot \ldots \cdot q_N))$ *such that* $|w| = N+1$ *and* $u \mid_{\mathsf{Z}(H)} w$.

Proof. Let $\mathcal{D} \in \mathbb{N}$ be the Davenport constant (cf. for example [**5**]) of $\widehat{H}^\times$. Then for every sequence $\varepsilon_1, \ldots, \varepsilon_{\mathcal{D}} \in \widehat{H}^\times$ there exists some non-empty subset $J \subset [1, \mathcal{D}]$ such that $\prod_{j \in J} \varepsilon_j = 1$. Set $E = 2\alpha(\mathcal{D}+1)$ and $N = \mathcal{D}$. Put $v_i = q_i u^{-1} = \varepsilon_i p_2^{\xi_i}$, where $\varepsilon_i \in \widehat{H}^\times$, $\xi_i \ge -2\alpha$ and let $\varnothing \ne J \subset [1, N]$ such that $\prod_{j \in J} \varepsilon_j = 1$. Then $\prod_{j \in J} v_j = p_2^t$, where

$t = \sum_{j \in J} \xi_j$. Since $t \geq -2\alpha\mathcal{D}$, we see that $\mathsf{V}_2(y) \geq 2\alpha$, where $y = qp_2^t$. Hence $y \in H$ and y is irreducible by Proposition 4.4. Now we set

$$w = u^{|J|} y \prod_j q_j,$$

where j varies over $[1, N] \setminus J$ and we are done. $\square$

Theorem 5.3 *Let H be a strongly ring-like finitely primary monoid with rank two. Then the monotone catenary degree $\mathsf{c}_{\mathrm{mon}}(H)$ of H is finite. In particular, $\sup\{\mathsf{c}_k(a) \mid a \in H, k \in \mathsf{L}(a)\}$ is finite.*

Proof. For $M \in \mathbb{N}$ set $\mathcal{B}(M) = \sup\{\delta_k(a), \mathsf{c}_k(a) \mid a \in H\,, k \in \mathbb{N}\,, k \leq \max \mathsf{L}(a) - M\} \in \mathbb{N}_0 \cup \{\infty\}$. Note that by [**17**], Proposition 4.1 the set $\{\min \mathsf{L}(a) \mid a \in H\}$ is bounded. Therefore Theorem 4.14 immediately implies that there exists some $M \in \mathbb{N}$ such that $\mathcal{B}(M)$ is finite. Pick $M \in \mathbb{N}$ with this property and let $K \in \mathbb{N}$ such that the assertions (1) and (2) of Corollary 4.16 are satisfied. Set $\widetilde{C} = 2\max\{M, K\}$. Let $N, E \in \mathbb{N}$ be suitable constants in Lemma 5.2. Put $L = \max\{E, \mu_1(\widetilde{C} + 1) + \alpha\}$ and let $\mathbb{B} = \{q \in \mathsf{A}(H) \mid \mathsf{V}_1(q) \leq L\,, \mathsf{V}_2(q) \leq L\}$. Clearly, $\mathbb{B}$ is a finite (and possibly empty) set. Denote by $H(\mathbb{B})$ the submonoid of H which is generated by $\mathbb{B}$. (Note that if $\mathbb{B} = \varnothing$ then $H(\mathbb{B}) = \{1\}$ and Case 1 below does not occur.) Then $H(\mathbb{B})$ is finitely generated and $\mathsf{A}(H(\mathbb{B})) = \mathbb{B}$. By Theorem 3.9 in [**8**], the monotone catenary degree $\mathsf{c}_{\mathrm{mon}}(H(\mathbb{B}))$ of $H(\mathbb{B})$ is finite. Define the constant $A \in \mathbb{N}$ by $A = \max\{\mathcal{B}(M), (1 + \mu_1)\widetilde{C} + N + \alpha, \mathsf{c}_{\mathrm{mon}}(H(\mathbb{B}))\}$.

Let $a \in H$, $z = z_1 \cdot \ldots \cdot z_n \in \mathsf{Z}(a)$ and $z' = z'_1 \cdot \ldots \cdot z'_{n'} \in \mathsf{Z}(a)$. We show by induction on $\max \mathsf{L}(a)$ that there exists a monotone A-chain in $\mathsf{Z}(a)$ which concatenates z and z'.

If $\max \mathsf{L}(a) \leq A$, then the trivial chain z, z' is a monotone A-chain. Hence assume for the rest of the proof that $\max \mathsf{L}(a) > A$. Assume first that $\min\{|z|, |z'|\} < \max \mathsf{L}(a) - \widetilde{C}$. If $\big||z| - |z'|\big| \leq \frac{\widetilde{C}}{2}$, then we have $\max\{|z|, |z'|\} \leq \min\{|z|, |z'|\} + \frac{\widetilde{C}}{2} < \frac{\widetilde{C}}{2} + \max \mathsf{L}(a) - \widetilde{C} \leq \max \mathsf{L}(a) - M$. Hence there exists some monotone $\mathcal{B}(M)$-chain from z to z'. If, on the other hand, $\big||z| - |z'|\big| > \frac{\widetilde{C}}{2}$, then there exists some monotone K-chain concatenating z and z' by Corollary 4.16. Hence we can assume in the following that $\min\{|z|, |z'|\} \geq \max \mathsf{L}(a) - \widetilde{C}$.

By Lemma 5.1 we have $\mathsf{V}_i(a) - \mu_i \max \mathsf{L}(a) \le \alpha$ for some $i \in [1,2]$. Without loss of generality we can assume that $i = 1$. We proceed with the following consideration. Let $w = w_1 \cdot \ldots \cdot w_l \in \mathsf{Z}(a)$ be an arbitrary factorization with $l \ge \max \mathsf{L}(a) - \widetilde{C}$. Then $\mathsf{V}_1(a) - \mu_1 l = \mathsf{V}_1(a) - \mu_1 \max \mathsf{L}(a) + \mu_1(\max \mathsf{L}(a) - l) \le \alpha + \mu_1 \widetilde{C}$. This estimate shows that $|\{j \in [1,l] \mid \mathsf{V}_1(w_j) \ne \mu_1\}| \le \alpha + \mu_1 \widetilde{C}$ since $\mathsf{V}_1(d) \ge \mu_1$ for every element $d \in H \setminus \{1\}$. Moreover, we see that $\mathsf{V}_1(q) \le \alpha + \mu_1(\widetilde{C}+1)$ for all $q \in \mathsf{A}(H)$ such that $q \mid w$. In particular, these considerations apply to z and z'. Since $\min\{|z|,|z'|\} \ge \max \mathsf{L}(a) - \widetilde{C} > A - \widetilde{C} \ge N + \alpha + \mu_1 \widetilde{C}$, we see that $\big|\{j \in [1,n] \mid \mathsf{V}_1(z_j) = \mu_1\}\big| > N$ and $\big|\{j \in [1,n'] \mid \mathsf{V}_1(z'_j) = \mu_1\}\big| > N$.

Set $U = \max\big(\{\mathsf{V}_2(z_j) \mid j \in [1,n]\} \cup \{\mathsf{V}_2(z'_j) \mid j \in [1,n']\}\big)$. Now we distinguish two cases.

Case 1. We have $U < E$. Then $z, z' \in \mathcal{F}(\mathbb{B})$. Hence there exists some monotone $\mathsf{c}_{\mathrm{mon}}(H(\mathbb{B}))$-chain from z to z' in $\mathsf{Z}_{H(\mathbb{B})}(a) \subset \mathsf{Z}_H(a)$.

Case 2. We have $U \ge E$. We assume (without restriction) that $U = \mathsf{V}_2(z_1)$. Since $\big|\{j \in [1,n'] \mid \mathsf{V}_1(z'_j) = \mu_1\}\big| \ge N+1 \ge 2$, an adjustment (if necessary) of the second valuation of two of the z'_j by Proposition 4.4 yields some factorization $\widetilde{z}' \in \mathsf{Z}(a)$ such that $\mathsf{d}(z', \widetilde{z}') \le 2$ and such that $u \mid_{\mathsf{Z}(H)} \widetilde{z}'$ for some $u \in \mathbb{A}$ with $\mathsf{V}_1(u) = \mu_1$. Assume (by reordering the atoms z_i if necessary) that $\mathsf{V}_1(z_j) = \mu_1$ holds for all $j \in [2, N+1]$. Set $b = \pi(z_1 \cdot \ldots \cdot z_{N+1}) \in H$. Then Lemma 5.2 (with $q = z_1$) implies that there exists some $w \in \mathsf{Z}(b)$ such that $|w| = N+1$ and $u \mid_{\mathsf{Z}(H)} w$. Set $\widetilde{z} = w z_{N+2} \cdot \ldots \cdot z_n$. Then $|z| = |\widetilde{z}|$ and $\mathsf{d}(z, \widetilde{z}) \le N+1 \le A$. Set $v = \widetilde{z} u^{-1}$, $v' = \widetilde{z}' u^{-1}$ and $c = a u^{-1}$. Then by the induction hypothesis there exists some monotone A-chain $v = v_0, v_1, \ldots, v_k = v'$ in $\mathsf{Z}(c)$. But then $z, v_0 u, v_1 u, \ldots, v_k u, z'$ is a monotone A-chain in $\mathsf{Z}(a)$. $\square$

6. Examples. For a commutative ring R we denote by $\mathfrak{z}(R)$ the set of zero divisors of R and we set $R^\bullet = R \setminus \mathfrak{z}(R)$. Furthermore, $\mathcal{Q}(R) = (R^\bullet)^{-1} R$ denotes the total quotient ring of R and $\overline{R}$ denotes the integral closure of R in $\mathcal{Q}(R)$. Let R be a semi-local ring. We denote by $\widehat{R}$ the completion of R with respect to the Jacobson radical-adic topology. Note that we have $R^\bullet \subset \widehat{R}^\bullet$ by flatness. Hence $\mathcal{Q}(R)$ naturally embeds into $\mathcal{Q}(\widehat{R})$.

Let R be a one-dimensional local reduced ring with minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Then $\mathcal{Q}(R) \cong \prod_{i=1}^n \mathcal{Q}(R/\mathfrak{p}_i)$ and $\overline{R} \cong \prod_{i=1}^n \overline{(R/\mathfrak{p}_i)}$, where each $\overline{(R/\mathfrak{p}_i)}$ is a semilocal principal ideal domain (cf. for example [**7**]). Assume that $\mathfrak{m}_{i,1}, \dots, \mathfrak{m}_{i,r_i}$ are the maximal ideals of $\overline{(R/\mathfrak{p}_i)}$. Then $\overline{(R/\mathfrak{p}_i)}_{\mathfrak{m}_{i,j}}$ is a discrete valuation ring and we denote by $\mathsf{V}_{i,j} : \mathcal{Q}(R/\mathfrak{p}_i) \to \mathbb{Z} \cup \{\infty\}$ the corresponding discrete valuation. If we put them together in the obvious way we obtain a map

$$\mathsf{V} : \mathcal{Q}(R) \longrightarrow \mathbb{Z}_\infty^s,$$

where $s = \sum_{i=1}^n r_i$ and $\mathbb{Z}_\infty = \mathbb{Z} \cup \{\infty\}$. We call V the valuation map of R. The submonoid $\mathsf{V}(R^\bullet) \subset \mathbb{N}_0^s$ is called the semigroup of values of R. Note that we have $\mathsf{V}(R^\bullet) \subset \mathbb{N}^s \cup \{\mathbf{0}\}$. To see this, let $x \in R^\bullet$ with $\mathsf{V}_{i,j}(x + \mathfrak{p}_i) = 0$ for some $i \in [1, n]$, $j \in [1, r_i]$. Then $x + \mathfrak{p}_i$ is a unit of $\overline{(R/\mathfrak{p}_i)}_{\mathfrak{m}_{i,j}}$, whence it is a unit of $R/\mathfrak{p}_i$. Hence x is a unit of R and therefore $\mathsf{V}(x) = \mathbf{0}$.

Proposition 6.1. *Let R be a one-dimensional local ring.*

(1) *If $\widehat{R}$ is reduced, then $R^\bullet$ is finitely primary with complete integral closure $\overline{R}^\bullet$.*

(2) *The natural map $j : R^\bullet/R^\times \to \widehat{R}^\bullet/\widehat{R}^\times$ is an isomorphism of monoids.*

(3) *Suppose that $\widehat{R}$ is reduced. With the natural injection $\iota : \mathcal{Q}(R) \to \mathcal{Q}(\widehat{R})$ we have a commutative diagram*

$$\begin{tikzcd} \mathcal{Q}(R) \arrow[rr, "\iota"] \arrow[dr, "\mathsf{V}"'] & & \mathcal{Q}(\widehat{R}) \arrow[dl, "\widehat{\mathsf{V}}"] \\ & \mathbb{Z}_\infty^s & \end{tikzcd}$$

where V and $\widehat{\mathsf{V}}$ denote the valuation maps of R and $\widehat{R}$, respectively. Furthermore, $\mathsf{V}(R^\bullet) = \widehat{\mathsf{V}}(\widehat{R}^\bullet)$.

Proof. (1) We clearly have $R^\bullet \subset \overline{R}^\bullet$ and $\overline{R}^\bullet$ is factorial. Since $\widehat{R}$ is reduced, $\overline{R}$ is a finitely generated R-module by [**23**], Theorem 10.2. Hence there exists some $f \in R^\bullet$ with $f\overline{R}^\bullet \subset R^\bullet$ and condition

(1) of Definition 2.2 is satisfied. But condition (2) of Definition 2.2 is equivalent to $\mathsf{V}(R^\bullet) \subset \mathbb{N}^s \cup \{\mathbf{0}\}$ which we have already proved.

(2) Since $R^\bullet \subset \widehat{R}^\bullet$, the map j is well defined. By [**24**], 18.4 j is injective. In order to show that j is surjective, let $\widehat{x} \in \widehat{R}^\bullet$. Let $x_n \in R$ be a sequence which converges to $\widehat{x}$. Since $\widehat{x}$ is a nonzero divisor of $\widehat{R}$ and since $\widehat{R}$ is one-dimensional, we have $\widehat{\mathfrak{m}}^k \subsetneq \widehat{x}\widehat{R}$ for some $k \in \mathbb{N}$, where $\widehat{\mathfrak{m}}$ denotes the maximal ideal of $\widehat{R}$. Let $l \in \mathbb{N}$ such that $\widehat{x} - x_l \in \widehat{\mathfrak{m}}^k$. Then $\widehat{x} - x_l = \widehat{\xi}\widehat{x}$, where $\widehat{\xi} \in \widehat{R}$. If $\widehat{\xi} \in \widehat{\mathfrak{m}}$, then $x_l = (1 - \widehat{\xi})\widehat{x}$ with the unit $1 - \widehat{\xi}$ and we are done. But the assumption $\widehat{\xi} \in \widehat{R}^\times$ yields the contradiction $\widehat{x}\widehat{R} = (\widehat{x} - x_l)\widehat{R} \subset \widehat{\mathfrak{m}}^k$.

(3) Since $\widehat{R}$ is reduced, we can form the valuation maps V of R and $\widehat{\mathsf{V}}$ of $\widehat{R}$. From $\mathcal{Q}(R) \subset \mathcal{Q}(\widehat{R})$ we see that $\overline{R} \subset \overline{\widehat{R}}$. By [**7**], Proposition 1.1 we know that $\overline{\widehat{R}}$ is actually the completion of $\overline{R}$, i.e. $\widehat{\overline{R}} = \overline{\widehat{R}}$. In particular, we have $\mathcal{Q}(\widehat{R}) = \mathcal{Q}(\overline{\widehat{R}}) = \mathcal{Q}(\widehat{\overline{R}})$ and $\mathcal{Q}(R) = \mathcal{Q}(\overline{R})$. Therefore it is enough to show commutativity of the diagram if $R = \overline{R}$, i.e. if R is a finite product of semi-local principal ideal domains. But in this case the assertion follows immediately from the fact that it is true for a single semi-local principal ideal domain. Now $\mathsf{V}(R^\bullet) = \widehat{\mathsf{V}}(\widehat{R}^\bullet)$ follows directly from (2). ◻

Proposition 6.2. (C. Lech) *Let $(R, \mathfrak{m})$ be a complete local ring. Then R is the completion of some local domain if and only if the following conditions are fulfilled:*

(1) *The prime ring* $\mathrm{Prim}(R)$ *of R is a domain and R is a torsion free* $\mathrm{Prim}(R)$*-module.*

(2) *Either $\mathfrak{m} = (0)$ or $\mathfrak{m}$ is not an associated prime of R.*

Proof. See [**22**].

In the following examples we show the existence of one-dimensional local domains D having certain arithmetical properties. By Proposition 6.1 and Proposition 6.2 it is enough to show these properties for the multiplicative monoid $R^\bullet$ of a complete, reduced one-dimensional local ring R (containing some field).

Example 6.3. There exists a one-dimensional local domain $(D, \mathfrak{m})$ having the following properties:

- $D/\mathfrak{m}$ is finite.
- $\overline{D}$ is a finitely generated D-module and has three maximal ideals.
- $\sup\{\mathsf{c}_k(a) | a \in D^\bullet, k \in \mathsf{L}(a)\}$ is infinite. In particular, $\mathsf{c}_{\mathrm{mon}}(D^\bullet) = \infty$.

Proof. Set $K = \mathbb{Z}/2\mathbb{Z}$ and let x, y, u, v, w be indeterminates. Define three ring homomorphisms $\varphi_1 : K[[x, y]] \to K[[u^2, u^3]]$, $\varphi_2 : K[[x, y]] \to K[[v]]$ and $\varphi_3 : K[[x, y]] \to K[[w^2, w^3]]$ by setting $\varphi_1(x) = u^2$, $\varphi_1(y) = u^3$, $\varphi_2(x) = v$, $\varphi_2(y) = v$, $\varphi_3(x) = w^3$ and $\varphi_3(y) = w^2$. Then the kernels of these homomorphisms are generated by $x^3 - y^2$, $x - y$ and $x^2 - y^3$, respectively. Set $\mathfrak{I} = \ker(\varphi_1) \cap \ker(\varphi_2) \cap \ker(\varphi_3)$ and let $R = K[[x, y]]/\mathfrak{I}$. Then R is a one-dimensional, complete, reduced local ring with minimal prime ideals $\mathfrak{p}_1 = (\overline{x}^3 - \overline{y}^2)$, $\mathfrak{p}_2 = (\overline{x} - \overline{y})$ and $\mathfrak{p}_3 = (\overline{x}^2 - \overline{y}^3)$, where bars denote residue classes modulo $\mathfrak{I}$. If we identify $\overline{x} = (u^2, v, w^3)$ and $\overline{y} = (u^3, v, w^2)$, then $R = K[[\overline{x}, \overline{y}]] \subset K[[u]] \times K[[v]] \times K[[w]]$, where the last product coincides with the integral closure of R. Let $\mathsf{V} : \mathcal{Q}(R) \to (\mathbb{Z}_\infty)^3$ denote the valuation map. Our first aim is to establish some properties of the semigroup of values of R.

Claim 1: $\{\mathsf{V}(\xi) \in \mathbb{N}^3 \mid \xi \in R^\bullet, \mathsf{V}_2(\xi) = 1\} = \{(2, 1, 3), (3, 1, 2)\}$.

Let $\xi \in R^\bullet$ with $\mathsf{V}_2(\xi) = 1$. Then $\xi = \xi_{0,0} + \xi_{1,0}\overline{x} + \xi_{0,1}\overline{y} + S$, where S is a sum of terms $\xi_{i,j}\overline{x}^i\overline{y}^j$ with $i + j \geq 2$. Since ξ is not a unit, we have $\xi_{0,0} = 0$. If $\xi_{1,0} = \xi_{0,1} = 1$, then $\mathsf{V}_2(\xi) \geq 2$, which is a contradiction. Hence $(\xi_{1,0}, \xi_{0,1}) \in \{(0, 1), (1, 0)\}$. But since $\mathsf{V}_2(S) \geq 2$ and $\mathrm{V}_i(S) \geq 4$ for $i \in \{1, 3\}$, the assertion follows.

Claim 2: $\{\mathsf{V}(\xi) \in \mathbb{N}^3 \mid \xi \in R^\bullet, \mathsf{V}_2(\xi) = 2\}$
$$= \{(2, 2, 2), (5, 2, 5), (4, 2, 4)\} \cup \{(4, 2, m), (m, 2, 4) \mid m \in \mathbb{N}_{\geq 6}\}.$$

Let $\xi \in R^\bullet$ with $\mathsf{V}_2(\xi) = 2$. As above we write $\xi = \xi_{1,0}\overline{x} + \xi_{0,1}\overline{y} + S$, where S is a sum of terms of degree at least two. If $\xi_{1,0} \neq \xi_{0,1}$, then $\mathsf{V}_2(\xi) = 1$ which is a contradiction.

Case 1: $\xi_{1,0} = \xi_{0,1} = 1$. Since $\mathsf{V}_i(S) \geq 4$ for $i \in \{1, 3\}$, we get $\mathsf{V}(\xi) = (2, 2, 2)$.

Case 2: $\xi_{1,0} = \xi_{0,1} = 0$. Then $\xi = \xi_{1,1}\overline{xy} + \xi_{2,0}\overline{x}^2 + \xi_{0,2}\overline{y}^2 + S'$, where S' is a sum of terms $\xi_{i,j}\overline{x}^i\overline{y}^j$ with $i+j \geq 3$. We must have $\xi_{1,1}+\xi_{2,0}+\xi_{0,2}=1$ in order to get $\mathsf{V}_2(\xi)=2$.

Case 2a: $\xi_{1,1}=1$ and $\xi_{2,0}=\xi_{0,2}=0$. Since $\mathsf{V}(\overline{xy})=(5,2,5)$ and since $\mathsf{V}_i(S') \geq 6$ for $i \in \{1,3\}$ we see that $\mathsf{V}(\xi)=(5,2,5)$ is the only possibility.

Case 2b: $\xi_{1,1}=\xi_{2,0}=\xi_{0,2}=1$. For the same reason as in Case 2a we get $\mathsf{V}(\xi)=(4,2,4)$.

Case 2c: $\xi_{2,0}=1$ and $\xi_{1,1}=\xi_{0,2}=0$. Then $\xi=(u^4,v^2,w^6)+S'$. Hence $\mathsf{V}(\xi)$ has the form $(4,2,m)$ with some $m \in \mathbb{N}_{\geq 6}$. Set $M=\{m \in \mathbb{N}_{\geq 6} \mid (4,2,m) \in \mathsf{V}(R^\bullet)\}$. We want to show that $M=\mathbb{N}_{\geq 6}$. Clearly, $6 \in M$. Let $\xi=(u^4,v^2,w^6)+\overline{y}^3+S''=(u^4+u^9,v^2+v^3,0)+S''$, where S'' is a sum

$$S''=\sum_{i,j}\xi_{i,j}\overline{xy} \tag{10}$$

such that the indices i and j satisfy the conditions $i+j \geq 3$ and $(i,j) \neq (0,3)$. Since $\{3i+2j \mid (i,j) \in \mathbb{N}_0^2, i+j \geq 3, (i,j) \neq (0,3)\}=\mathbb{N}_{\geq 7}$, we can choose suitable $\xi_{i,j}$ in (10) for every given $m \in \mathbb{N}_{\geq 7}$ to obtain an element with valuation $(4,2,m)$.

The remaining case $\xi_{0,2}=1$ and $\xi_{1,1}=\xi_{2,0}=0$ is symmetric to Case 2c. This completes the proof of Claim 2.

Since $\overline{R}$ is a finitely generated R-module, there exists some $f \in \mathbb{N}$ such that (u^f,v^f,w^f) is contained in the conductor $(R:\overline{R})$. Hence $(0,0,w^l) \in R$ for all $l \geq f+1$. Let $k \geq f$ be an integer. Then $q_k=\overline{x}^2+\overline{y}^3+(0,0,w^{k+4}+w^{k+9})=(u^4+u^9,v^2+v^3,w^{k+4}+w^{k+9}) \in R$. By symmetry, we also have $p_k=(u^{k+4}+u^{k+9},v^2+v^3,w^4+w^9) \in R$. Set $a_k=\overline{x}^{k-1}p_{k-1}=\overline{y}^{k-1}q_{k-1} \in R$. Claim 1 shows that $\overline{x},\overline{y}$ and p_k and q_k are irreducible elements of $R^\bullet$ if k is big enough. Hence $z_k=\overline{x}^{k-1}p_{k-1}$ and $z'_k=\overline{y}^{k-1}q_{k-1}$ are both factorizations of a_k with length k.

Let $\overline{\mathsf{V}}:\mathsf{Z}(R^\bullet) \to \mathcal{F}(\mathsf{V}(R^\bullet))$ be the extension of V which sends each $qR^\times \in \mathsf{A}(R^\bullet/R^\times)$ to $\mathsf{V}(q)$. We show that if k is large enough and z is any factorization of a_k with length k, then $\overline{\mathsf{V}}(z) \in \{\overline{\mathsf{V}}(z_k),\overline{\mathsf{V}}(z'_k)\}$. In particular we have $\mathsf{d}(z,z')=k$ if $z,z' \in \mathsf{Z}_k(a_k)$ with $\overline{\mathsf{V}}(z)=\overline{\mathsf{V}}(z_k)$ and $\overline{\mathsf{V}}(z')=\overline{\mathsf{V}}(z'_k)$. This implies $\mathsf{c}_k(a_k)=k$ for the catenary degree of a_k at length k.

Let $z = x_1 \cdot \ldots \cdot x_k \in \mathsf{Z}_k(a_k)$. Since $\mathsf{V}_2(a_k) = k+1$, we can assume that $\mathsf{V}_2(x_i) = 1$ for $i \in [1, k-1]$ and $\mathsf{V}_2(x_k) = 2$. For sufficiently large k we cannot have $\mathsf{V}(x_k) \in \{(2,2,2),(5,2,5),(4,2,4)\}$ because the distance between $\mathsf{V}(a_k) = (3k+1, k+1, 3k+1)$ and the plane $(2,1,3)\mathbb{Q}+(3,1,2)\mathbb{Q}$ is unbounded. Hence $\mathsf{V}(x_k) \in \{(4,2,m),(m,2,4) \mid m \in \mathbb{N}_{\geq 6}\}$ by the second claim. Suppose that $\mathsf{V}(x_k) = (4,2,m)$ for some m. Then there exist $\alpha, \beta, \gamma \in \mathbb{N}_0$ such that $\alpha(2,1,3) + \beta(3,1,2) + (4,2,\gamma) = (3k+1, k+1, 3k+1)$. This is a linear equation (in α, β and γ) with regular matrix. Hence there exists only the solution $\alpha = 0$, $\beta = k-1$ and $\gamma = k+3$. Thus $\overline{\mathsf{V}}(z) = \overline{\mathsf{V}}(z'_k)$. The same argument applies if $\mathsf{V}(x_k)$ is of the form $(m, 2, 4)$ for some m. □

Lemma 6.4. *Let H be a finitely primary monoid with rank two. Let $\mathsf{V} : \mathcal{Q}(H) \to \mathbb{Z}^2$ be the valuation map of H. Set $\nu_i = \min\{\mathsf{V}_i(h) \mid h \in H \backslash H^\times\}$ and $M_i = \{\mathsf{V}(h) \mid h \in H \backslash H^\times, \mathsf{V}_i(h) = \nu_i\}$ for $i \in [1,2]$ and assume that $1 < |M_i| < \infty$ for some $i \in [1,2]$. Then $\sup\{\delta_k(a) \mid a \in H, k \in \mathsf{L}(a)\}$ is infinite.*

Proof. Suppose without restriction that $1 < |M_1| < \infty$. Choose $p, q \in H$ such that their valuation is contained in M_1 and such that $\mathsf{V}(q) = \max M_1$ and $\mathsf{V}(p) < \mathsf{V}(q)$ (note that M_1 is totally ordered). Clearly, p and q are irreducible elements of H. Let $\alpha \in \mathbb{N}$ be an exponent of H and set $a_n = q^{n+\alpha} \in H$ for $n \geq 1$. Suppose that $z = z_1 \cdot \ldots \cdot z_{n+\alpha}$ is an arbitrary factorization of a_n with length $n+\alpha$. Then $\mathsf{V}(z_i) = \mathsf{V}(q)$ holds for every $i \in [1, n+\alpha]$ because of the properties of q. Clearly, $\xi_n = a_n p^{-n}$ is contained in H. Let ω_n be an arbitrary factorization of ξ_n. Then $|\omega_n| \leq \alpha$. Indeed, $|\omega_n| < \alpha$ for large n, since $\mathsf{V}_2(\xi_n)$ is unbounded and M_1 is finite. Set $v_n = \omega_n p^n \in \mathsf{Z}(a_n)$. Then $n \leq |v_n| < n+\alpha$ and $\mathrm{Dist}(\{v_n\}, \mathsf{Z}_{n+\alpha}(a_n)) \geq n$. This proves the lemma. □

Example 6.5. There exists a one-dimensional local domain $(D, \mathfrak{m})$ having the following properties:

(1) $D/\mathfrak{m}$ is finite.

(2) $\overline{D}$ is a finitely generated D-module and has two maximal ideals.

(3) $\sup\{\delta_k(a) \mid a \in D^\bullet, k \in \mathsf{L}(a)\}$ is infinite.

Proof. Let k be a field and set $R = k[[(t^2, u^3), (t^3, u^2)]] \subset k[[t]] \times k[[u]]$. Then $M = \{(2,2), (2,3), (3,2)\} \subset V(R^\bullet)$ and $V(R^\bullet)\backslash M \subset \mathbb{N}^2_{\geq 4}$. Hence the assertion follows from Lemma 6.4.

Remark 6.6. We have not been able to clarify whether there exists a one-dimensional analytically irreducible local domain D (i.e. $\overline{D}$ is a discrete valuation ring and finitely generated as a D-module) for which $\sup\{\delta_k(a) \mid a \in D^\bullet, k \in \mathsf{L}(a)\}$ or $\sup\{\mathsf{c}_k(a) \mid a \in D^\bullet, k \in \mathsf{L}(a)\}$ is infinite. Note that by [**8**], Theorem 3.9 such a ring D (if it exists) necessarily has infinite residue class field.

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Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstrasse 36, A-8010 Graz, Austria.
Email address: wolfgang.hassler@uni-graz.at