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An intrinsic characterization of the direct product of balls

By

Akio Kodama and Satoru Shimizu

Abstract

In this paper, we give a characterization of the direct product of balls by its holomorphic automorphism group. Using a result on the standardization of certain compact group actions on complex manifolds, we show that, for a connected Stein manifold M of dimension n, if its holomorphic automorphism group contains a topological subgroup that is isomorphic to the holomorphic automorphism group of the direct product $\mathbf{B}$ of balls in $\mathbf{C}^n$, then M itself is biholomorphically equivalent to $\mathbf{B}$.

Introduction

The well-known Riemann mapping theorem may be viewed as a kind of an intrinsic characterization of the unit disk in the complex plane. In fact, the Riemann mapping theorem asserts that the unit disk in the complex plane is characterized intrinsically as a simply connected domain in the complex plane that is hyperbolic in the sense of Kobayashi. Generalizing this fact to the higher-dimensional case, for example, characterizing intrinsically a single ball of dimension ≥ 2 or more generally the direct product of balls, is an interesting problem in several complex variables. It should be noted here that, as was shown by Poincaré, the direct product of balls can not be characterized by only topological conditions and hyperbolicity. So we need another approaches to the study of an intrinsic characterization of the direct product of balls. In this paper, directing our attention to holomorphic automorphism groups, we consider the following problem:

Let M be a connected complex manifold of dimension n and let $\mathbf{B}$ be a domain in $\mathbf{C}^n$ given as the direct product of balls. If the holomorphic automorphism group $\mathrm{Aut}(M)$ of M is isomorphic to that of $\mathbf{B}$ as topological groups, then is M itself biholomorphically equivalent to $\mathbf{B}$?

This is the problem of characterizing the direct product of balls intrinsically by using its holomorphic automorphism group. The purpose of this paper is to

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give some answer to this problem, and we show the following:

Theorem. *Let M be a connected Stein manifold of dimension $n \geq 2$ and let $\mathbf{B}$ be a domain in $\mathbf{C}^n$ given as the direct product $\mathbf{B} = B_{n_1} \times \cdots \times B_{n_s}$ of balls, where each B_{n_j} is the unit ball in $\mathbf{C}^{n_j}$ with $n_j > 1$ and $\sum_{j=1}^{s} n_j = n$. Assume that there exists a topological subgroup G of $\mathrm{Aut}(M)$ that is isomorphic to $\mathrm{Aut}(\mathbf{B})$ as topological groups, where the groups $\mathrm{Aut}(M)$ and $\mathrm{Aut}(\mathbf{B})$ are equipped with the compact-open topology. Then M is biholomorphically equivalent to $\mathbf{B}$.*

When $G = \mathrm{Aut}(M)$, this theorem gives a partial answer to the above problem. Note that, as for the particular case of a characterization of a single ball of dimension ≥ 2, the answer was already given in Isaev [1], Byun, Kodama, and Shimizu [2]. Note also that our theorem shows that if a Stein manifold M of dimension n is not biholomorphically equivalent to $\mathbf{B}$, then $\mathrm{Aut}(M)$ never contains a topolgical subgroup that is isomorphic to $\mathrm{Aut}(\mathbf{B})$.

The proof of our theorem is based on the methods developed in Kodama and Shimizu [3], [4], [5], and Byun, Kodama, and Shimizu [2]. In particular, the Generalized Standardization Theorem given in [5] plays a fundamental role in our study.

This paper is organized as follows. In Section 1, we recall the Generalized Standardization Theorem after making some preparations. In Section 2, we give a proof of the Theorem.

1. Generalized Standardization Theorem

We first recall some facts on Lie group actions, Reinhardt domains and torus actions. Let M be a complex manifold. An *automorphism* of M means a biholomorphic mapping of M onto itself. We denote by $\mathrm{Aut}(M)$ the topological group of all automorphisms of M equipped with the compact-open topology. When a topological subgroup G of $\mathrm{Aut}(M)$ has the structure of a finite-dimensional Lie group, we call G a *Lie group in* $\mathrm{Aut}(M)$.

Let G be a Lie group. When a continuous group homomorphism $\rho\colon G \to \mathrm{Aut}(M)$ of G into $\mathrm{Aut}(M)$ is given, the mapping

$$G \times M \ni (g, p) \longmapsto (\rho(g))(p) \in M$$

is of class C^ω, and we say that *G acts on M as a Lie transformation group through ρ.* Also, the action of G on M is called *effective* if ρ is injective. Note that if G is a Lie group in $\mathrm{Aut}(\mathbf{C}^n)$ and if $\iota\colon G \to \mathrm{Aut}(\mathbf{C}^n)$ is a natural inclusion mapping, then G acts effectively on $\mathbf{C}^n$ as a Lie transformation group through ι. Write $T^n = (U(1))^n$. The n-dimensional compact torus T^n acts as a group of automorphisms on $\mathbf{C}^n$ by the standard rule

$$\alpha \cdot z = (\alpha_1 z_1, \ldots, \alpha_n z_n) \text{ for } \alpha = (\alpha_1, \ldots, \alpha_n) \in T^n \text{ and } z = (z_1, \ldots, z_n) \in \mathbf{C}^n.$$

By definition, a *Reinhardt domain* D in $\mathbf{C}^n$ is a domain in $\mathbf{C}^n$ which is stable under the action of T^n, that is, such that $\alpha \cdot D \subset D$ for all $\alpha \in T^n$. Each element α of T^n then induces an automorphism π_α of D given by $\pi_\alpha(z) = \alpha \cdot z$, and the mapping ρ_D sending α to π_α is an injective continuous group homomorphism of the torus T^n into the topological group $\mathrm{Aut}(D)$. The subgroup $\rho_D(T^n)$ of $\mathrm{Aut}(D)$ is denoted by $T(D)$.

We now recall the Generalized Standardization Theorem given in Kodama and Shimizu [5].

Generalized Standardization Theorem. *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy and let K be a connected compact Lie group of rank n. Suppose that an injective continuous group homomorphism ρ of K into $\mathrm{Aut}(M)$ is given. Then there exists a biholomorphic mapping F of M onto a Reinhardt domain D in $\mathbf{C}^n$ such that*

$$F\rho(K)F^{-1} = U(n_1) \times \cdots \times U(n_s) \subset \mathrm{Aut}(D),$$

where each $U(n_j)$ is the unitary group of degree n_j and $\sum_{j=1}^{s} n_j = n$.

An application of the Generalized Standardization Theorem yields the following fact ([5, Theorem 2]), which is a key in our proof of the Theorem stated in the introduction.

Key Fact. *Let G be a connected Lie group in $\mathrm{Aut}(\mathbf{C}^n)$ and let m be an integer with $m \geq n$. Then G cannot be isomorphic to the holomorphic automorphism group $\mathrm{Aut}(B_m)$ of the unit ball B_m in $\mathbf{C}^m$, or there is no isomorphism of $\mathrm{Aut}(B_m)$ onto G as Lie groups.*

2. Proof of the Theorem

Let $\mathbf{B}$ be the direct product of balls given in the Theorem in the introduction. According to the direct decomposition $\mathbf{B} = B_{n_1} \times \cdots \times B_{n_s}$, we put

$$\begin{aligned}
K &= U(n_1) \times \cdots \times U(n_s),\\
K_j &= \left\{(\varphi_1, \ldots, \varphi_s) \in K \mid \varphi_j = E_{n_j}\right\},\ j = 1, \ldots, s,\\
L &= SU(n_1) \times \cdots \times SU(n_s),\\
L_j &= \left\{(\varphi_1, \ldots, \varphi_s) \in L \mid \varphi_j = E_{n_j}\right\},\ j = 1, \ldots, s,
\end{aligned}$$

where E_k denotes the unit matrix of degree k for a positive integer k. Also, for $j = 1, \ldots, s$, we regard $U(n_j)$ and $SU(n_j)$ naturally as subgroups of K and L, respectively.

Now, as in the Theorem stated in the introduction, let M be a connected Stein manifold of dimension n and assume that there exists an isomorphism $\Phi : \mathrm{Aut}(\mathbf{B}) \to G$ between the topological group $\mathrm{Aut}(\mathbf{B})$ and a topological subgroup G of $\mathrm{Aut}(M)$. Since K is a Lie subgroup of $\mathrm{Aut}(\mathbf{B})$, we have the natural injective continuous group homomorphism $\iota : K \to \mathrm{Aut}(\mathbf{B})$. Considering the composition of the mappings ι and Φ, we obtain an injective continuous group homomorphism $\Phi \circ \iota$ of the connected compact Lie group K of rank n into

$\mathrm{Aut}(M)$. By the Generalized Standardization Theorem, there exists a biholomorphic mapping F of M into $\mathbf{C}^n$ such that $D := F(M)$ is a Reinhardt domain in $\mathbf{C}^n$ and we have

$$F(\Phi \circ \iota)(K)F^{-1} = U(n'_1) \times \cdots \times U(n'_{s'}) \subset FGF^{-1} \subset \mathrm{Aut}(D),$$

where $\sum_{j'=1}^{s'} n'_{j'} = n$. Therefore, noting the conjugacy of the maximal tori $F(\Phi \circ \iota)(T(\mathbf{B}))F^{-1}$ and $T(D)$ in $U(n'_1) \times \cdots \times U(n'_{s'})$, we may assume that M is a Reinhardt domain D in $\mathbf{C}^n$ and we have an isomorphism $\Phi : \mathrm{Aut}(\mathbf{B}) \to G$ between the topological group $\mathrm{Aut}(\mathbf{B})$ and a topological subgroup G of $\mathrm{Aut}(D)$ such that $\Phi(K) = U(n'_1) \times \cdots \times U(n'_{s'})$ and $\Phi(T(\mathbf{B})) = T(D)$. Since the commutator group of $U(n_j)$ is $SU(n_j)$ and every $SU(n_j)$ is a simple Lie group by the assumption $n_j \geq 2$, after a suitable permutation of coordinates, we may assume further that

$$\Phi(K) = U(n_1) \times \cdots \times U(n_s) = K$$

and

$$\Phi(SU(n_j)) = SU(n_j) \quad \text{for every } j = 1, \ldots, s.$$

As a consequence of this, we have $\Phi(L) = L$ and $\Phi(L_j) = L_j$ for every $j = 1, \ldots, s$. Since D is a Stein manifold and invariant under the action of K, it follows that D is a complete Reinhardt domain and, putting $D_j = p_j(D)$ for $j = 1, \ldots, s$, we see that

$$D_j = B_{n_j}(r_j) \quad \text{and} \quad D \subset D_1 \times \cdots \times D_s,$$

where p_j is the projection given by

$$p_j : \mathbf{C}^n = \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_j} \times \cdots \times \mathbf{C}^{n_s} \ni (\mathbf{z}_1, \ldots, \mathbf{z}_j, \ldots, \mathbf{z}_s) \longmapsto \mathbf{z}_j \in \mathbf{C}^{n_j}$$

and $B_{n_j}(r_j)$ denotes the ball in $\mathbf{C}^{n_j}$ with radius $0 < r_j \leq +\infty$.

Given a subgroup Γ of $\mathrm{Aut}(\mathbf{B})$ and a subgroup Λ of G, we denote by $C(\Gamma)$ and $C^*(\Lambda)$ the centralizer of Γ in $\mathrm{Aut}(\mathbf{B})$ and the centralizer of Λ in G, respectively. Also, we write $Z(\Gamma) = [C(\Gamma), C(\Gamma)]$ and $Z^*(\Lambda) = [C^*(\Lambda), C^*(\Lambda)]$, where $[C(\Gamma), C(\Gamma)]$ and $[C^*(\Lambda), C^*(\Lambda)]$ are the commutator groups of $C(\Gamma)$ and $C^*(\Lambda)$, respectively. Then we have

$$\Phi(C(L_j)) = C^*(L_j) \text{ and } \Phi(Z(L_j)) = Z^*(L_j) \text{ for every } j = 1, \ldots, s, \tag{2.1}$$

because $\Phi(\mathrm{Aut}(\mathbf{B})) = G$ and $\Phi(L_j) = L_j$ for every $j = 1, \ldots, s$.

Lemma 2.1. *Let W be a pseudoconvex domain in $\mathbf{C}^n = \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_s}$ that is invariant under the action of $K = U(n_1) \times \cdots \times U(n_s) \subset \mathrm{Aut}(\mathbf{C}^n)$, and assume that $n_j \geq 2$ for every $j = 1, \ldots, s$. Put $W_j = p_j(W)$. Then every element φ of the centralizer $C_{\mathrm{Aut}(W)}(L_j)$ of L_j in $\mathrm{Aut}(W)$ has the form*

$$\begin{aligned} \varphi : W \ni (\mathbf{z}_1, \ldots, \mathbf{z}_s) \longmapsto \\ (\lambda_1(\mathbf{z}_j)\mathbf{z}_1, \ldots, \lambda_{j-1}(\mathbf{z}_j)\mathbf{z}_{j-1}, \varphi_j(\mathbf{z}_j), \lambda_{j+1}(\mathbf{z}_j)\mathbf{z}_{j+1}, \ldots, \lambda_s(\mathbf{z}_j)\mathbf{z}_s) \in W, \end{aligned}$$

where $\varphi_j \in \mathrm{Aut}(W_j)$ and $\lambda_i(\mathbf{z}_j)$, $i = 1, \ldots, j-1, j+1, \ldots, s$, are nowhere-vanishing holomorphic functions on W_j.

Proof. Note first that W is a complete Reinhardt domain. Write

$$\varphi(\mathbf{z}_1,\dots,\mathbf{z}_s) = (\varphi_1(\mathbf{z}_1,\dots,\mathbf{z}_s),\dots,\varphi_s(\mathbf{z}_1,\dots,\mathbf{z}_s)),$$

where $\varphi_i : W \to \mathbf{C}^{n_i},\ i = 1,\dots,s$, are holomorphic mappings. By the assumption that $\varphi \in C_{\mathrm{Aut}(W)}(L_j)$, we have

$$\begin{aligned}
&\varphi_i\left(A_1\mathbf{z}_1,\dots,A_{j-1}\mathbf{z}_{j-1},\mathbf{z}_j,A_{j+1}\mathbf{z}_{j+1},\dots,A_s\mathbf{z}_s\right) = A_i\varphi_i(\mathbf{z}_1,\dots,\mathbf{z}_s),\ i\neq j,\\
&\varphi_j\left(A_1\mathbf{z}_1,\dots,A_{j-1}\mathbf{z}_{j-1},\mathbf{z}_j,A_{j+1}\mathbf{z}_{j+1},\dots,A_s\mathbf{z}_s\right) = \varphi_j(\mathbf{z}_1,\dots,\mathbf{z}_s),
\end{aligned}$$

for every $A_i \in SU(n_i),\ i = 1,\dots,j-1,j+1,\dots,s$. Since $SU(n_i)$ has an orbit $SU(n_i)\cdot\mathbf{z}_i$ of dimension $2n_i - 1$ when $\mathbf{z}_i \neq 0$, it follows that, when $i \neq j$, φ_i depends only on the variables $\mathbf{z}_i$ and $\mathbf{z}_j$, and φ_j depends only on the variable $\mathbf{z}_j$, or that

$$\begin{aligned}
\varphi_i(\mathbf{z}_1,\dots,\mathbf{z}_s) &= \varphi_i(\mathbf{z}_i,\mathbf{z}_j),\ i\neq j,\\
\varphi_j(\mathbf{z}_1,\dots,\mathbf{z}_s) &= \varphi_j(\mathbf{z}_j).
\end{aligned}$$

As a consequence, we see that $\varphi_j \in \mathrm{Aut}(W_j)$.

We show that $\varphi_i(\mathbf{z}_i,\mathbf{z}_j)$ has the form $\varphi_i(\mathbf{z}_i,\mathbf{z}_j) = \lambda_i(\mathbf{z}_j)\mathbf{z}_i$, where $\lambda_i(\mathbf{z}_j)$ is a nowhere-vanishing holomorphic function on W_j. We verify this when $i = 1$. The verification of the case $i \neq 1$ is similar. Fix any point $\mathbf{z}_j^0$ of W_j. Since W is a complete Reinhardt domain and invariant under the action of $U(n_1) \subset K$, we can find a ball $B_{n_1}(r_1)$ in $\mathbf{C}^{n_1}$ such that

$$\{(\mathbf{z}_1,0,\dots,0,\mathbf{z}_j^0,0,\dots,0) \in \mathbf{C}^n \mid \mathbf{z}_1 \in B_{n_1}(r_1)\} \subset W.$$

Set $\tilde\varphi_1(\mathbf{z}_1) = \varphi_1(\mathbf{z}_1,\mathbf{z}_j^0)$. Then $\tilde\varphi_1(\mathbf{z}_1)$ is defined on $B_{n_1}(r_1)$, and we have

$$\begin{aligned}
\varphi(\mathbf{z}_1,0,\dots,0,\mathbf{z}_j^0,&0,\dots,0)\\
&= (\tilde\varphi_1(\mathbf{z}_1),\varphi_2(0,\mathbf{z}_j^0),\dots,\varphi_j(\mathbf{z}_j^0),\dots,\varphi_s(0,\mathbf{z}_j^0)).
\end{aligned}\tag{2.2}$$

Moreover, $\tilde\varphi_1(\mathbf{z}_1)$ satisfies

$$\tilde\varphi_1(A_1\mathbf{z}_1) = A_1\tilde\varphi_1(\mathbf{z}_1)\ \text{ for all } A_1 \in SU(n_1) \text{ and all } \mathbf{z}_1 \in B_{n_1}(r_1).\tag{2.3}$$

By (2.2), $\tilde\varphi_1(\mathbf{z}_1)$ is injective on $B_{n_1}(r_1)$. Indeed, if $\tilde\varphi_1(\mathbf{z}_1') = \tilde\varphi_1(\mathbf{z}_1'')$, then

$$\varphi(\mathbf{z}_1',0,\dots,0,\mathbf{z}_j^0,0,\dots,0) = \varphi(\mathbf{z}_1'',0,\dots,0,\mathbf{z}_j^0,0,\dots,0),$$

which implies that $\mathbf{z}_1' = \mathbf{z}_1''$ by the fact $\varphi \in \mathrm{Aut}(W)$. Since $\tilde\varphi_1(\mathbf{z}_1)$ is injective, it follows from (2.3) that $\tilde\varphi_1(0) = 0$ and $\tilde\varphi_1(\mathbf{z}_1)$ gives a biholomorphic mapping of $B_{n_1}(r_1)$ onto a ball in $\mathbf{C}^{n_1}$. Therefore $\tilde\varphi_1(\mathbf{z}_1)$ is a linear mapping. Again by (2.3), the matrix expression of $\tilde\varphi_1(\mathbf{z}_1)$ must be a scalar matrix. This shows that $\varphi_i(\mathbf{z}_i,\mathbf{z}_j)$ has the desired form. □

Applying Lemma 2.1 to $W = \mathbf{B}$, we obtain the following lemma immediately.

Lemma 2.2. *The groups $C(K_j)$ and $C(L_j)$ coincide, and they consist of all transformations of the form*

$$\mathbf{B} \ni (\mathbf{z}_1, \dots, \mathbf{z}_s) \longmapsto (\lambda_1 \mathbf{z}_1, \dots, \lambda_{j-1}\mathbf{z}_{j-1}, \varphi_j(\mathbf{z}_j), \lambda_{j+1}\mathbf{z}_{j+1}, \dots, \lambda_s \mathbf{z}_s) \in \mathbf{B},$$

where $\varphi_j \in \mathrm{Aut}(B_{n_j})$ and $\lambda_i \in U(1)$, $i = 1, \dots, j-1, j+1, \dots, s$. Consequently, the groups $Z(K_j)$ and $Z(L_j)$ coincide, and they consist of all transformations of the form

$$\mathbf{B} \ni (\mathbf{z}_1, \dots, \mathbf{z}_s) \longmapsto (\mathbf{z}_1, \dots, \mathbf{z}_{j-1}, \varphi_j(\mathbf{z}_j), \mathbf{z}_{j+1}, \dots, \mathbf{z}_s) \in \mathbf{B},$$

where $\varphi_j \in \mathrm{Aut}(B_{n_j})$. In particular, $Z(K_j)$ and $Z(L_j)$ are naturally identified with $\mathrm{Aut}(B_{n_j})$, and $Z^*(L_j) = \Phi(Z(L_j))$ is isomorphic to $\mathrm{Aut}(B_{n_j})$.

As previously stated, Φ maps $SU(n_j)$ onto $SU(n_j)$ for every $j = 1, \dots, s$. Using Lemmas 2.1 and 2.2, we can show a stronger fact.

Lemma 2.3. *The isomorphism Φ maps $U(n_j)$ onto $U(n_j)$ for every $j = 1, \dots, s$.*

Proof. To prove that $\Phi(U(n_j)) = U(n_j)$, put

$$\mathbf{T}_j = \left\{ \left(E_{n_1}, \dots, E_{n_{j-1}}, \alpha E_{n_j}, E_{n_{j+1}}, \dots, E_{n_s}\right) \in K \;\middle|\; \alpha \in U(1) \right\}.$$

Since $U(n_j) = \{\alpha A_j \,|\, \alpha \in U(1), A_j \in SU(n_j)\}$, it suffices to show that $\Phi(\mathbf{T}_j) \subset U(n_j)$. By the fact that $\Phi(T(\mathbf{B})) = T(D)$, the group $\Phi(\mathbf{T}_j)$ is a 1-dimensional subtorus of $T(D)$. Therefore $\Phi(\mathbf{T}_j)$ can be written as

$$\Phi(\mathbf{T}_j) = \{ (\Lambda_{n_1}(\alpha^{a_{11}}, \dots, \alpha^{a_{1n_1}}), \cdots, \Lambda_{n_s}(\alpha^{a_{s1}}, \dots, \alpha^{a_{sn_s}})) \in K \mid \alpha \in U(1) \},$$

where $\Lambda_{n_i}(\alpha^{a_{i1}}, \dots, \alpha^{a_{in_i}})$ is the diagonal matrix of degree n_i with diagonal entries $(\alpha^{a_{i1}}, \dots, \alpha^{a_{in_i}})$, $i = 1, \dots, s$, and a_{ik_i}, $i = 1, \dots, s$, $k_i = 1, \dots, n_i$, are integers. Since $\mathbf{T}_j$ is contained in the center of K and $\Phi(K) = K$, the group $\Phi(\mathbf{T}_j)$ is also contained in the center of K, which implies that, for every $i = 1, \dots, s$,

$$\Lambda_{n_i}(\alpha^{a_{i1}}, \dots, \alpha^{a_{in_i}}) A_i = A_i \Lambda_{n_i}(\alpha^{a_{i1}}, \dots, \alpha^{a_{in_i}}) \quad \text{for all } A_i \in U(n_i) \text{ and all } \alpha \in U(1).$$

It follows from this fact that, for every $i = 1, \dots, s$, we have $a_{i1} = \cdots = a_{in_i}$, and hence, denoting this value by a_i,

$$\Phi(\mathbf{T}_j) = \{ (\alpha^{a_1} E_{n_1}, \dots, \alpha^{a_s} E_{n_s}) \in K \mid \alpha \in U(1) \}.$$

To show that $\Phi(\mathbf{T}_j) \subset U(n_j)$, we have only to show that $a_i = 0$ for all $i \neq j$.

Suppose contrarily that $a_i \neq 0$ for some $i \neq j$. In the following, consider the case of $i = 1$. The argment for the case of $i \neq 1$ is similar. Take any element φ of $C^*(L_1)$. Applying Lemma 2.1 to $W = D$, we see that φ has the form

$$\varphi : D \ni (\mathbf{z}_1, \dots, \mathbf{z}_s) \longmapsto (\varphi_1(\mathbf{z}_1), \lambda_2(\mathbf{z}_1)\mathbf{z}_2, \dots, \lambda_s(\mathbf{z}_1)\mathbf{z}_s) \in D, \tag{2.4}$$

where $\varphi_1 \in \mathrm{Aut}(D_1)$ and $\lambda_i(\mathbf{z}_1)$, $i = 2, \ldots, s$, are nowhere-vanishing holomorphic functions on D_1. On the other hand, since $j \neq i = 1$, it follows from Lemma 2.2 that, for every $\psi \in C(L_1)$, we have $\tau \circ \psi = \psi \circ \tau$ for all $\tau \in \mathbf{T}_j$. Therefore, by (2.1), we have

$$\tau^* \circ \varphi = \varphi \circ \tau^* \quad \text{for all } \tau^* \in \Phi(\mathbf{T}_j). \tag{2.5}$$

Using (2.4) and (2.5), we see that, for every $i = 2, \ldots, s$,

$$\lambda_i(\alpha^{a_1}\mathbf{z}_1) = \lambda_i(\mathbf{z}_1) \quad \text{for all } \alpha \in U(1) \text{ and all } \mathbf{z}_1 \in D_1, \tag{2.6}$$

and that

$$\varphi_1(\alpha^{a_1}\mathbf{z}_1) = \alpha^{a_1}\varphi_1(\mathbf{z}_1) \quad \text{for all } \alpha \in U(1) \text{ and all } \mathbf{z}_1 \in D_1. \tag{2.7}$$

(2.6) shows that each λ_i is a constant function. Indeed, write $\mathbf{z}_1 = (z_{11}, \ldots, z_{1n_1})$, where $z_{11}, \ldots, z_{1n_1}$ are coordinate functions on $\mathbf{C}^{n_1}$. Then, by noting that D_1 is a complete Reinhardt domain, λ_i is represented as the Taylor series

$$\lambda_i(\mathbf{z}_1) = \sum_{\mu_1 \geq 0, \ldots, \mu_{n_1} \geq 0} c_\mu \mathbf{z}_1^\mu \quad \text{on } D_1,$$

where $\mu = (\mu_1, \ldots, \mu_{n_1})$ and $\mathbf{z}_1^\mu = z_{11}^{\mu_1} \cdots z_{1n_1}^{\mu_{n_1}}$. Therefore we see from (2.6) that

$$\sum_{\mu_1 \geq 0, \ldots, \mu_{n_1} \geq 0} c_\mu \alpha^{a_1(\mu_1 + \cdots + \mu_{n_1})} \mathbf{z}_1^\mu = \sum_{\mu_1 \geq 0, \ldots, \mu_{n_1} \geq 0} c_\mu \mathbf{z}_1^\mu \quad \text{for all } \alpha \in U(1).$$

If $c_\mu \neq 0$, then $\alpha^{a_1(\mu_1 + \cdots + \mu_{n_1})} = 1$ for all $\alpha \in U(1)$, which implies that $a_1(\mu_1 + \cdots + \mu_{n_1}) = 0$. Since $a_1 \neq 0$ and $\mu_1 \geq 0, \ldots, \mu_{n_1} \geq 0$, it follows that $\mu_1 = \cdots = \mu_{n_1} = 0$. This shows that λ_i is a constant function. By utilizing a similar argument for φ_1, (2.7) shows that φ_1 is given by a linear transformation $\varphi_1(\mathbf{z}_1) = A_1\mathbf{z}_1$, where $A_1 \in GL(n_1, \mathbf{C})$.

The result of the preceding paragraph yields that there exists a Lie subgroup G_1 of $GL(n_1, \mathbf{C})$ such that $Z^*(L_1) = [C^*(L_1), C^*(L_1)]$ consists of all transformations of the form

$$D \ni (\mathbf{z}_1, \ldots, \mathbf{z}_s) \longmapsto (A_1\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_s) \in D, \tag{2.8}$$

where $A_1 \in G_1$. Therefore $Z^*(L_1)$ can be regarded as a subgroup of $GL(n_1, \mathbf{C})$. This contradicts the Key Fact stated in Section 1, because $GL(n_1, \mathbf{C}) \subset \mathrm{Aut}(\mathbf{C}^{n_1})$ and $Z^*(L_1)$ is isomorphic to $\mathrm{Aut}(B_{n_1})$. $\square$

By Lemma 2.3, the restriction of Φ to $U(n_j)$ induces a Lie group automorphism of the Lie group $U(n_j)$ for every $j = 1, \ldots, s$. It follows from this fact that there exists an element σ_j of $U(n_j)$ such that

$$\Phi(A_j) = \sigma_j A_j \sigma_j^{-1} \quad \text{for all } A_j \in U(n_j)$$

or

$$\Phi(A_j) = \sigma_j \bar{A}_j \sigma_j^{-1} \quad \text{for all } A_j \in U(n_j),$$

where $\bar{A}_j$ denotes the complex conjugate of A_j. Consider the automorphism S of $\mathbf{C}^n = \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_s}$ given by

$$S : \mathbf{C}^n \ni (\mathbf{z}_1, \ldots, \mathbf{z}_s) \longmapsto (\sigma_1^{-1}\mathbf{z}_1, \ldots, \sigma_s^{-1}\mathbf{z}_s) \in \mathbf{C}^n,$$

and let Ψ be the topological group isomorphism of $\mathrm{Aut}(D)$ onto $\mathrm{Aut}(S(D))$ sending an element φ to $S \circ \varphi \circ S^{-1}$. Then $S(D)$ is a domain in $\mathbf{C}^n$ that is bihomorphic to D and invariant under the action of K. Moreover, $\Psi \circ \Phi$ is an isomorphism between the topological groups $\mathrm{Aut}(\mathbf{B})$ and $\Psi(G) \subset \mathrm{Aut}(S(D))$ such that $\Psi \circ \Phi$ maps K onto K by the rule that

$$K \ni (A_1, \ldots, A_s) \longmapsto (\tilde{A}_1, \ldots, \tilde{A}_s) \in K, \tag{2.9}$$

where $\tilde{A}_j = A_j$ or $\tilde{A}_j = \bar{A}_j$ for each $j = 1, \ldots, s$. Therefore, by replacing Φ by $\Psi \circ \Phi$ and D by $S(D)$ if necessary, we may assume from the beginning that Φ satisfies (2.9). Since $\Phi(K_j) = K_j$ by (2.9), we have

$$Z^*(K_j) = \Phi(Z(K_j)) = \Phi(Z(L_j)) = Z^*(L_j) = \Phi(\mathrm{Aut}(B_{n_j})).$$

Lemma 2.4. *For every $j = 1, \ldots, s$, the orbit $Z^*(L_j) \cdot O$ of $Z^*(L_j)$ through the origin O of $\mathbf{C}^n$ does not coincide with the origin O itself.*

Proof. Consider the case of $j = 1$. The proof for the case of $j \neq 1$ is similar. By regarding $\mathrm{Aut}(B_{n_1})$ naturally as a subgroup of $\mathrm{Aut}(\mathbf{B})$ and denoting by $\mathbf{T}^{s-1}$ the subgroup of $\mathrm{Aut}(\mathbf{B})$ consisting of all transformations of the form

$$\mathbf{B} \ni (\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_s) \longmapsto (\mathbf{z}_1, \lambda_2\mathbf{z}_2, \ldots, \lambda_s\mathbf{z}_s) \in \mathbf{B},$$

where $\lambda_i \in U(1)$, $i = 2, \ldots, s$, Lemma 2.2 shows that

$$\begin{aligned} &C(L_1) = \mathrm{Aut}(B_{n_1}) \cdot \mathbf{T}^{s-1} = \mathbf{T}^{s-1} \cdot \mathrm{Aut}(B_{n_1}) \quad \text{(direct product)}, \\ &Z(L_1) = [C(L_1), C(L_1)] = \mathrm{Aut}(B_{n_1}). \end{aligned}$$

Since $\Phi(\mathbf{T}^{s-1}) = \mathbf{T}^{s-1}$ by (2.9), it follows that

$$\begin{aligned} C^*(L_1) &= \Phi(\mathrm{Aut}(B_{n_1})) \cdot \mathbf{T}^{s-1} \\ &= \mathbf{T}^{s-1} \cdot \Phi(\mathrm{Aut}(B_{n_1})) \quad \text{(direct product)}, \end{aligned} \tag{2.10}$$

$$Z^*(L_1) = \Phi(Z(L_1)) = \Phi(\mathrm{Aut}(B_{n_1})). \tag{2.11}$$

Let Γ be the subgroup of K given by

$$\Gamma = \begin{pmatrix} 1 & O \\ 0 & U(n_1 - 1) \end{pmatrix} \times U(n_2) \times \cdots \times U(n_s).$$

Then, by (2.9), we have $\Phi(\Gamma) = \Gamma$ and $\Gamma \supset L_1$, and therefore

$$\Phi(C(\Gamma)) = C^*(\Gamma) \subset C^*(L_1), \tag{2.12}$$

$$\Phi(Z(\Gamma)) = Z^*(\Gamma) \subset Z^*(L_1). \tag{2.13}$$

Now suppose contrarily that $Z^*(L_1) \cdot O = O$. Since

$$O = Z^*(L_1) \cdot O = \Phi(\mathrm{Aut}(B_{n_1})) \cdot O = C^*(L_1) \cdot O$$

by (2.10) and (2.11), we see from (2.12) that

$$C^*(\Gamma) \cdot O = O. \tag{2.14}$$

In view of (2.13), we will derive a contradiction by showing that the group $Z(\Gamma) = [C(\Gamma), C(\Gamma)]$ is non-abelian, while the group $Z^*(\Gamma) = [C^*(\Gamma), C^*(\Gamma)]$ is abelian. Write

$$\mathbf{z}_1 = (z_{11}, z_{12}, \ldots, z_{1n_1}) = (u, v_1, \ldots, v_{n_1-1}) = (u, v),$$
$$\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_s) = (u, v, \mathbf{z}_2, \ldots, \mathbf{z}_s).$$

Let φ be any element of $C(\Gamma)$. A similar argument used in the proof of Lemma 2.1 yields that φ has the form

$$\varphi : \mathbf{B} \ni (u, v, \mathbf{z}_2, \ldots, \mathbf{z}_s) \longmapsto (\varphi_1(u), \alpha_1(u)v, \alpha_2(u)\mathbf{z}_2, \ldots, \alpha_s(u)\mathbf{z}_s) \in \mathbf{B},$$

where $\varphi_1 \in \mathrm{Aut}(B_1)$ and $\alpha_i(u)$, $i = 1, \ldots, s$, are nowhere-vanishing holomorphic functions on B_1. Since every element φ_1 of $\mathrm{Aut}(B_1)$ extends to an element $\tilde{\varphi}_1$ of $\mathrm{Aut}(B_{n_1})$ written in the form $\tilde{\varphi}_1(u, v) = (\varphi_1(u), \alpha_1(u)v)$ for $(u, v) \in B_{n_1}$, the mapping ρ sending φ to φ_1 is a group homomorphism of $C(\Gamma)$ onto $\mathrm{Aut}(B_1)$. Therefore we have

$$\rho(Z(\Gamma)) = [\mathrm{Aut}(B_1), \mathrm{Aut}(B_1)] = \mathrm{Aut}(B_1),$$

because $\mathrm{Aut}(B_1)$ is a simple Lie group. If $Z(\Gamma)$ is abelian, then $\mathrm{Aut}(B_1)$ must be abelian, which is a contradiction. We thus conclude that $Z(\Gamma)$ is non-abelian. It remains to show that $Z^*(\Gamma)$ is abelian. Let φ be any element of $C^*(\Gamma)$. Then, as is seen above, φ has the form

$$\begin{aligned}\varphi : D \ni (u, v, \mathbf{z}_2, \ldots, \mathbf{z}_s)& \\ &\longmapsto (\varphi_1(u), \alpha_1(u)v, \alpha_2(u)\mathbf{z}_2, \ldots, \alpha_s(u)\mathbf{z}_s) \in D,\end{aligned} \tag{2.15}$$

where $\varphi_1 \in \mathrm{Aut}(B_1(r))$ and $\alpha_i(u)$, $i = 1, \ldots, s$, are nowhere-vanishing holomorphic functions on $B_1(r)$. Note that $0 < r \leq +\infty$. Since $\varphi(0) = 0$ by (2.14), we have $\varphi_1(0) = 0$, which implies that φ_1 has the form

$$\varphi_1(u) = \alpha u \quad \text{for } u \in B_1(r), \tag{2.16}$$

where α is a non-zero complex constant. It follows from (2.15) and (2.16) that φ^{-1} is given by

$$\varphi^{-1}(u, v, \mathbf{z}_2, \ldots, \mathbf{z}_s) = \left(\frac{u}{\alpha}, \left(\alpha_1\left(\frac{u}{\alpha}\right)\right)^{-1} v, \left(\alpha_2\left(\frac{u}{\alpha}\right)\right)^{-1} \mathbf{z}_2, \ldots, \left(\alpha_s\left(\frac{u}{\alpha}\right)\right)^{-1} \mathbf{z}_s\right).$$

Therefore, if φ and ψ are any elements of $C^*(\Gamma)$ and if we write

$$\varphi(z) = (\alpha u, \alpha_1(u)v, \alpha_2(u)\mathbf{z}_2, \ldots, \alpha_s(u)\mathbf{z}_s)$$

and

$$\psi(z) = (\beta u, \beta_1(u)v, \beta_2(u)\mathbf{z}_2, \ldots, \beta_s(u)\mathbf{z}_s),$$

then $[\psi, \varphi] = \psi^{-1} \circ \varphi^{-1} \circ \psi \circ \varphi$ is given by

$$[\psi, \varphi](z) = \left(u, \frac{\beta_1(\alpha u)\alpha_1(u)}{\beta_1(u)\alpha_1(\beta u)} v, \frac{\beta_2(\alpha u)\alpha_2(u)}{\beta_2(u)\alpha_2(\beta u)} \mathbf{z}_2, \ldots, \frac{\beta_s(\alpha u)\alpha_s(u)}{\beta_s(u)\alpha_s(\beta u)} \mathbf{z}_s\right).$$

This shows that $Z^*(\Gamma)$ is abelian, and the proof of Lemma 2.4 is completed. $\square$

Lemma 2.4 has the following consequence.

Lemma 2.5. *For $j = 1, \ldots, s$, every element φ of $Z^*(L_j)$ has the form*

$$\varphi : D \ni (\mathbf{z}_1, \ldots, \mathbf{z}_s) \longmapsto (\mathbf{z}_1, \ldots, \mathbf{z}_{j-1}, \varphi_j(\mathbf{z}_j), \mathbf{z}_{j+1}, \ldots, \mathbf{z}_s) \in D,$$

where $\varphi_j \in \mathrm{Aut}(D_j)$.

Proof. Let φ be any element of $Z^*(L_j)$. By Lemma 2.1, φ has the form

$$\varphi : D \ni (\mathbf{z}_1, \ldots, \mathbf{z}_s) \longmapsto (\lambda_1(\mathbf{z}_j)\mathbf{z}_1, \ldots, \lambda_{j-1}(\mathbf{z}_j)\mathbf{z}_{j-1}, \varphi_j(\mathbf{z}_j), \lambda_{j+1}(\mathbf{z}_j)\mathbf{z}_{j+1}, \ldots, \lambda_s(\mathbf{z}_j)\mathbf{z}_s) \in D,$$

where $\varphi_j \in \mathrm{Aut}(D_j)$ and $\lambda_i(\mathbf{z}_j)$, $i = 1, \ldots, j-1, j+1, \ldots, s$, are nowhere-vanishing holomorphic functions on D_j. It suffices to show that each $\lambda_i(\mathbf{z}_j)$ is equal to 1 on D_j. Take any element ψ of $Z^*(L_i)$. Again by Lemma 2.1, ψ has the form

$$\psi : D \ni (\mathbf{z}_1, \ldots, \mathbf{z}_s) \longmapsto (\mu_1(\mathbf{z}_i)\mathbf{z}_1, \ldots, \mu_{i-1}(\mathbf{z}_i)\mathbf{z}_{i-1}, \psi_i(\mathbf{z}_i), \mu_{i+1}(\mathbf{z}_i)\mathbf{z}_{i+1}, \ldots, \mu_s(\mathbf{z}_i)\mathbf{z}_s) \in D,$$

where $\psi_i \in \mathrm{Aut}(D_i)$ and $\mu_k(\mathbf{z}_i)$, $k = 1, \ldots, i-1, i+1, \ldots, s$, are nowhere-vanishing holomorphic functions on D_i. Since $i \neq j$, Lemma 2.2 shows that $f \circ g = g \circ f$ for all $f \in Z(L_i)$ and all $g \in Z(L_j)$. Therefore we have $\varphi \circ \psi = \psi \circ \varphi$. Comparing the i-th components of $\varphi \circ \psi$ and $\psi \circ \varphi$, we see that

$$\lambda_i(\mu_j(\mathbf{z}_i)\mathbf{z}_j)\psi_i(\mathbf{z}_i) = \psi_i(\lambda_i(\mathbf{z}_j)\mathbf{z}_i) \quad \text{for all } (\mathbf{z}_1, \ldots, \mathbf{z}_s) \in D.$$

By putting $\mathbf{z}_i = 0$, it follows that

$$\lambda_i(\mu_j(0)\mathbf{z}_j)\psi_i(0) = \psi_i(0) \quad \text{for all } \mathbf{z}_j \in D_j. \tag{2.17}$$

Now, by Lemma 2.4, we can take an element ψ of $Z^*(L_i)$ such that $O \neq \psi(O) = (0, \dots, 0, \psi_i(0), 0, \dots, 0)$. An application of (2.17) to this ψ yields that

$$\lambda_i(\mu_j(0)\mathbf{z}_j) = 1 \quad \text{for all } \mathbf{z}_j \in D_j,$$

because $\psi_i(0) \neq 0$. By noting that $\mu_j(0) \neq 0$, this implies that $\lambda_i(\mathbf{z}_j)$ is equal to 1 on D_j, and the proof of Lemma 2.5 is completed. $\square$

We are now in a position to complete the proof of our theorem. By Lemma 2.5, we have the mapping τ of $Z^*(L_j)$ into $\mathrm{Aut}(D_j)$ sending φ to φ_j. Since $Z^*(L_j)$ is isomorphic to $\mathrm{Aut}(B_{n_j})$ and since $D_j = B_{n_j}(r_j)$, it follows that τ gives an injective continuous group homomorphism of $\mathrm{Aut}(B_{n_j})$ into $\mathrm{Aut}(B_{n_j}(r_j))$. If $r_j = +\infty$, or $B_{n_j}(r_j) = \mathbf{C}^{n_j}$, then $\tau(Z^*(L_j))$ is a Lie group in $\mathrm{Aut}(\mathbf{C}^{n_j})$ that is isomorphic to $\mathrm{Aut}(B_{n_j})$, which contradicts the Key Fact stated in Section 1. Therefore we have $r_j < +\infty$. This implies that

$$\tau(Z^*(L_j)) = \mathrm{Aut}(B_{n_j}(r_j)), \tag{2.18}$$

because $\mathrm{Aut}(B_{n_j})$ and $\mathrm{Aut}(B_{n_j}(r_j))$ are connected and have the same dimension. Using (2.18), we see that

$$\begin{aligned} B_{n_1}(r_1) \times \cdots \times B_{n_s}(r_s) &= Z^*(L_1) \cdots Z^*(L_s) \cdot O \\ &\subset D \subset D_1 \times \cdots \times D_s = B_{n_1}(r_1) \times \cdots \times B_{n_s}(r_s), \end{aligned}$$

which shows that $D = B_{n_1}(r_1) \times \cdots \times B_{n_s}(r_s)$. We thus conclude that D is biholomorphic to $\mathbf{B} = B_{n_1} \times \cdots \times B_{n_s}$, and the proof of our theorem is completed.

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Division of Mathematical and Physical Sciences
Graduate School of Natural Science and Technology
Kanazawa University, Kanazawa, 920-1192
Japan
e-mail: kodama@kenroku.kanazawa-u.ac.jp

Mathematical Institute
Tohoku University
Sendai, 980-8578
Japan
e-mail: shimizu@math.tohoku.ac.jp

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