A note on strong unique continuation for normal elliptic systems with Gevrey coefficients

By

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Abstract

In this paper, we consider the strong unique continuation for normal elliptic systems whose coefficients are Gevrey class. By using Lerner’s lemma, we prove the Carleman estimate with some weight function.

1. Introduction

Let $\Omega$ be an open neighbourhood of the origin in $\mathbb{R}^2$. If $v \in C^\infty(\Omega)$ satisfies

\[ \partial^\alpha v(0) = 0, \quad \forall \alpha \in \mathbb{Z}_{\geq 0}^2 \]

then we say that $u$ is flat at the origin, and $C^\infty(\Omega)$ is the space of functions in $C^\infty(\Omega)$ which are flat at the origin.

We suppose that

\[ P(x, \partial) = \partial_1 + N(x)\partial_2 + R(x) \]

is an elliptic differential operator in $\Omega$ where $N$ and $R$ are $l \times l$ matrix valued functions with the entries which are in $C^\infty(\Omega)$. For a differential operator $P$, we shall adopt the following

Definition 1.1. We say that $P$ has the strong unique continuation property at the origin if whatever $Pu = 0$ in $\Omega$ and $u \in C^\infty_\flat(\Omega; \mathbb{C}^l)$, then $u \equiv 0$ in a neighbourhood of the origin.

In [1], Ōkaji proved a result of strong unique continuation property for a class of elliptic systems of normal type in two independent variables. His result is, roughly speaking, that if $N$ is a normal elliptic matrix, and if there exists some complex number $\zeta$ such that

\[ \text{all the eigenvalues of } N(0) \text{ are equal to } \zeta \text{ or } \bar{\zeta}, \]

then $P$ has the strong unique continuation property at the origin.
In this paper we are interested in the strong unique continuation property for elliptic systems of normal type which do not satisfy the assumption of (1.2). As in [2], [4], studying the case in which the coefficients of $P$ are in Gevrey class of appropriate index, we prove the strong unique continuation for elliptic systems of normal type which do not satisfy the assumption in (1.2).

The proof is based on Carleman's method, and is based on the technique used in [2]. As in [1], there is no regularity assumptions on the eigenvalues of $N$. Therefore we can’t use a usual smooth diagonalization approach, so to prove the Carleman inequality for $P(x, \partial)$ with some weight function, we use the result proved in [5] which is an extension of Hörmander’s pseudo convex condition in [3] to systems of partial differential operator. This method can make us compute the Gevrey index of the coefficients of $P$ for which we can prove a Carleman estimate.

2. Statement of result

Let $G^s(\Omega)$ denote the set of all functions of Gevrey class of order $s > 1$. We set $P(x, \partial) = \partial_1 + N(x)\partial_2 + R(x)$ where $N, R$ are $l \times l$ matrices whose entries are in $G^s(\Omega)$. Moreover we shall assume that

1. $N(x)$ is normal for any $x \in \Omega$, that is,

\begin{equation}
N(x)^* N(x) = N(x) N(x)^* \quad x \in \Omega,
\end{equation}

where $N(x)^*$ is the conjugate transposed matrix of $N(x)$.

2. $P$ is elliptic in $\Omega$, that is,

\begin{equation}
\det P_1(x, \xi) \neq 0, \quad x \in \Omega, \quad \xi \in \mathbb{R}^2 \setminus \{0\},
\end{equation}

where $P_1(x, \xi)$ is the principal symbol of $P$, that is,

\begin{equation}
P_1(x, \xi) = \xi_1 I_l + \xi_2 N(x),
\end{equation}

where $I_l$ is the $l \times l$ unit matrix.

**Remark 1.** The ellipticity of $P$ is satisfied if and only if

\begin{equation}
\text{Im} \lambda_j(x) \neq 0 \quad j = 1, 2, \ldots, l, \quad x \in \Omega,
\end{equation}

where $\lambda_j(x)$ are the eigenvalues of $N(x)$.

Our main theorem is the following;

**Theorem 2.1.** Let $P(x, \partial) = \partial_1 + N(x)\partial_2 + R(x)$ be a differential operator with $N$ and $R$ in $G^s(\Omega; M_l(\mathbb{C}))$ which satisfies (2.1), (2.2). Set $\nu = \frac{1}{2s}$, and $\lambda_j = \lambda_j(0)$ where $\lambda_j(x)$, $j = 1, 2, \ldots, l$ are the eigenvalues of $N(x)$. If the quadratic polynomials in $z = (z_1, z_2) \in \mathbb{R}^2$

\begin{equation}
q_j(\nu, z) = (\nu + 1 - |\lambda_j|^2)z_1^2 + 2(\nu + 1)(\text{Re} \lambda_j)z_1 z_2 + \{(\nu + 1)|\lambda_j|^2 - 1\}z_2^2
\end{equation}

are positive definite for all $j = 1, 2, \ldots, l$, then $P$ has the strong unique continuation property at the origin.
Remark 2. If $\text{Re} \lambda_j = 0$, $j = 1, 2, \ldots, l$, the condition of $q_j$ is equivalent to the following:

\[ s < 1 + \frac{1}{\max_{1 \leq j \leq l} (|\lambda_j|^2, |\lambda_j|^{-2}) - 1}. \]

If $N(x)$ satisfies Okaji’s assumption (1.2), by the change of variables we may assume that

\[ \lambda_j = i \text{ or } -i \quad \forall j. \]

Then for any Gevrey index $s$, our theorem holds.

3. Proof

We begin with the key lemma for the choice of the weight function used in Carleman estimate for $P$.

Lemma 3.1 ([4]). If $u \in G^s(\Omega)$ is flat at zero, then there exists a function $v \in C^\infty(\Omega)$ flat at zero, such that

\[ u = \exp(-r^{-\nu})v, \quad r = |x| \]

provided $1 + \nu^{-1} > s$.

By this lemma and by Gevrey hypoellipticity of $P$, if $u \in C^\infty(\Omega; \mathbb{C}^l)$ is solution to

\[ Pu = 0 \quad \text{in } \Omega, \]

then $u$ satisfies

\[ \partial^\alpha u(x) = o(\exp(r^{-\nu})) \quad r \to 0, \quad \forall \alpha, \]

so we can use the weight function $\exp(r^{-\nu})$ ($0 < \alpha < \nu$) in the Carleman estimate of $P$. We now state the Carleman estimate for the operator $P$ with respect to the singular weight

\[ \varphi(x) = r^{-\alpha} \]

with $\nu_0 < \alpha < \nu$ where $\nu_0 = \inf\{\nu : q_j(\nu, x) \text{ is positive definite for all } j\}$. We denote the $L^2$-norm by $\| \cdot \|$.

Proposition 3.1. There exist some constants $\tau_0 > 0$, $\varepsilon > 0$ such that for $\tau > \tau_0$ we have the estimate

\[ \tau \| e^{\varphi} u \|^2 \leq C \| e^{\varphi} P(x, \partial) u \|^2, \quad u \in C^\infty_0 (B_\varepsilon(0) \setminus \{0\}), \]

where $C$ is the positive constant depending only on $P$. 

Using this proposition, Theorem 2.1 can be proved by a standard manner. In fact, let \( \chi \) be a smooth function supported in \( B_{\epsilon}(0) \) and equal to 1 in \( B_{\epsilon}^c(0) \) and let us put, for \( j \in \mathbb{N} \), \( \theta_j(x) = \theta(jx) \), where \( \theta \in C^\infty(\mathbb{R}^2) \), is such that

\[
\theta(x) = \begin{cases} 
0 & \text{ if } |x| \leq \frac{1}{2} \\
1 & \text{ if } |x| \geq 1.
\end{cases}
\]

Let now \( u \in C^\infty_c(\Omega) \) be a solution to

\[
Pu = 0 \quad \text{in } \Omega.
\]

Then we apply the Proposition 3.1 to \( \theta_j \chi u \), and by (3.2), when \( j \to \infty \), we have

\[
\tau^{\frac{1}{2}} \| e^{\tau \phi} \chi u \| \leq C \| e^{\tau \phi} P(x, \partial)(\chi u) \|.
\]

Moreover since \( u \) is a solution to \( Pu = 0 \), we have

\[
P(x, \partial)(\chi u) = [P(x, \partial), \chi]u,
\]

where the commutator is supported in \( \{ x \in \mathbb{R}^2 : \frac{1}{2} < |x| < \epsilon \} \). Therefore we have the inequality

\[
\tau^{\frac{1}{2}} \| e^{\tau \phi} \chi u \| \leq C e^{\tau \phi(\xi)} \| [P(x, \partial), \chi]u \|.
\]

When \( \tau \to \infty \), this shows that \( u = 0 \) in \( B_{\frac{1}{2}}^c(0) \), we have thus completed the proof of Theorem 2.1. Hence it remains to prove Proposition 3.1. For that purpose, we shall need the following Theorem which is the special case of Theorem 4.1.

Let \( \Omega_0 \) be a bounded open subset in \( \mathbb{R}^2 \) with \( \Omega_0 \subset \Omega \). We suppose that \( Q(x, \partial) = \partial_1 + N(x) \partial_2 + R(x) \) is a partial differential operator with \( N \) and \( R \) in \( C^\infty(\Omega_0; M_1(\mathbb{C})) \). Set \( Q_1(x, \xi) = \xi_1 I_1 + \xi_2 N(x) \).

**Theorem 3.1.** Let \( \psi \in C^\infty(\Omega_0; \mathbb{R}) \) with \( \psi'(x) \neq 0 \) on \( \Omega_0 \). If \( Q_1(x, \xi) \) satisfies

1. \( \det Q_1(x, \xi) \neq 0, \quad (x, \xi) \in \Omega_0 \times (\mathbb{R}^2 \setminus \{0\}) \).
2. For any \( (x, \xi) \in \Omega_0 \times (\mathbb{R}^2) \), the Hermitian matrix

\[
\frac{1}{2\epsilon} \| [Q_{1, \psi}, Q_{1, \psi}] - \{Q_{1, \psi}, Q_{1, \psi}\} \| (x, \xi)
\]

is positive definite on \( \ker Q_{1, \psi}(x, \xi) \), where

\[
Q_{1, \psi}(x, \xi) = Q_1(x, \xi + i\psi'(x))
\]

and \( \{ , \} \) denotes the Poisson bracket for matrix valued functions.

3. \( N(x)N(x)^* = N(x)^* N(x) \), \( x \in \Omega_0 \), then there exists \( \tau_0 > 0 \) such that for any \( \tau > \tau_0 \) we have

\[
\sum_{|\gamma| \leq 1} \tau^{2(1-|\gamma|)-1} \| e^{\tau \phi} \partial^\gamma v \|^2 \leq C \| e^{\tau \phi} Q(x, \partial) v \|^2, \quad v \in C^\infty_c(\overline{\Omega_0}; \mathbb{C}^l),
\]

where \( C \) is a positive constant independent of \( v, \tau \).
However, because the weight function \( \varphi \) is singular at 0, Theorem 3.1 can’t be applicable to the proof of Proposition 3.1 directly. So following [2], we use a dyadic decomposition for \( \mathbb{R}^2 \),

\[(3.11) \quad \bigcup_{k \in \mathbb{Z}} A_k = \mathbb{R}^2 \setminus \{0\},\]

where

\[(3.12) \quad A_k = \{2^{k-1} < |x| < 2^{k+1}\}.
\]

And we take the smooth partition of unity

\[(3.13) \quad 1 = \sum_{k \in \mathbb{Z}} \chi_k \text{ on } \mathbb{R}^2 \setminus \{0\}\]

with \( \chi_k \in C_0^\infty (A_k) \).

In order to prove Proposition 3.1 we shall derive a Carleman estimate on \( A_k \). Then we use the following lemma.

**Lemma 3.2.** There exists some negative integer \( k_0 \) such that for any negative integer \( k < k_0 \) and any \( \tau > 1 \) we have

\[(3.14) \quad \sum_{|\gamma| \leq 1} |\gamma|^{2n}|v^{\sigma \varphi \partial^\gamma v}|^2 \leq C\|e^{\sigma \varphi} P_1 (2^k y, \partial) v\|^2, \quad v \in C_0^\infty (A_0),\]

where \( \sigma = \tau 2^{-\alpha k} \) and \( C \) is a constant independent of \( v, k, \tau \).

To prove this lemma, the following lemma is useful.

**Lemma 3.3.** Suppose that \( \varphi \) and \( P_1 (\partial) = P_1 (0, \partial) \) satisfy the conditions of Theorem 3.1 in \( \overline{A_0} \). Then there exists some \( \varepsilon_0 > 0 \) such that if \( 0 < \varepsilon < \varepsilon_0 \), \( \varphi \) and \( P (\varepsilon x, D) \) don’t satisfy the condition 2 in Theorem 3.1. When we define

\[(3.15) \quad P_{1, \varepsilon, \varphi} (x, \xi) = P_1 (\varepsilon x, \xi + i\varphi' (x)),\]

then we can take \((x_j, \xi_j) \in \overline{A_0} \times \mathbb{R}^2\) such that

\[(3.16) \quad \det P_{1, \varepsilon_j, \varphi} (x_j, \xi_j) = 0,\]

and

\[(3.17) \quad M_{\varepsilon_j} (x_j, \xi_j) \text{ is not positive definite in ker } P_1 (\varepsilon_j x_j, \xi_j + i\varphi' (x_j))\]

where

\[(3.18) \quad M_{\varepsilon_j} (x, \xi) = \frac{1}{2i} [P_{1, \varphi} (\varepsilon_j \cdot, \cdot)^*, P_{1, \varphi} (\varepsilon_j \cdot, \cdot)] - [P_{1, \varphi} (\varepsilon_j \cdot, \cdot)^*, P_{1, \varphi} (\varepsilon_j \cdot, \cdot)] (x, \xi).\]
By (3.17) there exists \( \{ w_j \} \subset \ker P_1(\varepsilon_j x_j, \xi_j + i \varphi'(x_j)) \) with \( |w_j| = 1 \) such that

\[
(M_{\varepsilon_j}(x_j, \xi_j)w_j, w_j) \leq 0, \quad j = 1, 2, \ldots
\]

Now we may assume that the sequences \( \{ x_j \}, \{ w_j \} \) are convergent to \( \tilde{x}, \tilde{w} \) where \( \tilde{x} \in \mathcal{A}_0, \tilde{w} \in \mathbb{R}^2 \) with \( |\tilde{w}| = 1 \). If \( \{ \xi_j \} \) has no subsequence which is convergent, then \( \{ \xi_j \} \) is not bounded, and we can choose the subsequence of \( \{ \xi_{jk} \} \) which satisfy

\[
\lim_{k \to \infty} \xi_{jk} = \infty \quad \text{and} \quad \frac{\xi_{jk}}{|\xi_{jk}|} \text{ is convergent to } \tilde{\eta}.
\]

Then, by (3.16), we have

\[
\det P_1(0, \tilde{\eta}) = \lim_{k \to \infty} \det P_1(\varepsilon_{jk} x_{jk}, \xi_{jk} + i \varphi'(x_{jk})) = \lim_{k \to \infty} |\xi_{jk}|^{-1} \det P_1(\varepsilon_{jk} x_{jk}, \xi_{jk} + i \varphi'(x_{jk})) = 0.
\]

But this contradicts with the ellipticity of \( P_1 \), so we may assume that \( \{ \xi_j \} \) is bounded, and convergent to the \( \tilde{\xi} \). Then \( \tilde{x}, \tilde{\xi}, \) and \( \tilde{w} \) satisfy

\[
\det P_1(0, \tilde{\xi} + i \varphi'(\tilde{x})) = 0,
\]

\[
(M_0(\tilde{x}, \tilde{\xi})\tilde{w}, \tilde{w}) \leq 0,
\]

\[
\tilde{w} \in \ker P_1(0, \tilde{\xi} + i \varphi'(\tilde{x})).
\]

But this contradicts with the assumption that \( P_1(\partial) = P_1(0, \partial) \) and \( \varphi \) satisfy the condition of Theorem 3.1 in \( \mathcal{A}_0 \). We have thus proved the Lemma 3.3.

**Proof of Lemma 3.2.** By Lemma 3.3, it remains that we check the condition (2) in Theorem 3.1 for \( P_1(\partial) \) in order to prove Lemma 3.2. Put \( p_j(\xi) = \xi_1 + \lambda_j \xi_2 \). Because we may assume that \( N(0) \) is a diagonal matrix, we have

\[
P(0, \xi + i \varphi(x)) = \text{diag}[p_{j,\varphi}(x, \xi), \ldots, p_{l,\varphi}(x, \xi)],
\]

where \( p_{j,\varphi}(x, \xi) = p_j(\xi + i \varphi'(x)) \). Therefore \( \varphi \) and \( P_1(\partial) \) satisfy the condition (2) of Theorem 3.1 in \( \mathcal{A}_1 \) if and only if for all \( j \)

\[
(3.20) \quad \frac{1}{i} \{ p_{j,\varphi}, p_{j,\varphi} \}(x, \xi) > 0, \quad \text{if } p_{j,\varphi}(x, \xi) = 0.
\]

By the calculation, we have

\[
(3.21) \quad \frac{1}{i} \{ p_{j,\varphi}, p_{j,\varphi} \}(x, \xi) = 2\alpha |x|^{-\alpha-4}q_j(\alpha, x),
\]

therefore the condition (3.20) is equivalent to

\[
(3.22) \quad 2\alpha |x|^{-\alpha-4}q_j(\alpha, x) > 0, \quad x \in \mathcal{A}_0,
\]
thus we have completed the proof of Lemma 3.2. □

Set $\varepsilon_0 = 2^{k_0-2}$. For $u \in C_0^\infty(B_\varepsilon(0)\setminus\{0\})$ with $0 < \varepsilon < \varepsilon_0$, we define

$$u_k = \chi_k u.$$ 

Then we have

$$u = \sum_{k \leq k_0} u_k, \quad (3.23)$$

$$(\forall u \in C_0^\infty(B_\varepsilon(0)\setminus\{0\}))$$

$$\|u\|^2 \leq \sum_{k \leq k_0} \|u_k\|^2 \leq C_2 \|u\|^2. \quad (3.24)$$

Let us apply Lemma 3.2 to $v = u_k(2^k x) \in C_0^\infty(A_0)$, then for any negative integer $k < k_0$ and any $\tau > 1$ we have

$$\sum_{|\gamma| \leq 1} \sigma^{1-2|\gamma|}\|e^{\sigma \varphi} \partial^\gamma v\|^2 \leq C \|e^{\sigma \varphi} P_1(2^k y, \partial) v\|^2, \quad v \in C_0^\infty(A_0). \quad (3.25)$$

LHS of (3.25) =

$$\sum_{|\gamma| \leq 1} \sigma^{1-2|\gamma|}\int e^{2\sigma \varphi(x)} |\partial^\gamma v(x)|^2 dx$$

$$= \sum_{|\gamma| \leq 1} (\sigma^{2^{-\alpha_k}})^{1-2|\gamma|} \int e^{2\sigma \varphi(y)} |\partial^\gamma u_k(y)|^2 dy$$

$$= \sum_{|\gamma| \leq 1} \tau^{1-2|\gamma|} 2^{k(2\alpha + 1)|\gamma|} \|e^{\sigma \varphi} \partial^\gamma u_k\|^2. \quad (3.26)$$

On the other hand,

RHS of (3.25) = $C \|e^{\sigma \varphi} P_1(x, \partial) u_k\|^2$

$$\leq C \{\|\chi_k e^{\sigma \varphi} P_1(x, \partial) u_k\|^2 + \|e^{\sigma \varphi}[P_1(x, \partial), \chi_k] u\|^2\}$$

$$\leq C \|\chi_k e^{\sigma \varphi} P_1(x, \partial) u_k\|^2$$

$$+ 2^{-2k} C \{\|e^{\sigma \varphi} u_{k-1}\|^2 + \|e^{\sigma \varphi} u_k\|^2 + \|e^{\sigma \varphi} u_{k+1}\|^2\}$$

since $[P_1(\cdot, \partial), \chi_k]$ is a function belonging to $C_0^\infty(A_k)$, whose $L^\infty$-norm is $O(2^{-k})$. Therefore we have the following inequality:

$$2^{-\alpha_k \tau} \|e^{\sigma \varphi} u_k\|^2 \leq 2^{-k} C \|\chi_k e^{\sigma \varphi} P_1 u\|^2$$

$$+ C \{\|e^{\sigma \varphi} u_{k-1}\|^2 + \|e^{\sigma \varphi} u_k\|^2 + \|e^{\sigma \varphi} u_{k+1}\|^2\}, \quad k \leq k_0.$$ 

So, summing up over $k$, if supp $u \subset \{|x| < 2^{k_0-1}\}$ and $\tau$ is large, we obtain

$$\tau \|e^{\sigma \varphi} u\|^2 \leq C \|e^{\sigma \varphi} P_1 u\|^2. \quad (3.26)$$

If $\tau$ is so large, we can substitute $P$ for $P_1$ in (3.26), therefore we have thus proved Proposition 3.1. 
4. Appendix

In this section, we give a general theorem of the Carleman estimates for systems of partial differential operators. The following theorem is considered as the extension of the Hörmander’s theorem about the Carleman estimates (Theorem 8.3.1 in [3]) to systems of elliptic partial differential operators.

Let $U$ be a bounded open set in $\mathbb{R}^n$. We suppose that $Q(y, D) = \sum_{|\alpha| \leq m} A_\alpha(y) D^\alpha y$ is a partial differential operator whose coefficients $A_\alpha$ are in $C^\infty(U; M_t(C))$. $Q_m(y, \eta)$ is the principal symbol of $Q(y, D)$, that is,

$$Q_m(y, \eta) = \sum_{|\alpha|=m} A_\alpha(y) \eta^\alpha.$$  

**Theorem 4.1.** Let $\psi$ be in $C^2(U; \mathbb{R})$ with $\psi'(x) \neq 0$ on $U$. If $Q_m(y, \eta)$ satisfies

1. $\det Q_m(y, \eta) \neq 0$, $y \in \overline{U}, \eta \in \mathbb{R}^n \setminus \{0\}$,
2. For any $(y, \eta) \in \overline{U} \times \mathbb{R}^n$, the Hermitian matrix

$$\frac{1}{2\pi} \{[Q_{m,\psi}, Q_{m,\psi}^*] - \{Q_{m,\psi}, Q_{m,\psi}^*\}\}(y, \eta)$$

is positive definite on $\ker Q_{m,\psi}(y, \eta)$, where $Q_{m,\psi}(y, \eta) = Q_m(y, \eta + i\psi'(y))$.
3. $A_\alpha(y)^* A_\beta(y) = A_\beta(y) A_\alpha(y)^*$, $y \in \overline{U}$, $|\alpha| = |\beta| = m$

then there exist some $C > 0$, $\tau_0 > 0$ such that for $\tau > \tau_0$

$$\sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \| e^{\tau \psi} D^\alpha v \|^2 \leq C \| e^{\tau \psi} Q(y, D) v \|^2, \quad v \in C_0^\infty(\overline{U}),$$

where $C$ is independent of $v$, $\tau$.

**Remark 3.** For $P(y, \eta) = (p_{i,j}(y, \eta))_{1 \leq i, j \leq l}$, $Q(y, \eta) = (q_{i,j}(y, \eta))_{1 \leq i, j \leq l}$ in $C^\infty(U \times \mathbb{R}^n; M_t(C))$, $\{P, Q\}(y, \eta)$ is the Poisson bracket of $P$ and $Q$, that is,

$$\{P, Q\}(y, \eta) = \sum_{j=1}^n \left( \partial_{\eta_j} P \partial_{y_j} Q - \partial_{y_j} P \partial_{\eta_j} Q \right)(y, \eta),$$

where $\partial P = (\partial p_{i,j})_{1 \leq i, j \leq l}$, $\partial Q = (\partial q_{i,j})_{1 \leq i, j \leq l}$.

**Remark 4.** The constant $C$ in (4.3) depends on $\| \partial^\gamma A_\alpha \|_\infty$ with $|\gamma| \leq m$.

This theorem can be proved by the following facts.

**Lemma 4.1** (Lemma 8.3.1 in [3]). Let $Q$ be a differential operator with coefficients in $L^\infty(\Omega, M_t(C))$, and assume that every $x \in \overline{U}$ has an open neighbourhood $U_x$ such that (4.3) is valid for some constant $K_x$ when $v \in C_0^\infty(U \cap U_x; C_l)$ and $\tau$ is sufficiently large. Then one can find a constant $K$ such that (4.3) is valid for all $v \in C_0^\infty(\overline{U}; C_l)$ when $\tau$ is large enough.
Theorem 4.2 ([5]). If $Q_m(y, \eta)$ satisfies

1. $\det Q_m(0, \eta) \neq 0, \quad \eta \in \mathbb{R}^n \setminus \{0\}$,
2. $\frac{1}{2\pi}[\{Q^*_{m,\psi}, Q_{m,\psi}\} - \{Q_{m,\psi}, Q^*_{m,\psi}\}](0, \eta)$ is positive definite on $\ker Q_{m,\psi}(0, \eta)$,
3. $A_{\alpha}(y)^* A_{\beta}(y) = A_{\beta}(y) A_{\alpha}(y)^*, \quad x \in U, |\alpha| = |\beta| = m$,

then there exist some $C > 0, \tau_0 > 0, \text{ and the neighborhood of the origin, } V \subset U$ such that for $\tau > \tau_0$

$$
\sum_{|\alpha| \leq m} \tau^{2(m-|\alpha|)-1} \|e^{\tau \varphi} D^\alpha v\|^2 \leq C \|e^{\tau \varphi} P(x, D)v\|^2,
$$

where $C$ is independent of $v, \tau$.

References


