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Gabor multipliers for weighted Banach spaces on locally compact abelian groups

By

S. S. Pandey*

Abstract

We use a projective groups representation ρ of the unimodular group $\mathcal{G}\times\hat{\mathcal{G}}$ on $L^2(\mathcal{G})$ to define Gabor wavelet transform of a function f with respect to a window function g, where $\mathcal{G}$ is a locally compact abelian group and $\hat{\mathcal{G}}$ its dual group. Using these transforms, we define a weighted Banach $\mathcal{H}_w^{1,\rho}(\mathcal{G})$ and its antidual space $\mathcal{H}_w^{1^\sim,\rho}(\mathcal{G})$, w being a moderate weight function on $\mathcal{G}\times\hat{\mathcal{G}}$. These spaces reduce to the well known Feichtinger algebra $S_0(\mathcal{G})$ and Banach space of Feichtinger distribution $S_0'(\mathcal{G})$ respectively for $w\equiv 1$. We obtain an atomic decomposition of $\mathcal{H}_w^{1,\rho}(\mathcal{G})$ and study some properties of Gabor multipliers on the spaces $L^2(\mathcal{G})$, $\mathcal{H}_w^{1,\rho}(\mathcal{G})$ and $\mathcal{H}_w^{1^\sim,\rho}(\mathcal{G})$. Finally, we prove a theorem on the compactness of Gabor multiplier operators on $L^2(\mathcal{G})$ and $\mathcal{H}_w^{1,\rho}(\mathcal{G})$, which reduces to an earlier result of Feichtinger [Fei 02, Theorem 5.15 (iv)] for $w=1$ and $\mathcal{G}=R^d$.

1. Introduction

H.G. Feichtinger, in a resent paper [Fei 02, § 4], has initiated the study of multiplier operators associated with expansions of arbitrary functions in terms of translations and modulations of an analyzing vector with coefficients as Gabor transform of these functions. According to his definition, if Λ is a TF-lattice in $R^d\times\hat{R}^d$, $\{m(\lambda)\}_{\lambda\in\Lambda}$ a complex-valued sequence on Λ and g_1,g_2 are any two L^2-functions, then the Gabor multiplier G_m associated with the triple (g_1,g_2,Λ) and upper symbol m is given by

$$G_m(f)\equiv G_{g_1,g_2,\Lambda,m}(f)=\sum_{\lambda\in\Lambda} m(\lambda)\langle f,\pi(\lambda)g_1\rangle\pi(\lambda)g_2,$$

where $\pi(\lambda)$ denotes a time-frequency shift operator for a $\lambda\in\Lambda$, which is a point in the time-frequency plane $R^d\times\hat{R}^d$.

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Feichtinger [Fei 02], in fact, has paved a new way to move from function space theory towards operator theory associated with Gabor expansions and laid the foundation of the theory of Gabor multipliers, which arise from pointwise multiplications of Gabor coefficients. He has discussed in detail the boundedness properties of Gabor multipliers between the function space $L^2(R^d)$, Feichtinger algebra $S_0(R^d)$ and its dual space $S_0'(R^d)$.

More recently, Feichtinger and Nowak [FN 03, Chapter 5] have given the first systematic and extensive survey of Gabor multipliers. As pointed out by Feichtinger and Ströhmer [FS 98, p.18], "the essential ingredients of Gabor theory are the commutative (= abelian) group of translations in combination with another commutative group, the so called dual group of modulation operators. Hence it is possible to extend Gabor theory to the general setting of locally compact abelian groups $\mathcal{G}$, which includes all settings discussed above."

The present paper is an outcome of the motivation provided by the above remark of Feichtinger and Ströhmer. Since translation and modulation do not commute, it is convenient to use projective group representation for the study of time-frequency analysis on the phase space $\mathcal{G} \times \hat{\mathcal{G}}$, where $\mathcal{G}$ is a locally compact abelian group and $\hat{\mathcal{G}}$ its dual group consisting of all continuous characters on $\mathcal{G}$. Christensen [Chr 96] has employed the method of projective group representation to generalize the well known FG-theory of atomic decomposition of coorbit spaces.

In this paper, following Christensen [Chr 96], we use the concept of projective representation of $\mathcal{G} \times \hat{\mathcal{G}}$ on $L^2(\mathcal{G})$ to study time-frequency analysis on the phase space $\mathcal{G} \times \hat{\mathcal{G}}$. In section 2, we present the notations and basic concepts for use in the sequel. In section 3, we define a projective representation ρ of $\mathcal{G} \times \hat{\mathcal{G}}$ on $L^2(\mathcal{G})$ and use it to obtain the basic properties of the corresponding Gabor transform of a function f with respect to a window function g. Section 4 deals with the weighted Banach space $\mathcal{H}_w^{1,\rho}(\mathcal{G})$ and the space of its anti-functionals $\mathcal{H}_w^{1^\sim,\rho}(\mathcal{G})$, where w is a moderate weight function on the phase space $\mathcal{G} \times \hat{\mathcal{G}}$. In case $w \equiv 1$, these spaces reduce to the well known Feichtinger algebra $S_0(\mathcal{G})$ and the Banach space of Feichtinger distributions $S_0'(\mathcal{G})$ respectively. Also, in this section, we obtain an atomic decomposition of $\mathcal{H}_w^{1,\rho}(\mathcal{G})$ and use it to prove the theorem (Theorem 8.1) on Gabor multipliers corresponding to that of Feichtinger and Zimmermann [FZ 98, p.144] on the Euclidean space R^d. In the last section, we prove a theorem to characterize the compactness property of Gabor multiplier operators on the space $L^2(\mathcal{G})$ and $\mathcal{H}_w^{1,\rho}(\mathcal{G})$. Our results in Lemma 9.2 can be reduces to the corresponding result of Dörfler, Feichtinger and Gröchenig [DFG 02, Theorem 2] for $w \equiv 1$, $\mathcal{G} = R^d$ and $\hat{\mathcal{G}} = \hat{R}^d$, while Theorem 9.1 provides a generalization to an earlier result of Feichtinger [Fei 02, Theorem 5.15 (iv)].

2. Notations and basic concepts

Let $\mathcal{G}$ be a separable locally compact abelian group with the dual group $\hat{\mathcal{G}}$ consisting of all continuous characters on $\mathcal{G}$. It is well known that $\mathcal{G} \times \hat{\mathcal{G}}$ is a locally compact group with respect to the product topology and the composition:

$$(x, \alpha)(y, \beta) = (x + y,\ \alpha\beta);$$

for all $x, y \in \mathcal{G}$ and $\alpha, \beta \in \hat{\mathcal{G}}$. Here we use the same notations as in [Ru 67, p 6-7.] for (x, α). We assume that the group $\mathcal{G}$ and $\hat{\mathcal{G}}$ are $\sigma-$compact, so that all index sets of $\mathcal{G}$ and $\hat{\mathcal{G}}$, partitions of unity and coverings are countable.

Also, it is known that $\Gamma \equiv \mathcal{G} \times \hat{\mathcal{G}}$ is unimodular, i.e., the Haar measure on Γ is both left and right invariant and this measure is the product measure. A non-negative and locally integrable function on Γ is known as a weight function. A weight function v on Γ is called submultiplicative provided that

$$v(\xi_1 .\ \xi_2) \leq v(\xi_1)v(\xi_2) \quad \text{for all } \ \xi_1, \xi_2 \ \text{ in } \Gamma.$$

A weight function w on Γ is termed as $v-$moderate, if

$$w(\xi_1 \ \ \xi_2) \leq c\ v(\xi_1)\ w\ (\xi_2); \forall\ \xi_1, \xi_2 \in \Gamma,$$

c being a positive constant. We assume that $w(\xi) \geq 1, \forall \xi \in \Gamma$.

Let $L^p_w(\Gamma),\ 1 \leq p < \infty$, be the Banach space of all Lebesgue measurable functions F on Γ with respect to the norm

$$\|F\|_{p,w} = \left(\int_{\hat{\mathcal{G}}} \left(\int_{\mathcal{G}} |F(x, \alpha)|^p\ w^p(x, \xi) dx \right) d\alpha \right)^{1/p} < \infty. \tag{2.1}$$

In case that $p = \infty$, then the corresponding p-norm is replaced by the essential supremum such that

$$\|F\|_{\infty,w} = \text{ess sup}_{(x,\alpha)\in\mathcal{G}\times\hat{\mathcal{G}}} \{|F(x, \alpha)|\ w(x, \alpha)\} < \infty. \tag{2.2}$$

Let $C^0(\mathcal{G} \times \hat{\mathcal{G}})$ be the space of all continuous functions on $\mathcal{G} \times \hat{\mathcal{G}}$ vanishing at infinity.

We denote the translation (time-shift) operators on $\mathcal{G}$ by $\tau_x,\ \ x \in \mathcal{G}$, such that

$$\tau_x\ f(y) = f(y - x); \ \ \forall x, y \in \mathcal{G}.$$

The modulation (frequency-shift) operators $M_\alpha, \alpha \in \hat{\mathcal{G}}$, are define by

$$\begin{aligned} M_\alpha f(y) &= \alpha(y)\ f(y) \\ &= (y, \alpha)\ f(y); \forall y \in \mathcal{G} \ \text{ and } \ \alpha \in \hat{\mathcal{G}}. \end{aligned}$$

By virtue of these definitions, it can be easily seen that

$$(\tau_x\ f)^\wedge = M_{-\alpha}\ \hat{f}$$

and

$$(M_\alpha\ f)^\wedge = \tau_\alpha\ \hat{f},$$

where $\hat{f}$ denotes the Fourier transform of f on $\mathcal{G}$ given by

$$\hat{f}(\alpha) = \int_{\mathcal{G}} (-x,\ \alpha)\ f(x)\ dx.$$

Also, we have

$$\begin{aligned} \tau_x\ M_\alpha\ f(y) &= (M_\alpha\ f)\ (y-x) \\ &= (y-x,\ \alpha)\ f\ (y-x) \\ &= (-x,\ \alpha)\ (y,\ \alpha)\ f\ (y-x) \\ &= (-x,\ \alpha)\ (y,\ \alpha)\ \tau_x\ f\ (y) \\ &= (-x,\ \alpha)\ M_\alpha\ \tau_x\ f\ (y) \\ \Rightarrow \tau_x\ M_\alpha &= (-x,\ \alpha)\ M_\alpha\ \tau_x, \end{aligned}$$

which entails that the operators τ and M are non-commutative.

The convolution $F * G$ of any two functions F and G is defined by

$$(F * G)\ (\xi) = \left(\int_\Gamma F(\eta)\ L_\eta\ G\ d\eta \right)\ (\xi);\ \forall \xi, \eta \in \Gamma,$$

provided the above integral exists, where L_η is the left translation such that $L_\eta\ G(\xi) = G(\eta^{-1}\ \xi)$.

It can be verified that $L^1_w(\Gamma)$ is a Banach convolution algebra, not necessarily commutative, and $L^p_w(\Gamma)$ is a Banach convolution module over $L^1_w(\Gamma)$, i.e.,

$$L^p_w(\Gamma) * L^1_w(\Gamma) \subseteq L^p_w(\Gamma),\ \ 1 \le p \le \infty,$$

and

$$\|F * G\|_{p,w} \le \|F\|_{p,w}\ \|G\|_{1,w}$$

for all $F\ \in L^p_w(\Gamma)$ and $G \in L^1_w(\Gamma)$.

We denote by $C^0(\Gamma)$ and $C_c(\Gamma)$ the spaces of all continuous functions on Γ vanishing at infinity and with compact support, respectively.

We denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators $T : X \to Y$ with the operator norm $|||T|||_{\mathcal{L}(x,y)}$, $X$and Y being any two Banach spaces of functions or distributions. In case $X = Y$, we simple write $\mathcal{L}(X, X)$ as $\mathcal{L}(X)$.

3. Projective group representations and Gabor transforms

Let us denote by $\mathcal{U}(L^2(\mathcal{G}))$, the set of all unitary operators on $L^2(\mathcal{G})$. On the lines of Christensen [Chr 93, p. 67], we say that a projective representation of $\mathcal{G} \times \hat{\mathcal{G}}$ on $L^2(\mathcal{G})$ is a mapping $\rho : \mathcal{G} \times \hat{\mathcal{G}} \to \mathcal{U}(L^2(\mathcal{G}))$ satisfying the following conditions:

(i) $\rho(0(e,1)) = 1,$

where e is the identity element in $\mathcal{G}, 1 \in \hat{\mathcal{G}}$ and $O(e,1)$ is the unit element in $\mathcal{G} \times \hat{\mathcal{G}} = \Gamma$.

(ii) There exists a continuous function $c : \mathcal{G} \times \hat{\mathcal{G}} \to \mathcal{C}$ such that

$$\rho[(x,\alpha)\ (x',\alpha')] \\ = c\ (x,\alpha)\ (x',\alpha')\ \rho(x,\ \alpha)\ \rho(x'\ \alpha');\ \forall\, x, x' \in \mathcal{G} \text{ and } \alpha, \alpha' \in \hat{\mathcal{G}},$$

where $\mathcal{C}$ is the set of all complex numbers and c is a positive constant such that $|c(\xi.\eta)| = 1$ and $c[\xi.O(e,1)] = c[O(e,1).\xi] = 1,\ \ \forall\, \xi, \eta \in \Gamma$.

(iii) The mapping

$$(x,\alpha) \to \ \langle \rho(x,\alpha) f_1,\ f_2 \rangle$$

is a Borel function on $\mathcal{G} \times \hat{\mathcal{G}}$ for all f_1, f_2 in $L^2(\mathcal{G})$.

In case $c = 1$, then ρ is a unitary representation.

Throughout this paper ρ denotes a projective representation of $\mathcal{G} \times \hat{\mathcal{G}}$ on $L^2(\mathcal{G})$ such that

$$\begin{aligned} [\rho(x,\alpha)\ f]\ (y) &= \tau_x\ M_\alpha\ f(y) \\ &= \tau_x\ (\alpha f)\ (y) \\ &= (y - x, \alpha)\ f\ (y - x) \end{aligned} \tag{3.1}$$

for all $x, y \in \mathcal{G}$ and $\alpha \in \hat{\mathcal{G}}$.

We denote the Gabor wavelet transform of $f \in L^2(\mathcal{G})$ with respect to a window function $g \in L^2(\mathcal{G})$ by

$$V_g : L^2(\mathcal{G}) \to C(\mathcal{G})$$

such that

$$\begin{aligned} V_g\ f(x,\alpha) &= \langle \rho\ (x,\alpha) g,\ f \rangle \\ &= \int_{\mathcal{G}} \int_{\hat{\mathcal{G}}} \rho(x,\alpha)\ g(x) \overline{f(x)}\ d\alpha\ dx, \end{aligned} \tag{3.2}$$

where $C(\mathcal{G})$ is the space of all complex-valued functions on $\mathcal{G}$. Hence, as in [Chr 93, p.18], we have

$$\int_{\mathcal{G}} \int_{\hat{\mathcal{G}}} |\langle \rho(x,\alpha)\ g, f \rangle|^2\ d\alpha\ dx = \|f\|^2\ \|g\|^2 \tag{3.3}$$

for all $f, g \in L^2(\mathcal{G})$.

This ensures that ρ is irreducible, if $g \in L^2(\mathcal{G}) \setminus \{0\}$.

Next, let $f \perp$ span $\rho(x,\alpha)g$ for all $x \in \mathcal{G}$ and $\alpha \in \hat{\mathcal{G}}$. Then it is clear that

$$\begin{aligned} V_g\ f(x,\alpha) &= 0 \\ \Rightarrow \|V_g\ f\|_2 &= 0. \\ \Rightarrow \|f\|_2\ \|g\|_2 &= 0. \\ \Rightarrow f &= 0. \end{aligned}$$

Throughout this paper we assume that ρ is an irreducible, unitary, continuous and square-integrable projective representation of $\mathcal{G} \times \hat{\mathcal{G}}$ on $L^2(\mathcal{G})$.

The Gabor transform as described in (3.2) is a linear mapping from the Hilbert space $L^2(\mathcal{G})$ into the space of bounded and continuous functions on $\mathcal{G} \times \hat{\mathcal{G}} = \Gamma$, because for $f, g \in L^2(\mathcal{G})$, we have

$$\begin{aligned} |V_g\ f(\gamma) - V_g\ f(\gamma\,\gamma')| &= |V_g\ f(x,\alpha) - V_g\ f(x,\alpha)\ (x',\alpha')| \\ &= |V_g\ f(x,\alpha) - \langle \rho(x,\alpha)\ (x',\alpha')\ g,\ f\rangle| \\ &= |V_g\ f(x,\alpha) - \langle c(x,\alpha)\ (x',\alpha')\ \rho(x,\alpha)\ \rho(x',\alpha')\ g,\ f\rangle| \\ &= |V_g\ f(x,\alpha)[1 - c(x,\alpha)\ (x',\alpha')\ \rho(x',\alpha')]| \\ &\quad \to 0 \quad \text{as } (x'\alpha') \to 0(e,1). \end{aligned}$$

The following proposition will be useful in the sequel:

Proposition 3.1.

(i) *The operator V_g satisfies the interwining property*:

$$V_g(\rho(\gamma)\ f) = L_\gamma(V_g\ f),\ \forall\ \gamma \in \Gamma, \tag{3.4}$$

where L_γ is the left translation operator.

(ii) *If $*$ denotes the involution such that*

$$V_g\ f^*(\gamma) = \overline{V_g\ f(\gamma^{-1})},$$

then

$$V_g\ f^* = V_f\ g. \tag{3.5}$$

(iii) *If $f_1, f_2, g_1, g_2 \in L^2(\mathcal{G})$, then*

$$V_{g_1}\ f_1\ *\ V_{g_2}\ f_2\ = \langle g_1, f_2\rangle\ V_{g_2}\ f_1.$$

In case $f_1 = f,\ f_2 = g_1 = g_2 = g$ and $\|g\|_2 = 1$, then the reproducing formula:

$$V_g\ f\ *\ V_g\ g = V_g\ f \tag{3.6}$$

holds true.

Proof. (i) From the definition of Gabor transform, we have

$$\begin{aligned} V_g(\rho(\gamma)f)(\gamma') &= \langle \rho(\gamma')g,\ \rho(\gamma)f\rangle \\ &= \langle \rho(\gamma^{-1}\,\gamma')g,\ f\rangle \\ &= V_g f(\gamma^{-1}\,\gamma') \\ &= L_\gamma(V_g\ f)(\gamma') \\ \Rightarrow V_g(\rho(\gamma)f) &= L_\gamma(V_g f). \end{aligned}$$

(ii) We have

$$\begin{aligned} V_g f^*(\gamma) &= \overline{V_g f(\gamma^{-1})} \\ &= \langle \rho(\gamma^{-1})g,\ f\rangle \\ &= \langle \rho(\gamma)f,\ g\rangle \\ &= V_f g(\gamma) \\ \Rightarrow V_g f^* &= V_f g. \end{aligned}$$

(iii) From the definition of convolution, we have

$$\begin{aligned} (V_{g_1}\ f_1 * V_{g_2} f_2)(\gamma) &= \langle L_\gamma(V_{g_2} f_2)^*, V_{g_1} f_1\rangle \\ &= \langle L_\gamma(V_{f_2} g_2), V_{g_1} f_1\rangle \\ &= \langle V_{f_2}(\rho(\gamma)g_2), V_{g_1} f_1\rangle \\ &= \langle V_{g_1}^{-1} V_{f_2}(\rho(\gamma)g_2),\ f_1\rangle \\ &= \langle g_1,\ f_2\rangle\langle \rho(\gamma)g_2,\ f_1\rangle \\ &= \langle g_1,\ f_2\rangle V_{g_2} f_1(\gamma) \\ \Rightarrow V_{g_1} f_1 * V_{g_2} f_2 &= \langle g_1,\ f_2\rangle V_{g_2} f_1. \end{aligned}$$

Putting $f_1 = f,\ f_2 = g_1 = g_2 = g$ and $\|g\|_2 = 1$, we get

$$V_g f * V_g f = V_g f$$

□

This completes the proof.

4. Atomic decomposition of weighted Banach spaces

We suppose that $(\rho, L^2(\mathcal{G}))$ is a projective representation of $\mathcal{G}\times\hat{\mathcal{G}}$ on $L^2(\mathcal{G})$ and define

$$A_w^\rho(\mathcal{G}) = \{g \in L^2(\mathcal{G}) : V_g\ g \in L_w^1(\mathcal{G}\times\hat{\mathcal{G}})\} \neq \{0\}.$$

Now, fixing a $g \in A_w^\rho(\mathcal{G})$, we define

$$\mathcal{H}_w^{1,\rho}(\mathcal{G}) = \{f \in L^2(\mathcal{G}) :\ V_g f \in L_w^1(\mathcal{G}\times\hat{\mathcal{G}})\} \tag{4.1}$$

and equip it with the norm

$$\|f|\mathcal{H}_w^{1;\rho}(\mathcal{G})\| = \|V_g\, f|L_w^1(\mathcal{G}\times\hat{\mathcal{G}})\|. \tag{4.2}$$

On the lines of Feichtinger and Gröchenig [FG 89, p.317], it can be verified that $\mathcal{H}_w^{1;\rho}(\mathcal{G})$ is a ρ-invariant Banach space. In case $w = 1$, the space $\mathcal{H}_w^{1;\rho}(\mathcal{G})$ reduce to the well known Feichtinger algebra $S_0(\mathcal{G})$, which satisfies a number of highly useful functorial properties.

Let U be a neighbourhood of the unit element $O(e,1)$ in $\mathcal{G}\times\hat{\mathcal{G}}$. Since $\Gamma = \mathcal{G}\times\hat{\mathcal{G}}$ is equipped with the product topology, we can choose neighbourhoods$M$ of $O(e)$ in $\mathcal{G}$ and N of $O(1)$ in $\hat{\mathcal{G}}$ such that

$$M\times N \subseteq U.$$

A collection $\{x_i\}_{i\in I}\in\mathcal{G}$ is called M-dense provided the family $\{x_iM\}_{i\in I}$ covers $\mathcal{G}$, i.e.,

$$\bigcup_{i\in I} x_iM = \mathcal{G}.$$

The collection $\{x_i\}_{i\in I}$ in $\mathcal{G}$ is called V-separated, if for some relatively compact neighbourhood V of the identity $O(e)$ in $\mathcal{G}$ the sets $(x_iV)_{i\in I}$ are pairwise disjoint, i.e.,

$$x_iV\cap x_jV = \phi \ \text{ for all } \ i\neq j.$$

The collection $\{x_i\}_{i\in I}\in\mathcal{G}$ is called relatively separated, if it is the finite union of V-separated collections. A collection $\{x_i\}_{i\in I}$ is called well-spread in $\mathcal{G}$ provided it is both M-dense and relatively separated.

Throughout this paper we assume that the family $\{x_i\}_{i\in I}$ is well-spread in $\mathcal{G}$. We now choose a finite N-dense collection $\{\alpha_j\}_{j=1}^n$ such that

$$\{(x_i,\ \alpha_j)\}\in\mathcal{G}\times\hat{\mathcal{G}}.$$

Hence it is clear that the collection $\{(x_i,y_j)\}$ is U-dense and relatively separated in $\mathcal{G}\times\hat{\mathcal{G}}$.

Let U be a compact neighbourhood of the identity $O(e,1)$ in $\mathcal{G}\times\hat{\mathcal{G}}$. A family $\Psi = (\psi_{i,j})_{i\in I}$ in $C^0(\mathcal{G}\times\hat{\mathcal{G}})$ is called a bounded uniform partition of unity of size U (U-BUPU) provided the following conditions hold true:

(i) $0\le\psi_{i,j}\ (x,\alpha)\le 1\ ;\ \forall i\in I, j=1,\dots,n,\ x\in\mathcal{G}\ \text{ and }\ \alpha\in\hat{\mathcal{G}}$.

(ii) There exists a well-spread family $\{(x_i,\ \alpha_j)\}$ in $\mathcal{G}\times\hat{\mathcal{G}}$ such that

$$\sup\psi_{i,j}\subseteq(x_i,\alpha_j)U$$

for all $i\in I$ and $j=1,\dots,n$.

(iii)

$$\sum_{i\in I, J=1,\dots,n}\psi_{i,j}(x,\ \alpha) = 1$$

We define the U-oscillation of $V_g\ g = G$ by

$$G_U^\sharp(x,\alpha) = \sup_{(y,\beta)\in U} |G(y,\beta)\ (x,\alpha) - G(x,\alpha)|.$$

We suppose that T is the convolution operator on $L^1_w(\Gamma)$ such that

$$T\ F = F * G,$$

where

$$F = V_g\ f \ \text{ and } \ G = V_g\ g.$$

Next, on the lines of Feichtinger and Gröchenig [FG 89, p.329], we define an approximation operator T_Ψ such that

$$T_\Psi : F \to \sum_{i\in I}\sum_{j=1}^{n} \langle \psi_{i,j}, F\rangle L_{(x_i,\ \alpha_j)} G,$$

which is composed of a coefficient mapping

$$F \to \langle \psi_{i,j}, F\rangle_{i\in I,\ j\in 1,\ldots,n}$$

and a convolution operator:

$$\{(\lambda_{i,j})_{i\in I}\}_{j=1}^{n} \to \sum_{i\in I}\sum_{j=1}^{n} \lambda_{i,j}\ L_{(x_i,\ \alpha_j)}\ G$$
$$= \left(\sum_{i\in I}\sum_{j=1}^{n} \lambda_{i,j}\ \delta(x_i,\ \alpha_j)\right) * G,$$

where $\delta(x_i,\ \alpha_j)$ is the point measure at $(x_i,\ \alpha_j)$.

Our aim is to prove the following two theorems:

Theorem 4.1. *If T_Ψ i a net of U-BUPUs and the condition* (4.2) *holds true, then*

$$\lim_{\Psi\to\infty} |||(T_\Psi - T)|L^1_w(\Gamma)\ ||| = 0.$$

Theorem 4.2. *If $0 \neq g \in A_w^{1,\rho}(G)$, then there exists a neighbourhood U_0 of the identity $O(e,1)$ in $\mathcal{G}\times\hat{\mathcal{G}}$ and a constant $c > 0$, both depending on g, such that for every well-spread $(x_i,\ \alpha_j)$ in $\mathcal{G}\times\hat{\mathcal{G}}$ any function $f \in \mathcal{H}_w^{1,\rho}(\mathcal{G})$ may be expressed in the form*

$$f = \sum_{i\in I}\sum_{j=1}^{n} \lambda_{i,j}(f)\rho(x_i,\alpha_j)g \tag{4.3}$$

with

$$\sum_{i\in I}\sum_{j=1}^{n}|\lambda_{i,j}|w(x_i,\alpha_j)\le C\|f|\mathcal{H}_w^{1;\rho}(\mathcal{G})\| \tag{4.4}$$

and the series in (4.3) is absolutely convergent in the norm topology of $\mathcal{H}_w^{1;\rho}(\mathcal{G})$, *where*

$$\lambda_{i,j}=\langle\psi_{i,j},\ T_\Psi^{-1}\,V_g f\rangle.$$

5. Necessary Lemmas

We shall use the following lemmas in the proof of our theorems:

Lemma 5.1. *If G is continuous, then we have*

$$\lim_{U\to O(e,1)}\|G_U^\sharp|L_w^1(\Gamma)\|=0. \tag{5.1}$$

Proof. Let U be a neighbourhood of the identity element in Γ such that $U\subseteq U_0$.

$$\implies\ G_U^\sharp\le G_{U_0}^\sharp.$$

Hence, for any $\epsilon>0$, there exists a compact set K in Γ such that

$$\int_{\Gamma\setminus K}G_U^\sharp(\xi)\,d\xi<\epsilon/2; \tag{5.2}$$

for all $U\subseteq U_0$.

Since K is compact, G is uniformly continuous on it. Hence there exists a neighbourhood $U_1\subseteq U_0$ such that

$$G_{U_1}^\sharp(\xi)<\frac{\epsilon}{2\mu(K)w} \tag{5.3}$$

holds true for all ξ in K, where $\mu(K)$ is the Haar measure of K and

$$w=\sup_{\gamma\in K}\ w(\gamma).$$

Thus, on account of (5.3), we obtain

$$\int_K G_U^\sharp(\gamma)\,w(\gamma)\,d\gamma<\epsilon/2 \tag{5.4}$$

for all $U\subseteq U_1$.

Now, combining (5.2) and (5.4), we get

$$\|G_U^\sharp|L_w^1(\Gamma)\|<\epsilon$$

for all $U\subseteq U_1$.

$$\Rightarrow\lim_{U\to O(e,1)}\|G_U^\sharp|L_w^1(\Gamma)\|=0.$$

□

Lemma 5.2. *If $\eta \in \xi\, U$, then*

$$|L_\eta\, G - L_\xi\, G| \leq L_\eta\, G^\sharp_U$$

holds true pointwise, i.e.,

$$|G(\eta^{-1}\zeta) - G(\xi^{-1}\zeta)| \leq G^\sharp_U(\eta^{-1}\zeta)$$

for all $\zeta \in \Gamma$.

Proof. We have

$$\begin{aligned}|G(\eta^{-1}\zeta) - G(\xi^{-1}\zeta)| &\leq \sup_{u\in U} |G(\eta^{-1}\zeta) - G(u\eta^{-1}\zeta)| \\ &= G^\sharp_U(\eta^{-1}\zeta),\end{aligned}$$

for $\eta \in \xi\, U \Rightarrow \xi^{-1} = u\, \eta^{-1}$. □

Lemma 5.3. *If $G \in L^1_w(\Gamma)$, $X = \{(x_i,\ \alpha_j)_{i\in I}\}_{j=1}^n = (\xi_{i,j})$ is any U-dense family in Γ and $\Lambda = \{(\lambda_{i,j})_{i\in I}\}_{j=1}^n\}$ is defined by $\lambda_{i,j} = (\langle \psi_{i,j}, F\rangle)$, then*

$$F = \sum_{i\in I}\sum_{j=1}^n \lambda_{i,j}\ L_{(x_i,\alpha_j)}\ G \in L^1_w(\mathcal{G}\times\hat{\mathcal{G}})$$

if and only if $\Lambda \in l^1_{w(x_i,\alpha_j)}$.

Proof. Let $\xi,\ \eta,\ \zeta$ be any three element of Γ. Since w is a v-moderate weight function on Γ, we have

$$w(\eta-\zeta) \leq\ c\, v(\zeta)\, w(\eta).$$

We put $\eta - \zeta = \xi_{i,j} = (x_i.\alpha_j),\ \eta \in \xi_{i,j}U$

and $C_0 = \sup_{\zeta\in U} v(\zeta)$.

Then we have

$$\begin{aligned}w(\xi_{i,j}) &\leq C\ v(\zeta)\ w(\eta) \\ &\leq C_0\ w(\eta).\end{aligned}$$

Hence we see that

$$|\langle \psi_{i,j}, F\rangle|\ w(\xi_{i,j}) \leq C_0\langle \psi_{i,j},\ w(\eta)|F|\rangle,$$

which ensures that

$$\begin{aligned}\|\Lambda|l^1_w\| &= \sum_{i\in I}\sum_{j=1}^n |\langle \psi_{i,j}, F\rangle| w(\xi_{i,j}) \\ &\leq C_0 \sum_{i\in I}\sum_{j=1}^n \langle \psi_{i,j}, w|F|\rangle \\ &= C_0\|F|L^1_w(\Gamma)\|.\end{aligned}$$

Conversely, we have

$$\begin{aligned}\|\sum_{i\in I}\sum_{j=1}^{n}\lambda_{i,j}L_{\xi_{i,j}}G|L^1_w(\Gamma)\| &\le \sum_{i\in I}\sum_{j=1}^{n}|\lambda_{i,j}|\ \|L_{\xi_{i,j}}G|L^1_w(\Gamma)\|\\ &\le \sum_{i\in I}\sum_{j=1}^{n}|\lambda_{i,j}|w(\xi_{i,j})\|G|L^1_w(\Gamma)\|\\ &\le C\|\Lambda|l^1_w(\Gamma)\|.\end{aligned}$$

□

Hence the lemma holds true.

Lemma 5.4. *The set of operators $\{T_\Psi\}$, where Ψ runs through the family of U-BUPUs, acts uniformly bounded on $L^1_w(\Gamma)$.*

Proof. Let $F \in L^1_w(\Gamma)$. Then we have

$$\begin{aligned}\|T_\Psi F|L^1_w\| &= \|\sum_{i\in I}\sum_{j=1}^{n}\langle\psi_{i,j},F\rangle L_{\xi_{i,j}}G|L^1_w(\Gamma)\|\\ &= \|\left(\sum_{i\in I}\sum_{j=1}^{n}\langle\psi_{i,j},F\rangle\delta(\xi_{i,j})*G\right)|L^1_w(\Gamma)\|\\ &\le \sum_{i\in I}\|\left(\int_{\xi_{i,j}U}F(\eta)\psi_{i,j}(\eta)L_{\xi_{i,j}}G\,d\eta\right)|L^1_w(\Gamma)\|\\ &\le \sum_{i\in I}\sum_{j=1}^{n}\int_{\xi_{i,j}U}|F(\eta)|\psi_{i,j}(\eta)\|\left(L_{\xi_{i,j}}G\right)|L^1_w(\Gamma)\|\,d\eta\\ &\le \sum_{i\in I}\sum_{j=1}^{n}\sup_{u\in U}\|L_{\xi_{i,j}}G(u)|L^1_w(\Gamma)\|\langle\psi_{i,j},|F|\rangle\\ &\le C_0\sum_{i\in I}\sum_{j=1}^{n}w(\xi_{i,j})\langle\psi_{i,j},|F|\rangle\\ &\le C_0\|F|L^1_w(\Gamma)\|\\ &\le C_0\|\Lambda|l^1_w\| \qquad \text{by Lemma 5.3.}\end{aligned}$$

□

6. Proof of Theorem 4.1

For $F \in L^1_w(\Gamma)$, we have

$$
\begin{aligned}
\|(T_\Psi F - TF)|L^1_w(\Gamma)\| &= \| \left(\sum_{i\in I}\sum_{j=1}^{n} (\langle \psi_{i,j}, F\rangle \delta(\xi_{i,j}) - F\psi_{i,j}) * G \right) |L^1_w(\Gamma)\| \\
&\le \sum_{i\in I}\sum_{j=1}^{n} \| \int_{\xi_{i,j}U} \left(L_{\xi_{i,j}}G - L_\eta G\right) F(\eta)\psi_{i,j}(\eta) d\eta |L^1_w(\Gamma)\| \\
&\le \sum_{i\in I}\sum_{j=1}^{n} \int_{\xi_{i,j}U} \| \left(L_{\xi_{i,j}}G - L_\eta G\right) |L^1_w(\Gamma)|F(\eta)|\psi_{i,j}(\eta) d\eta \\
&\le \sum_{i\in I}\sum_{j=1}^{n} \sup_{u\in U} \|L_{\xi_{i,j}}G - L_{\xi_{i,j}u}G|L^1_w(\Gamma)\langle \psi_{i,j}, |F|\rangle \\
&\le \sup_{u\in U} \|(G - L_uG)|L^1_w\| \sum_{i\in I}\sum_{j=1}^{n} w(\xi_{i,j})\langle \psi_{i,j}, |F|\rangle \quad \text{for} \quad w(\xi) \ge 1 \\
&\le \omega_U(G) C_0 \|F|L^1_w\|,
\end{aligned}
$$

where

$$\omega_U(G) = \sup_{u\in U} \|(G - L_uG)|L^1_w(\Gamma)\|$$

is the modulus of continuity of G with respect to the norm $\|\cdot\|_{L^1_w}$.

Hence, choosing U sufficiently small, we obtain

$$
\begin{aligned}
\|\|(T_\psi - T)|L^1_w\|\| &\le C_0 \;\; \omega_U(G) \\
&\longrightarrow 0 \quad \text{as} \quad U \to O(e,1).
\end{aligned}
$$

Hence the theorem holds true. □

7. Proof of Theorem 4.2

Let g be a non-zero fixed element of $A^\rho_w(\mathcal{G})$, which is normalized by

$$\|g\|_2 = 1.$$

Then we have

$$\|F * G\|_{1,w} \le \|F\|_{1,w} \;\; \|G\|_{1,w},$$

and, by virtue of the relation

$$V_g f = V_g f * V_g g,$$

we have

$$F \in L^1_w * G$$

if and only if $F = V_g f$ for a fixed element $f \in \mathcal{H}^1_w(\mathcal{G})$.

Hence the convolution $L_w^1 * G$ is a bounded projection from $L_w^1(\Gamma)$ onto the closed subspaces of $L_w^1 * G$.

Since from the definition

$$TF = F * G,$$

the operator T acts as an identity operator on $L_w^1 * G$.

Hence there exists a net $\{T_\Psi\}$ of U-BUPUs, which is norm convergent to T (by Theorem 4.1) and we have

$$\lim_{\Psi \to \infty} |||(T_\Psi - T)|L_w^1 * G||| = 0.$$

Thus we may choose $a > 0$ such that

$$|||(T - T_\Psi)|L_w^1 * G||| < a < 1$$

in a sufficiently small neighbourhood U of $O(e, 1)$, which implies that

$$|||T_\Psi^{-1}||| \leq (1 - a)^{-1}.$$

Hence we see that

$$\begin{aligned} V_g f &= F \\ &= T_\Psi(T_\Psi^{-1} F) \\ &= \sum_{i \in I} \sum_{j=1}^{n} \langle \psi_{i,j}, T_\Psi^{-1} F \rangle L_{\xi_{i,j}} G \\ \Rightarrow f &= \sum_{i \in I} \sum_{j=1}^{n} \langle \psi_{i,j}, T_\Psi^{-1} F \rangle V_g^{-1}(L_{\xi_{i,j}} G) \\ &= \sum_{i \in I} \sum_{j=1}^{n} \langle \psi_{i,j}, T_\Psi^{-1} F \rangle V_g^{-1}(L_{\xi_{i,j}} V_g g) \\ &= \sum_{i \in I} \sum_{j=1}^{n} \langle \psi_{i,j}, T_\Psi^{-1} F \rangle \rho(\xi_{i,j}) g \qquad \text{by (3.4)}. \\ &= \sum_{i \in I} \sum_{j=1}^{n} \lambda_{i,j} \rho(x_i, \alpha_j) g. \end{aligned}$$

Next, since

$$T_\Psi^{-1} F \in L_w^1 * G,$$

we have

$$
\begin{aligned}
\|\Lambda\|_{1,w} &= \sum_{i\in I}\sum_{j=1}^{n} \lambda_{i,j} w(x_i,\alpha_j) \\
&= \sum_{i\in I}\sum_{j=1}^{n} \langle \psi_{i,j}, T_\Psi^{-1}F\rangle w(x_i,\alpha_j) \\
&\le C_0 \sum_{i\in I}\sum_{j=1}^{n} \langle \psi_{i,j}, w|T_\Psi^{-1}F|\rangle \\
&= C_0 \|T_\Psi^{-1}F\|_{1,w} \\
&\le C\|f|\mathcal{H}_w^{1,\rho}(\mathcal{G})\|, \\
\text{for}\quad |||T_\Psi^{-1}||| &\le (1-a)^{-1} < \infty \quad \text{and} \\
F &= V_g f.
\end{aligned}
$$

□

This completes the proof of the theorem.

8. Gabor multipliers for Banach spaces

Feichtinger [Fei 02], in a recent paper, has initiated the study of Gabor multipliers on the spaces $L^2(R^d)$, $S_0(R^d)$ and $S_0'(R^d)$, R^d being the d-dimensional Euclidean space. In this section we apply the concept of Gabor multipliers to the spaces $L^2(\mathcal{G})$, $\mathcal{H}_w^{1,^\rho}(\mathcal{G})$ and $\mathcal{H}_w^{1^{\sim},\rho}(\mathcal{G})$, where $\mathcal{H}_w^{1^{\sim},\rho}(\mathcal{G})$ is the space of all continuous conjugate linear functionals on $\mathcal{H}_w^{1,^\rho}(\mathcal{G})$. On account of these definitions, it can be easily verified that the continuous embeddings

$$\mathcal{H}_w^{1,^\rho}(\mathcal{G}) \hookrightarrow L^2(\mathcal{G}) \hookrightarrow \mathcal{H}_w^{1,\sim\rho}(\mathcal{G})$$

holds true.

We suppose that a function f has a formal expansion of the form (4.4). Then, as in [Fei 02, p.116], if g,h are any two functions in $L^2(\mathcal{G})$, a complex-valued sequence $\{m(\xi_{i,j})_{i\in I}\}_{j=1}^n$ is called a Gabor wavelet multiplier associated with the triple (g,h,X) provided

$$G_m(f) \equiv G_{g,h,X,m}(f) = \sum_{i\in I}\sum_{j=1}^{n} m(x_i,\alpha_j)\lambda_{i,j}(f)\rho(x_i,\alpha_j)h, \tag{8.1}$$

where $X = \{(x_i,\alpha_j)_{i\in I}\}_{j=1}^n$ is a well spread family in $\mathcal{G}\times\hat{\mathcal{G}}$. and $\lambda_{i,j}(f) = \langle f, (\rho(x_i,\alpha_j)g\rangle$.

In case $g=h$, we simply write

$$G_m(f) \equiv G_{g,X,m}.$$

We prove the following:

Theorem 8.1. *Let G_m be a linear operator defined by* (8.1). *Then the following results hold true*:

(i) *If $f, g \in L^2(\mathcal{G}),\ h \in \mathcal{H}_w^{1,\rho}(\mathcal{G})$ and $m \in l^1(X)$, then $G_m \in \mathcal{L}(L^2(\mathcal{G}))$ with*

$$\|\|G_m|\mathcal{L}(L^2(\mathcal{G}))\|\| \le C_X \|h|\mathcal{H}_w^{1,\rho}(\mathcal{G})\|\ \|m|l^1(X)\| \cdot \|g|L^2(\mathcal{G})\|, \tag{8.2}$$

C_X being a positive constant depending on X.

(ii) *For $f, g \in \mathcal{H}_w^{1,\rho}(\mathcal{G}),\ h \in L^2(\mathcal{G})$ and $m \in L^2(X)$, we have $G_m \in \mathcal{L}(\mathcal{H}_w^{1,\rho}(\mathcal{G}))$ with*

$$\|\|G_m|\mathcal{L}(\mathcal{H}_w^{1,\rho}(\mathcal{G}))\|\| \le C_X\ \|h\|_2\ \|g|\mathcal{H}_w^{1,\rho}(\mathcal{G})\|\ \|m|l^2(X)\|. \tag{8.3}$$

(iii) *If $g \in \mathcal{H}_w^{1,\rho}(\mathcal{G});\ f, h \in \mathcal{H}_w^{1^\sim,\rho}(\mathcal{G})$ and $m \in l_w^1(X)$, then $G_m \in \mathcal{L}(\mathcal{H}_w^{1^\sim,\rho}(\mathcal{G}))$ with*

$$\|\|G_m|\mathcal{L}(\mathcal{H}_w^{1^\sim,\rho}(\mathcal{G}))\|\| \le C_X\ \|h|\mathcal{H}_w^{1^\sim,\rho}(\mathcal{G})\|\ \|m|l_w^1(X)\|\ \|g|\mathcal{H}_w^{1,\rho}(\mathcal{G})\|. \tag{8.4}$$

Proof. (i) Using the definition (8.1), we have

$$\begin{aligned}
|G_m f| &= |\sum_{i\in I}\sum_{j=1}^{n} m(x_i,\alpha_j)\lambda_{i,j}\ (f)\rho(x_i,\alpha_j)h| \\
&= |\sum_{i\in I}\sum_{j=1}^{n} m(x_i,\alpha_j)\langle\psi_{i,j}, T_\Psi^{-1}V_g f\rangle\rho(x_i,\alpha_j)h| \\
&\le \sup_{i\in I, j=1,2,\dots,n} |\ \langle\psi_{i,j}, T_\Psi^{-1}V_g f\rangle\ | \sum_{i\in I}\sum_{j=1}^{n} |\rho(x_i,\alpha_j)h|\ |m(x_i,\alpha_j)| \\
&\le C_X\|f\|_2\ \ \|g\|_2\ \ \|h|\mathcal{H}_w^{1,\rho}(\mathcal{G})\| \sum_{i\in I}\sum_{j=1}^{n} ||m(x_i,\alpha_j)| \\
&\le C_X\ \|\ f\ \|_2\|\ g\ \|_2\|\ m\ |\ l^1(X)\ \|
\end{aligned}$$

$$\Longrightarrow G_m \in \mathcal{L}(L^2(\mathcal{G}))$$

and the inequality in (8.2) holds true.

(ii) We have

$$\begin{aligned}
|G_m f| &= |\sum_{i\in I}\sum_{j} m(x_i,\alpha_j)\langle\psi_{i,j}, T_\Psi^{-1}V_g f\rangle\rho(x_i,\alpha_j)h| \\
&\le \sup_{i\in I, j=1,2,\dots,n} \langle\psi_{i,j}, T_\Psi^{-1}V_g f\rangle \sum_{i\in I}\sum_{j} |m(x_i,\alpha_j)|\ \rho(x_i,\alpha_j)h| \\
&\le C_X\ \|g|\mathcal{H}_w^{1,\rho}(\mathcal{G})\|\ \|f|\mathcal{H}_w^{1^\sim,\rho}(\mathcal{G})\|m|l^2(X)\|\ \|h\|_2 \\
&\le C_X\ \|g|\mathcal{H}_w^{1,\rho}(\mathcal{G})\|\ \|f|\mathcal{H}_w^{1^\sim,\rho}(\mathcal{G})\|m|l^2(X)\|\ \|h\|_2 \\
&\qquad\qquad \text{for } \mathcal{H}_w^{1,\rho}(\mathcal{G}) \hookrightarrow \mathcal{H}_w^{1^\sim,\rho}(\mathcal{G})
\end{aligned}$$

$$\Longrightarrow\ G_m \in \mathcal{L}(\mathcal{H}_w^{1,\rho}(\mathcal{G}))$$

and the inequality in (8.3) follows.

(iii) As above, we have

$$\begin{aligned}|G_m f| &\le \sup_{i\in I, j=1,2,\ldots,n} |\langle \psi_{i,j}, T_\Psi^{-1} V_g f\rangle| \sum_{i\in I}\sum_j |m(x_i,\alpha_j)|\cdot|\rho(x_i,\alpha_j)h| \\ &\le C_X \, \|g|\mathcal{H}_w^{1,\rho}(\mathcal{G})\| \, \|f|\mathcal{H}_w^{1^\sim,\rho}(\mathcal{G})\| \, \|m|l_w^1\| \, \|h|\mathcal{H}_w^{1^\sim,\rho}(\mathcal{G})\| \\ \Longrightarrow G_m &\in \mathcal{L}(\mathcal{H}_w^{1^\sim,\rho}(\mathcal{G}))\end{aligned}$$

and the inequality in (8.4) is valid. □

9. Compactness of Gabor multipliers

If Y is a Banach space, then it is known that (cf. [La 71], p. 249) any linear operator in $\mathcal{L}(Y)$ is compact provided it maps norm bounded subsets of Y into compact subsets.

As in [DFG 02, pp. 38], we say that a bounded set $\{V_g f : f \in S\}$ is tight in $L^2(\mathcal{G}\times\hat{\mathcal{G}})$ provided, for any given $\epsilon > 0$, there exists a compact set $U \subseteq \mathcal{G}\times\hat{\mathcal{G}}$ such that

$$\sup_{f\in S}\left(\int_{U^c} | V_g f(x,\alpha) |^2 \, dx d\alpha\right)^{1/2} < \infty,$$

U^c being the complementary set of U.

In this section we prove the following theorem on the compactness of Gabor multiplier operators:

Theorem 9.1. *If $g, h \in \mathcal{H}_w^{1,\rho}(\mathcal{G})$, and S is any bounded closed subset of $L^2(\mathcal{G})$ such that $\{V_g f : f \in S\}$ is tight in $L^2(\mathcal{G}\times\hat{\mathcal{G}})$, then G_m is a compact operator on $L^2(\mathcal{G})$ and $\mathcal{H}_w^{1,\rho}(\mathcal{G})$.*

In case $g = h$, $w = 1$ and $\mathcal{G} = R^d$, then this theorem reduces to an earlier result of Feichtinger [Fei 02, Theorem 5.15(iv)] *for Gabor multipliers.*

We shall use the following lemma in the proof of our theorem, which is an extension of an analogous result of Dörfler, Feichtinger and Gröchenig [DFG 02, Theorem 2] on d-dimensional Euclidean space:

Lemma 9.2. *A closed and bounded set $S \subseteq L^2(\mathcal{G})$ is compact if and only if the set $\{V_g f : f \in S\}$ is tight in $L^2(\mathcal{G}\times\hat{\mathcal{G}})$.*

Proof. The proof follows on the lines of Dörfler, Feichtinger and Gröchenig [DFG 02]. Since our setting is on a locally compact abelian group, it is necessary to give a proof.

We suppose that $S \subseteq L^2(\mathcal{G})$ is compact. Thus there exists a finite number of functions $f_1, f_2, \ldots, f_j$, say, such that

$$\min_{j=1,2,\ldots,n} \|f - f_j\|_2 < \frac{\epsilon}{2}, \qquad \forall\, f \in S.$$

Next, since $V_g f_j \in L^2(\mathcal{G} \times \hat{\mathcal{G}})$, we may find a compact set $U \subseteq (\mathcal{G} \times \hat{\mathcal{G}})$ such that

$$\int_{U^c} |V_g f_j(x,\alpha)|^2 \, dx < \frac{\epsilon^2}{4}$$

for $j = 1, 2, \ldots, n$, where U^c is the complementary set of U in $\mathcal{G} \times \hat{\mathcal{G}}$.

Hence, $\forall f \in S$, we have

$$\begin{aligned}\left(\int_{U^c} |V_g f(x,\alpha)|^2 \, dx \, d\alpha\right)^{1/2} &\leq \min_{j=1,2,\ldots,n} \left[\left(\int_{U^c} |V_g(f-f_j)(x,\alpha)|^2 \, dx \, d\alpha\right)^{1/2}\right. \\ &\qquad \left. + \left(\int_{U^c} |V_g f_j(x,\alpha)|^2 \, dx \, d\alpha\right)^{1/2}\right] \\ &\leq \min_{j=1,2,\ldots,n} \|f - f_j\|_2 \; + \; \epsilon/2. \\ &\leq \epsilon.\end{aligned}$$

Conversely, we assume that the set $\{V_g \; f : f \in S\}$ is tight in $L^2(\mathcal{G} \times \hat{\mathcal{G}})$. Hence, for any given $\varepsilon > 0$, there exists a set $U \subseteq \mathcal{G} \times \hat{\mathcal{G}}$ such that

$$\int_{U^c} |V_g \; f(x,\alpha)|^2 \, dx \, dx < \epsilon^2, \; \forall \; f \in S. \tag{9.1}$$

Let $\{f_m\}_{m \in N}$ be a sequence of functions in S. Since S is bounded in $L^2(\mathcal{G})$, it is weakly compact in $L^2(\mathcal{G})$. Thus the sequence $\{f_m\}$ has a weakly convergent subsequence $\{f_{m_\nu}\} = \{f_\nu\}$, say, such that

$$\begin{aligned} f_\nu \to f \qquad & \text{as } \; \nu \to \infty. \\ \Rightarrow \langle f_\nu, h\rangle \to \langle f, h\rangle \quad & \text{as } \; \nu \to \infty, \; \forall \; h \in L^2(\mathcal{G}).\end{aligned}$$

Now, putting $h = \rho(x,\alpha)g$, we obtain

$$V_g \; f_\nu(x,\alpha) \to V_g \; f(x,\alpha) \;\; \text{as} \;\; \nu \to \infty; \; \forall x \in \mathcal{G} \;\; \text{and} \;\; \alpha \in \hat{\mathcal{G}} \tag{9.2}$$

Also, applying Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}|V_g(f - f_\nu)(x,\alpha)| &\leq C \; \|f - f_\nu\|_2 \\ &\leq C \sup_\nu (\|f_\nu\|_2 + \|f\|_2) \\ &\leq C,\end{aligned}$$

where C is a positive constant not necessarily the same at each occurrence.

Thus the restriction of $|V_g \; (f - f_\nu)|^2$ to U is dominated by $C^2 \mathcal{X}_U \in \; L^1(\mathcal{G} \times \hat{\mathcal{G}})$, where $\mathcal{X}_U$ is the characteristic function of U.

Hence, on account of (9.2) and the dominated convergence theorem, we have

$$\int_U |V_g(f - f_\nu) \; (x,\alpha)|^2 dx \; d\alpha \to 0 \quad \text{as} \;\; \nu \to \infty \tag{9.3}$$

for all $f \in S$.

Finally, combining (9.1) and (9.3), we obtain

$$\begin{aligned}\overline{\lim_{\nu\to\infty.}}\|f-f_\nu\|_2 &\le \frac{1}{C}\ \overline{\lim_{\nu\to\infty}}\|V_g(f-f_\nu)\|_2\\ &\le C^{-1}\left[\overline{\lim_{\nu\to\infty}}\left(\int_U |V_g\,(f-f_\nu)\,(x,\alpha)|^2dx\,d\alpha\right)^{1/2}\right.\\ &\qquad\left.+\overline{\lim_{\nu\to\infty}}\left(\int_{U^c}|V_g\,(f-f_\nu)\,(x,\alpha)|^2dx\,d\alpha\right)^{1/2}\right]\\ &\le C^{-1}(0+2\epsilon).\end{aligned}$$

Making $\epsilon \to 0$, we get

$$\overline{\lim_{\nu\to\infty.}}\|f-f_\nu\|_2 = 0.$$

$\Rightarrow$ Every sequence $\{f_m\}$ in S has a convergent subsequence.
$\Rightarrow$ S is compact. □

Proof of Theorem 9.1. By virtue of the above lemma, $\mathcal{H}_w^{1,p}(\mathcal{G})$ is a compact subset of $L^2(\mathcal{G})$. Hence, on account of Theorem 8.1(i), any operator $G_m \in \mathcal{L}(L^2(\mathcal{G}))$ maps norm bounded subsets of $L^2(\mathcal{G})$ onto $\mathcal{H}_w^{1,p}(\mathcal{G})$ for all $m(x_i,\alpha_j)\to 0$ as $i\to\infty$.
$\Rightarrow G_m$ is compact on $L^2(\mathcal{G})$.

Also, from Theorem 8.1 (ii), it follows that $G_m \in \mathcal{L}(\mathcal{H}_w^{1,p}(\mathcal{G}))$ is compact. This complete the proof of the theorem. □

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Department of Mathematics
R.D.University, Jabalpur
India
e-mail: sheelpandey@hotmail.com

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