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The Theorem of BERTINI on Linear Systems in Modular Fields.

By

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O. Zariski[1] has clarified algebraically the proof of the so-called theorem of Bertini concerning the reducible linear systems on algebraic varieties in the projective space over the ground field of characteristic zero. We wish to investigate it over general ground field of any characteristic.

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Following A. Weil[2], we fix once for all the universal domain Ω, of the given characteristic p. We understand under the "extension of a field k" the subfield of Ω which is finitely generated over k by a set of quantities.

§ I Preliminaries (I)

Let a field Σ be an extension of a field k; the derivations of Σ over k form a Σ-module and we denote it by $\mathfrak{D}(\Sigma/k)$. Since Σ is an extension of k, the rank of $\mathfrak{D}(\Sigma/k)$ with respect to Σ is finite.

Let V^r be a Subvariety of a projective n-space (F-Appendix I) and assume that it is everywhere relatively normal with reference to some field of definition k for V. (F-Appendix II). Let V_α be a representative of V and M or M_α be a generic Point of V or V_α respectively over k. Then it holds $\Sigma = k(M) = k(M_\alpha)$ and the ring $k[M_\alpha]$ is integrally closed in Σ. It is well-known that the (r−1)-dimensional irreducible Subvariety of V over k (prime

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1) O. Zariski. Pencils on an algebraic variety and a new proof of a theorem of Bertini. Trans. Amer. Math. Soc. 1941. We indicate this by (Z).

2) A. Weil. Foundations of algebraic geometry. We indicate this by F-..

rational V-divisor over k)[3] defines in Σ' a prime divisor of the first kind with respect to some representative V_α, corresponding to the discrete (r−1)-dimensional valuation of rank 1 in Σ.

Conversely, to any prime divisor of the first kind with respect to V_α, there corresponds the uniquely determined prime rational V-divisor Γ, called the center of the valuation defined by that prime divisor.

We fix one of the representatives, say V_α. Let P be a field such that we have $\Sigma \supset P \supset k$, $\dim_k P=1$ and $(u_{ij}^{(\alpha)})(1 \leqq i \leqq r-1 : 1 \leqq j \leqq n)$ be $(r-1)n$ independent variables over the field Σ, and put $\eta_{i,\alpha}=\sum_{j=0,\neq\alpha} u_{ij}^{(\alpha)}\xi_{j\alpha}(1 \leqq i \leqq r-1)$, where $(\xi_{0,\alpha},\dots,\xi_{n,\alpha})$ is a generic point of V_α. Then $(u^{(\alpha)},\eta_{(\alpha)})$ is a set of algebraically independent variables over P. The field $\Sigma_\alpha^*=\Sigma(u^{(\alpha)})$ is of dimension 1 over the field $k_\alpha^*=k(u^{(\alpha)},\eta_{(\alpha)})$ and the field $P_\alpha^*=P(u^{(\alpha)},\eta_{(\alpha)})$ is also of dimension 1 over k_α^*. Therefore $\Sigma(u^{(\alpha)})/P_\alpha^*$ is an algebraic extension (cf. (Z) § 3). Heretofore, we omit to write the suffix α.

The prime divisor $\mathfrak{p}$ of P over k is uniquely extended to the prime divisor $\mathfrak{p}^*$ of P^* over k^*, under the conditions that it does map each u, η into itself and so the $\mathfrak{p}^*$-residues of them are algebraically independent over the residue field of $\mathfrak{p}$.

If
$$\mathfrak{p}^*=\mathfrak{P}^{*h_1}_1\dots\mathfrak{P}^{*h_m}_m \text{ in } \Sigma^*,$$
then
$$\mathfrak{p} =\mathfrak{P}^{h_1}_1\dots\mathfrak{P}^{h_m}_m \text{ in } \Sigma ,$$
where $\mathfrak{P}_i^*$ induces in Σ the prime divisor $\mathfrak{P}_i$ and $\mathfrak{P}_i$ is a prime divisor of the first kind with respect to some V_β. Thus we find the mode of the decomposition of $\mathfrak{p}^*$ into the power product of $\mathfrak{P}_i^*$ is independent of the choice of the representative V_α. (cf. (Z), § 4, § 5, and § 6).

By the degree of $\mathfrak{p}$, we understand the degree $[\varDelta(u,\eta):k^*]$ where $\varDelta$ is the residue field of $\mathfrak{p}$. $\varDelta(u,\eta)$ is nothing but the residue field of $\mathfrak{p}^*$, since $\mathfrak{p}^*$ satisfies the above mentioned conditions.

By the degree of $\mathfrak{P}_i$, we understand the degree $[\varDelta_i^*:k^*]$, where $\varDelta_i^*$ is the residue field of $\mathfrak{P}_i^*$. Since the prime divisor we consider is of the first kind, the following formula holds:

$$(1) \qquad [\Sigma^*:P^*]\ deg\ (\mathfrak{p})=\textstyle\sum_{i=1}^m h_i\ deg\ (\mathfrak{P}_i)^{(4)}$$

3) We include in the V-cycles, multiple components.

4) By deg ($\mathfrak{p}$) or deg ($\mathfrak{P}$), we mean the degree of $\mathfrak{p}$ or $\mathfrak{P}$ respectively.

§ 2. Preliminaries (II).

We find in Weil's book the following

LEMMA 1. *Let a field* Σ *be an extension of a field* k. *If rank* $(\mathfrak{D}(\Sigma/k))=r$, *then we can find elements* $u_1,\ldots,u_r$ *from* Σ *in such a way that* $\Sigma/k(u)$ *is separably algebraic. Moreover,* r *is the minimal number having this property.*

LEMMA 2. *Let* $\Lambda=k(x)$ *be an extension of a field* k. *Then if* $u\in\Lambda, \bar{\in} k(x^p)$, *there exists at least one derivation* D *in* $\mathfrak{D}(\Lambda/k)$ *such that* $Du\neq 0$[5]. *And rank* $(\mathfrak{D}(\Lambda/k))=log_p[\Lambda:k(x^p)]$, *here we assume that the characteristic* p *of* k *is different from zero.*

THEOREM 2. 1 *If a field* Σ *is an extension of dimension* r *over a field* k, *then it is separably generated over* k *if and only if*

$$rank\ (\mathfrak{D}(\Sigma/k))=r.$$

REMARK The first part of Lemma 2 does hold even when Λ is not finitely generated over k. We can prove it by using Zorn's lemma in the same way as in the ordinary case.

THEOREM 2. 2 *Let a field* $\Sigma=k(M)$ *be a separably generated extension of a field* k *and* P *an intermediary field between* Σ *and* k, *having dimension* s *over* k. *Then* Σ/k *is separably generated if and only if* $[P(M^p):k(M^p)]=p^s$. *Here we assume that the characteristic* p *of* k *is different from zero.*

PROOF. Put $rank\ (\mathfrak{D}(\Sigma/P))=t$. Then by Lemma 2, $[\Sigma:P(M^p)]=p^t$ and $[\Sigma:k(M^p)]=p^r$ where $r=dim_k\ \Sigma$. Since P contains k, we have

$$[P(M^p):k(M^p)]=p^{r-t}.$$

This proves that $t=rank(\mathfrak{D}(\Sigma/P))=r-s$ and $[P(M^p):k(M^p)]=p^s$ imply each other. Our theorem is thereby proved.

COROLLARY 1. *Let a field* $\Sigma=k(M)$ *be a separably generated extension of a field* k *and* P *an intermediary field between* Σ *and* k, *having dimension* 1 *over* k. *Then* Σ/P *is separably generated if and only if* $P\not\subset k(M^p)$

PROOF. By the above theorem, Σ/P is separably generated if and only if

$$[P(M^p):k(M^p)]=p.$$

5) This was for the first time obtained by Baer. Cf. R. Baer. Algebraische Theorie der differentiierbaren Funktionenkörper. I. Sitz. Heid. 1927.

If $P \subset k(M^p)$, $P(M^p) \subset k(M^p)$ and hence $P(M^p)=k(M^p)$ since P contains k. Therefore Σ/k is not separably generated. If $P \not\subset k(M^p)$, $P(M^p)$ contains $k(M^p)$ properly, and hence

$$[P(M^p) : k(M^p)]=p^m \text{ with } m \geqq 1.$$

But, by Lemma 1, $[\Sigma : P(M^p)] \geqq p^{r-1}$, where $r=dim_k \Sigma$. This implies that we have

$$[P(M^p) : k(M^p)] \leq p.$$

This completes our proof.

COROLLARY 2.[6] *Let a field $\Sigma=k(M)$ be an extension of a perfect field k and P an intermediary field between Σ and k, having dimension 1 over k such that there exists no purely inseparable element in Σ over P other than that of P. Then Σ/P is separably generated.*

PROOF. Since k is perfect, Σ/k is separably generated. By our assumption on P, there exists an element u in P such that its p-th root does not belong to Σ. Then u cannot be an element of $k(M^p)$, since k is perfect. Consequently

$$P \not\subset k(M^p).$$

By the above corollary, therefore, Σ/P is separably generated.

THEOREM 2.3 *Let $\Sigma=k(M)$ be a separably generated extension of a field k of dimension r. Let u_{ij} $(1 \leqq i \leqq r-1 ; 1 \leqq j \leqq n)$ be $(r-1)n$ independent variables over Σ, and $M=(\xi_1,\ldots,\xi_n)$, $\eta_i=\sum u_{ij}\xi_j$ $(1 \leqq i \leqq r-1)$. Further let P be an intermediary field between Σ and k, such that $dim_k P=1$. Then $\Sigma(u)$ is separably algebraic over $P(u,\eta)$ if Σ is separably generated over P. And Σ is always separably generated over $k(u,\eta)$.*

PROOF. Since (u) is free over k with respect to Σ, it follows that each of $\Sigma(u)/k(u)$, $\Sigma(u)/P(u)$ and $P(u)/k(u)$ is separably generated. Moreover, $\Sigma(u)$ and $P(u)$ have dimensions r and 1 respectively over $k(u)$ and hence $rank(\mathfrak{D}(\Sigma(u)/P(u))=r-1$. Let $\{D_1,\ldots,D_{r-1}\}$ be *a* basis for the $\Sigma(u)$-module $\mathfrak{D}(\Sigma(u)/P(u))$ over $\Sigma(u)$. Since $\mathfrak{D}(\Sigma(u)/P(u,\eta))$ is contained in $\mathfrak{D}(\Sigma(u)/P(u))$ any element D of $\mathfrak{D}(\Sigma(u)/P(u,\eta))$ can be written in the form

$$D=\textstyle\sum_{j=1}^{r-1} a_j D_j,$$

6) This result is not new. Cf. MacLane. Modular fields I, Separating transcendence basis. Duke Math. J. vol. 5. 1939.

with $a_j \in \Sigma(u)$. Since $\eta_i = \sum_l u_{il}\xi_l$ and since D is a derivation over $P(u, \eta)$, it follows that

$$D\eta_i = \sum_{j=1}^{r-1} a_j (\sum_{l=1}^{n} u_{il} D_j \xi_l) \quad (1 \leqq i \leqq r-1).$$

Then $$|\sum_l u_{il} D_j \xi_l| = |(u_{il})(D_j \xi_l)^t| \quad (1 \leqq i, j \leqq r-1).$$

As, by our assumption on $D_1, \dots, D_{r-1}$, the rank of the matrix $(D_j\xi_l)$ is $r-1$ and u's are independent variables over Σ, it follows that the above determinant is not zero. This implies that $a_j = 0$ for every j, and consequently D must be a trivial one. This proves that $\Sigma(u)$ is separably algebraic over $P(u, \eta)$. Since (u, η) is free over k with respect to P (cf. (Z) § 3), and $k(u, \eta)$ is a pure transcendental extension of k, this implies that P and $k(u, \eta)$ are linearly disjoint over k with respect to each other. As Σ is separably generated over k, and $\Sigma \supset P \supset k$, Σ is linearly disjoint over k with respect to $k^{p^{-m}}$, for all integer $m > 0$. Therefore P is linearly disjoint over k with respect to $k^{p^{-m}}$ for all integer $m > 0$. This shows that P is separably generated over k. By the linear disjointness of P and $k(u, \eta)$ over k, it follows that $P(u, \eta)$ is separably generated over $k(u, \eta)$.

But the field P such that Σ is separably generated over it, surely exists by virtue of cor. 1, th. 2.2. This completes our proof.

THEOREM 2.4 *Let $\Sigma = k(M)$ be a regular extension of dimension $\geqq 2$ of a field k with infinitely many elements. Let x and y be two algebraically independent elements of Σ over k. If x or y is not in $k(M^p)$, then but for a finite number of constants c in k, Σ is regular over $k(x+cy)$. (If $p=0$, then this theorem holds for any x and y. This will be clear from the proof given below).*

PROOF. By our assumption, x or y is not in $k(M^p)$, say $y \bar{\in} k(M^p)$. There exists at least one derivation D in $\mathfrak{D}(\Sigma/k)$ such that we have

$$Dy \neq 0.$$

If $D(x+cy) = 0$, then $Dx + cDy = 0$ and therefore $c = -Dx/Dy$. This shows that if x or y is not in $k(M^p)$, there is at most one constant c in k such that we have $x+cy \in k(M^p)$. Avoiding this special constant, if necessary, we conclude that Σ is separably generated over $\Lambda = k(x+cy)$ (cf. cor. 1, th. 2.2), and hence Σ is linearly disjoint over Λ with respect to $\Lambda^{p^{-m}}$ for every positive integer m. Let T_c be a field consisting of all algebraic elements of Σ over $k(x+cy)$. Then T_c must be a separably algebraic extension of Λ

since Σ and hence T_c is linearly disjoint over Λ with respect to Λ^{p-m} for every positive integer m.

Put $\Sigma_c = T_c(y)$, then as y is an independent variable over T_c, it follows that Σ_c is a separably algebraic extension of $k(x, y)$. Let Σ_1 be the field consisting of all separable elements in Σ over $k(x, y)$. Then Σ_c is contained in Σ_1. There are only a finite number of intermediary fields between Σ_1 and $k(x, y)$. Therefore, except for a finite number of constants in k, there exists a constant $d \neq c$ such that we have

$$\Sigma_c = \Sigma_d$$

As we have $\Sigma \supset T_d \supset k$ and Σ is regular over k, it follows that T_d is regular over k. Since $x+cy$ is linearly disjoint over k with respect to T_d, the field

$$\Sigma_d = T_d(y) = T_d(x+cy)$$

is regular over $k(x+cy)$. As we have $\Sigma_c = \Sigma_d$, this proves that $T_c = k(x+cy)$ and so $k(x+cy)$ is algebraically closed in Σ. Therefore our assertion is completely proved.

§ 3. Theorem of Bertini for Pencils.

Let V^r or V_α be as stated in § 1, and let M or M_α be its generic Point over k. Put

$$\Sigma = k(M) = k(M_\alpha).$$

Let P be a field such that $\Sigma \supset P \supset k$ and $dim_k\ P = 1$. If $\mathfrak{p}$ is a prime divisor of P over k, it decomposes in Σ in the following way to the divisors

$$\mathfrak{p} = \mathfrak{P}_1^{h_1} \cdots\cdots \mathfrak{P}_m^{h_m}$$

where each $\mathfrak{P}_i$ is of the first kind with respect to some V_α. As we have seen in § 1, there corresponds to $\mathfrak{P}_i$ on V a prime rational V-divisor Γ_i over k. To each $\mathfrak{p}$ we now associate the rational V-divisor over k of the form

$$W_{\mathfrak{p}} = \Sigma_i h_i \Gamma_i$$

and as $\mathfrak{p}$ varies in the set of all the prime divisors of P over k, the totality of $W_{\mathfrak{p}}$ is called the pencil on V defined by the field P over k.

DEFINITION. *A pencil* $\{W\}$ *defined by the field* P *is called non-composite, if* Σ *is regular over the field* P.

This definition implies that a pencil defined by the field P is non-composite, if and only if P is maximally algebraic in Σ and $P \not\subset k(\boldsymbol{M}^p)$.

If a pencil $\{\boldsymbol{W}\}$ defined by the field P is not non-composite, we shall say that it is composite (composite with a certain other pencil).

Assume that a pencil $\{\boldsymbol{W}\}$ defined by the field P is composite. This implies that Σ is not regular over P.

(i) If P is not maximally algebraic in Σ and if Σ is regular over the algebraic closure P' of P in Σ, then

$$\boldsymbol{W}_{\mathfrak{p}}=\rho_1\boldsymbol{W}'_{\mathfrak{p}'_1}+\dots+\rho_s\boldsymbol{W}'_{\mathfrak{p}'_s}$$

where $\{\mathfrak{p}\}$ is a prime divisor of P over k and

$$\mathfrak{p}=\mathfrak{p}_1'^{\rho_1}\dots\dots\mathfrak{p}_s'^{\rho_s} \quad \text{in } P' \text{ (cf. (Z), § 9)},$$

and where $\{\boldsymbol{W}'\}$ is the pencil defined on $\boldsymbol{V}$ by the field P' over k.

(ii) Assume now that Σ is not regular over the algebraic closure P' of P in Σ. In this case, k is imperfect by virtue of cor. 2, th. 2.2. Let k_1 be the smallest perfect field containing k and put

$$P'k_1=P_1', \quad \Sigma_1=\Sigma k_1.$$

Since Σ is regular over k, Σ_1 is regular over k_1. Since Σ is not regular over P' and P' is maximally algebraic in Σ, it follows that Σ is not separably generated over P'. This shows that

$$P'\subset k(\boldsymbol{M}^p). \qquad \text{(cf. cor. 1, th. 2.2).}$$

Therefore

$$P_1'\subset k_1(\boldsymbol{M}^p).$$

This shows that P_1' is not maximally algebraic in Σ_1, since k_1 is perfect.

Let $\boldsymbol{V}_1$ be a Subvariety of a projective space and such that it is derived from $\boldsymbol{V}$ by normalization with reference to k_1. Let $\{\boldsymbol{W}_1\}$ and $\{\boldsymbol{W}_2\}$ be pencils on $\boldsymbol{V}_1$, defined respectively by the field $P_1=Pk_1$ and its algebraic closure P_2 in Σ_1. Since $P_2 \supsetneqq P_1' \supset P_1$, we have, by what have been proved already above,

$$\boldsymbol{W}_{1\mathfrak{p}_1}=\rho_1\boldsymbol{W}_{2\mathfrak{p}_{21}}+\dots+\rho_s\boldsymbol{W}_{2\mathfrak{p}_{2s}}$$

THEOREM 3.1 (*Theorem of Bertini for pencils*). *If a pencil* $\{\boldsymbol{W}\}$, *free from fixed components, is non-composite, then all but a* *finite number of members of the pencil are irreducible.*

PROOF. For some representative V_α of V we take the auxiliary fields Σ_α^*, P_α^* and k_α^*; for brevity let us omit to write the suffix α; i. e.

$$\Sigma^*=\Sigma(u),\ P^*=P(u,\eta) \text{ and } k^*=k(u,\eta),$$

then $dim_{k^*}P^*=1$. By th. 2.3, Σ^* is a separably algebraic extension of P^*, and Σ^* is separably generated over k^*, since by our assumption, Σ is regular over k.

Let ζ be a primitive element of Σ^* with respect to P^*, and let

$$F(u,\eta;\zeta)=0 \text{ with } F(U,Y;Z)\ \epsilon P[U,Y,Z]$$

be an irreducible equation for ζ over P, of degree ν, where $\nu=[\Sigma^*:P^*]$.

Since Σ is regular over P and since (u) is a set of independent variables over Σ, Σ^* is regular over P. It follows that F is absolutely irreducible.

Let $\Phi(u,\eta;\zeta)$ be the polynomial with general coefficients a, b,... of the same degree as F, and let $a_0, b_0,\ldots$, be the corresponding coefficients of F, i. e. $a_0, b_0,\ldots$, are all in P. Then there exists a finite number of finite sets of polynomials in a, b,... with rational coefficients, say

$$\{G_{i_1}(a,b\ldots),\ldots,G_{ij(i)}(a,b,\ldots)\}\quad i=1,2,\ldots,\rho$$

with the following property; if $F'(u,\eta;\zeta)$ is a polynomial with coefficients $a',b',\ldots$ in a certain field T, a necessary and sufficient condition that F' be reducible in some extension of T is that, for some i,

$$G_{ij}(a',b',\ldots)=0 \text{ for every } j.$$

Since F is absolutely irreducible, at least one of

$$G_{ij}(a_0,b_0,\ldots)\qquad 1\leqq i\leqq j(i)$$

is not zero for every i.

There exists only a finite number of prime divisors in P over k, which maps at least one of $a_0, b_0,\ldots$ to ∞. If we avoid moreover, a finite number ot prime divisors in P over k, it follows that

$$G_{ij}(a_0,b_0,\ldots)\quad 1\leqq i\leqq j(i)$$

will not be mapped all to zero for every i.

Let $\mathfrak{p}$ be one of such a prime divisor of P over k, and $\mathfrak{p}^*$ be its uniquely determined extension in P^* over k^*.

Let

$$\mathfrak{p}^* = \mathfrak{P}_1^{*h_1} \dots \mathfrak{P}_m^{*h_m}$$

be its decomposition in Σ^*. Take one of its components, say $\mathfrak{P}_1^*$ to which the prime divisor $\mathfrak{P}_1$ in Σ corresponds. Denote by $(-)$ the $\mathfrak{P}^*$-residues, where $\mathfrak{P}_1^* = \mathfrak{P}^*$.

If Δ is the residue field of $\mathfrak{p}$, then $\Delta(u, \eta)$ is the residue field of $\mathfrak{p}^*$, by the condition that $\mathfrak{p}^*$-residues of u, η are $\neq \infty$ and are algebraically independent over Δ.

By Δ^* we shall mean the residue field of $\mathfrak{P}^*$. Then by what we have already seen above,

$$\bar{F}(u, \eta; \bar{\zeta}) = 0$$

is absolutely irreducible and in particular irreducible over Δ.

If the additional finite number of prime divisors of P over k is avoided, the degree of $\bar{F}$ in $\bar{\zeta}$ remains ν. Assume that $\mathfrak{p}$ is already chosen in such a way. Then we may see easily

$$[\Delta^* : \Delta(u, \eta)] = \nu.$$

By the formula (1), we conclude that

$$\mathfrak{p}^* = \mathfrak{P}^* \qquad \text{i.e.} \qquad \mathfrak{p} = \mathfrak{P}.$$

This proves that $W_{\mathfrak{p}}$ is prime rational over k and our theorem is therefore proved.

THEOREM 3.2 *If a pencil* $\{W\}$, *defined by the field* P *over* k, *with infinitely many elements, is non-composite and free from fixed components, then a member* $W_{\mathfrak{p}}$ *of* $\{W\}$ *is absolutely reducible if the degree of that prime divisor* $\mathfrak{p}$ *is greater than* 1. *While with a finite number of exceptions, all* $W_{\mathfrak{p}}$ *corresponding to* $\mathfrak{p}$ *whose degree are* 1 *are absolutely irreducible.*

PROOF. Let $W = \sum_i h_i \Gamma_i$ be a member of $\{W\}$, corresponding to the prime divisor $\mathfrak{p}$ of degree >1. We shall show that even when $h_1 = 1$, $h_2 = \dots = h_m = 0$, Γ_1 is absolutely reducible.

The field Δ' of rational functions on Γ_1 is the residue field of the prime divisor defined by Γ_1. Since Δ' contains the residue field of the prime divisor $\mathfrak{p}$, which is a proper algebraic extension of k, therefore Δ' is not regular over k. Thus Γ_1 is absolutely reducible.

Now assume that the degree of $\mathfrak{p}$ is 1. Using the notations in the proof of th. 3.2, $\Delta=k$ and $\Delta(u,\eta)=k^*$ since the degree of $\mathfrak{p}$ is 1. Therefore avoiding a finite number of prime divisors in P over k, Δ^* is regular over k, since $\bar{F}(u,\eta;\bar{\zeta})$ is absolutely irreducible. But since the residue field Δ^* of $\mathfrak{P}^*$ contains the residue field of $\mathfrak{P}$, it follows that the residue field of $\mathfrak{P}$ is regular over k. This shows that Γ_1 is absolutely irreducible. q. e. d.

THEOREM 3.3 *Let V be defined and normal over a finite field k with $q=p^n$ elements. Let $\{W\}$ be a pencil defined on V by the field P over k, and assume that it is non-composite and has no fixed components. Then there exists a field k' which is an algebraic extension of k, having the property; there exists at least one element in the pencil defined by Pk' such that it is absolutely irreducible.*

PROOF. In this case, V has no multiple Subvariety of dimension $r-1$ by F-Appendix II prop. 2, since k is perfect. Therefore V is absolutely normal by F-Appendix II, prop. 5.

Let $\bar{k}$ be the algebraic closure of k. By our assumption, Σ is regular over P. Therefore $\Sigma\bar{k}$ is regular over $\bar{k}$. By th. 3.1, and th. 3.2, there exist infinitely many $W_{\bar{\mathfrak{p}}}$ in the pencil defined by $P\bar{k}$ over $\bar{k}$ such that it is irreducible over $\bar{k}$.

As $\Sigma\supset P\supset k$ and Σ is regular over k, it follows that P is regular over k. Therefore there exists a set of quantities $(x, y_1,\dots, y_m)$ in P such that $P=k(x,y)$ and the ring $k[x,y]$ is integrally closed in P, where x is a variable over k. Then $P\bar{k}=\bar{k}(x,y)$ and the ring $\bar{k}[x,y]$ is integrally closed in $P\bar{k}$ by F-Appendix II, prop. 5, since P is a regular extension of a perfect field k. We may assume, without loss of generality, that (x,y) has a finite $\bar{\mathfrak{p}}$-residue $(c, c_1,\dots, c_m)$, where c and c_i are in $\bar{k}$.

Consider the field $k'=k(c, c_1,\dots, c_m)$, then k' is an extension of k and the ring $k'[x,y]$ is integrally closed in Pk'. (F-Appendix II, prop. 5). Let $\mathfrak{p}'$ be the prime divisor which $\bar{\mathfrak{p}}$ induces in Pk'. Then it is clear that $\mathfrak{p}'$ is of degree 1 and as $W_{\bar{\mathfrak{p}}}$ is irreducible over $\bar{k}$, $W'_{\mathfrak{p}'}$ which corresponds to $\mathfrak{p}'$ in the pencil defined on V by Pk' is absolutely irreducible. q. e. d.

§ 4. Theorem of Bertini for linear systems.

Let 1, $t_1,\dots, t_r$ be a set of functions on V defined over k, and assume that it is linearly independent over k. For arbitrary constants $c_0,\dots, c_r$ in k, not all zero, let

$$(c_0+c_1t_1+\ldots+c_rt_r)=\mathfrak{A}_{(c)}/\mathfrak{A}_{(\infty)}$$

where $\mathfrak{A}_{(c)}$ and $\mathfrak{A}_{(\infty)}$ are integral divisors of the first kind in Σ over k with respect to V and where $\mathfrak{A}_{(\infty)}$ is independent of the choice of the c's. As in § 3, $\mathfrak{A}_{(c)}$ determines on V a rational V-divisor over k of the form

$$W_{(c)}=\sum_i h_i\Gamma_i$$

As (c) varies freely in k, the totality of $W_{(c)}$ is called the linear system of dimension r defined on V by $(1, t_1,\ldots, t_r)$ over k. If $r=1$, it is called the linear pencil.

DEFINITION. *A linear system* $|W|$ *is said to be composite with a pencil* $\{Z\}$ *if each member of* $|W|$ *decomposes into a certain number of members of* $\{Z\}$.

Let a linear system $|W|$ be defined on V, by the set of functions $(1, t_1,\ldots, t_r)$ in Σ. Let k' be a field containing k and algebraic over k. Let V' be a Subvariety of a projective space and such that it is derived from V by normalization with reference to k'. If a Point M' of V is generic over k', then $k'(M')=k'(M)$ by the definition of normalization. When V is without multiple Subvariety of dimension $r-1$ (this is the case in particular when k is perfect), then it is also such over k', and therefore we may assume that $M=M'$, $V=V'$ (cf. F-Appendix II, prop. 5).

There exists the uniquely determined integer $e\geq 0$ such that we have for every i, $t_i\epsilon k(M^{p^e})$ but for certain j, $t_j\bar{\epsilon} k(M^{p^{e+1}})$. Put $k'=k^{p^{-e}}$ and let t'_i be a function in $k'(M')$ defined by $(t'_i)^{p^e}=t_i$. Then (t') has the following property: $t'_j\bar{\epsilon}k'(M^p)=k'(M'^p)$ for certain j.

DEFINITION. *A linear system* $|W|$ *defined by* $(1, t_1,\ldots, t_r)$ *on* V *over* k *is said to be reducible, if each member of the linear system defined on* V' *by* $(1, t_1', t_2',\ldots, t'_r)$ *over* k' *is reducible over* k', *when* k *has infinitely many elements. When* k *is a finite field, then* $|W|$ *is said to be reducible if the linear system defined by* $(1, t_1',\ldots, t_r')$ *on* V *is reducible over the algebraic closure* $\bar{k}$ *of* k.

THEOREM 4.1 *Let* $|W|$ *be a linear system defined by the functions* $(1, t_1,\ldots, t_r)$ *over* k *and free from fixed components. If* $r>1$ *and* $dim_k(t)=1$, *then* $|W|$ *is composite with a pencil.*

We omit the proof. Cf. (Z) § 14.

THEOREM 4.2 (*Theorem of Bertini for linear systems*). *Let* $|W|$ *be a linear system defined over a field* k, *which is free from fixed components. Then if* $|W|$ *is absolutely reducible,* $|W|$ *is* *composite with a peucil.*

PROOF. Let $|W|$ be defined by the functions $(1, t_1, \dots, t_r)$ where each t is an element of Σ. We may assume, by the definition of the reducibility that $t_i \in k(M)$ for all i, but $t_j \bar{\in} k(M^p)$ for certain j[7]. After applying on (t) linear homogeneous transformations, we may also assume that $t_1 \bar{\in} k(M^p)$ and $(t_1) = \mathfrak{A}_{(0)}/\mathfrak{A}_{(\infty)}$.

(i) k is not a finite field. Consider the linear pencil $\{W\}$, defined over k by $(1, t_1 + ct_i)$ contained in the linear system. Then but for a finite number of constants c in k, it is free from fixed components. Therefore it coincides with the pencil determined by $k(t_1 + ct_i)$, but it contains only those which correspond to prime divisors whose degrees are 1. Since $|W|$ is absolutely reducible, members of $\{W\}$ are also absolutely reducible, and consequently, by th. 3.1 and th. 3.2, $\Sigma = k(M)$ cannot be regular over $k(t_1 + ct_i)$. But by th. 2.4, since $t_1 \bar{\in} k(M^p)$ for non special constants c in k, Σ is regular over $k(t_1 + ct_i)$ if $dim_k\ (t_1, t_i) = 2$. Therefore t_i must be algebraic over $k(t_1)$, i.e. $dim_k\ (t) = 1$. Our theorem then follows from th. 4.1.

(ii) k is a finite field. Replacing k by $\bar{k}$ in (i), we conclude that $dim_{\bar{k}}\ (t) = 1$ and a fortiori $dim_k\ (t) = 1$. Our theorem then follows also from th. 4.1. q. e. d.

7) When the characteristic p of k is zero, considerations whether t is in $k(M^p)$ or not is not necessary.