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On Generalized Spaces which admit given Holonomy Groups.

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Introduction

The properties of generalized spaces (in the sense of E. Cartan) admitting various holonomy groups have been investigated mainly in concrete examples by many authors. The present purpose is to establish, by applying the Cartan's theory of continuous groups, a general theory which includes their results as its special cases or which may avail to treat systematically concrete cases.

In order to attain this, it will be necessary to extend the *fundamental theorem* of holonomy groups which demands us to take a very restricted *moving frame of reference* (repère mobile).

The first section will be concerned with the equations which are to be satisfied by Pfaffian forms of connection in the extended fundamental theorem. In the second section we shall mainly deal with the structure of the space with general connection whose holonomy group is intransitive or imprimitive.

I. Ricci's Relations of a subgroup.

1. Ricci's families. Let $\mathfrak{G}=\{T_a\}$ be a transitive Lie group of transformations which have been defined on a Klein space $\boldsymbol{E}^n$ of points ξ^i. Let $a^1,\dots,a^r$ be the parameters of $\mathfrak{G}$ and $T_0=T_{a=0}$ the identical transformation.

Let R_a be a moving frame of $\mathfrak{G}$ and $\omega^1(a,da),\dots,\omega^r(a,da)$ its relative components: the symbol of an infinitesimal transformation $T_a^{-1}T_{a+da}$ can be written as

$$\omega^s(a,da)X_s,$$

where $(X_1,\dots,X_r)$ is a fixed set of r independent operators of $\mathfrak{G}$ taken so that $X_{n+1},\dots,X_r$ may generate the subgroup which fixes the origin of the frame R_0.

Let $\mathfrak{H}=\{T_u\}$ be a subgroup of $\mathfrak{G}$ depending upon ρ parameters $u^1,\dots,u^\rho(\rho<r)$ such that $T_{u=0}=T_0$, and let $\bar{\omega}^1(u,du),\dots,\bar{\omega}^r(u;du)$ denote the relative components of the family $\mathfrak{H}R_0$ of frames. When we express by

$$l_p^m\bar{\omega}^p=0 \quad (m=1,\dots,r-\rho;\ p=1,\dots,r) \tag{1}$$

the equations which are to be satisfied by $\bar{\omega}^p$, the subgroup $\mathfrak{H}$ can be characterized by the constants l_p^m having certain properties.

DEFINITION. A continuous subfamily $\mathfrak{F}$ of $\{R_a\}$ is called a *Ricci's family* of $\mathfrak{H}$ if $T_u\mathfrak{F}=\mathfrak{F}$ for every $T_u\in\mathfrak{H}$, and a system of equations which characterizes the relative components of $\mathfrak{F}$ is called a *Ricci's relation* of $\mathfrak{H}$ or of $\mathfrak{F}$.

We assume that $\mathfrak{F}$ depend upon $\rho+k$ parameters.

For a transfomation $T\in\mathfrak{G}$, $TR_0\in\mathfrak{F}$ implies that $\mathfrak{H}TR_0\subset\mathfrak{F}$. Hence we can write as

$$\mathfrak{F}=\sum_y\mathfrak{H}T_yR_0,$$

where $\{T_y\}$ is a family of transformations of $\mathfrak{G}$ depending upon k parameters $y^1,\dots,y^k$.

We may always suppose that $R_0\in\mathfrak{F}$ and $T_{y=0}=T_0$ without loss of generality.

Since

$$(T_uT_y)^{-1}(T_{u+du}T_{y+dy})=T_y^{-1}(T_u^{-1}T_{u+du})T_y(T_y^{-1}T_{y+dy}),$$

denoting by ω^s and by $\omega^{*s}(y\,dy)$ the relative components of $\mathfrak{F}$ and $\{T_y\}$ respectively, we obtain the equations

$$\omega^s=a_p^s(y)\bar{\omega}^p+\omega^{*s}(y,dy),\ l_p^m\bar{\omega}^p=0 \tag{2}$$

as a Ricci's relation of $\mathfrak{H}$, where $\|a_p^s\|$ is a matrix of the linear adjoint group of $\mathfrak{G}$ corresponding to T_y^{-1}.

The relation (2) may be regarded as $r-\rho$ independent linear homogeneous equations of $r+k$ indetrminates ω^s, dy^j with coefficients of functions of y^j.

Each coset $\mathfrak{H}T_yR_0$ of $\mathfrak{F}$ is invariant under every transformation of $\mathfrak{H}$, and so the parameters y^j which determine a coset of $\mathfrak{F}$ will be called the *invariants of the Ricci's relation.* They may be regarded as functions defined on $\mathfrak{F}$.

2. Ricci's relations. Taking into account the Lie's equations of structure, we can easily verify that $a_p^s(y)$, $\omega^{*s}(y, dy)$ in (2) satisfy the equations

$$da_p^s + C_{qt}^s a_p^q \omega^{*t} = 0, \quad a_p^s(0) = \delta_p^s. \tag{3}$$

Conversely, we have the following:

LEMMA 1. *Let $\omega^{*s}(y, dy)$ be r Pfaffian forms of $y^1, \dots, y^k$ satisfying the Cartan's equations of structure and $a_p^s(y)$ be r^2 functions of y^j satisfying* (3). *Then* (2) *is a Ricci's relation of $\mathfrak{F}$, if ω^s are $(\rho+k)$-independent in* (2).

In fact: since $\omega^{*s}(y, dy)$ satisfy the Cartan's equations, we have a unique k-parameter family $\{T_y\}$ whose relative components are $\omega^{*s}(y, dy)$ and such that $T_{y=0} = T_0$. The Cartan's equations of ω^{*s} and the wellknown relations between the constants of structure C_{pq}^s assure immediately that a system of Pfaffian equations

$$da_p^s + C_{qt}^s a_p^q \omega^{*t} = 0$$

is completely integrable in the (r^2+k)-dimensional space (a_p^s, y^j). Hence it has a unique solution $a_p^s(y)$ reducing to δ_p^s at $y^j = 0$, and so $\| a_p^s(y) \|$ coincides with the matrix of the linear adjoint group of $\mathfrak{G}$ corresponding to T_y^{-1}. Moreover the $(\rho+k)$-independence of ω^s asserts that $T_y T_{y'}^{-1} \bar{\in} \mathfrak{H}$ for $(y'^j) \neq (y^j)$. (2) is therefore a Ricci's relation of $\sum\limits_y \mathfrak{H} T_y R_0$.

Let us give up the last condition in Lemma 1, and suppose that ω^s are $(\rho+k')$-independent in (2), $(k' < k)$.
Changing parameters y^j conveniently if necessary, we can still make ω^s $(\rho+k')$-independent even after setting $y^{k'+1} \equiv 0, \dots, y^k \equiv 0$ in (2). Thus we may have a Ricci's relation whose k' invariants are $y^1, \dots, y^{k'}$.

In this case, to have set $y^{k'+1} \equiv 0, \dots, y^k \equiv 0$ means to have taken only one transformation $T_{\bar{y}}$ belonging to each family $\{T_y\} \cap \mathfrak{H} T_y$, and to construct a Ricci's relation of the $(\rho+k')$-parameter family $\sum\limits_{\bar{y}} \mathfrak{H} T_{\bar{y}} R_0$.

This fact may furnish us an actual process to construct a relation of the least Ricci's family including a given family $\{T_y R_0\}$.

3. A geometrical interpretation. Let us now consider a class of objects Ω, on which $\mathfrak{G}$ operates transitively, equivalent to the class of integral varieties of $l_p^m \omega^p(a, da) = 0$, and let $g_1, \dots, g_{r-\rho}$

denote the components of objects Ω.

On the transformation group $\mathfrak{G}(\Omega)$, we can express the symbol of $T_a^{-1}T_{a+da}$ by the form

$$\omega^s(a, da)\,Y_s\Big(g, \frac{\partial}{\partial g}\Big)$$

where ω^s are the relative components of $\mathfrak{G}$. The operators Y_s are not always r-independent. Let Ω_0 be the object such that $T_u\Omega_0=\Omega_0$ for every $T_u\in\mathfrak{H}$, and let g_m, g_m+dg_m denote its components referring to the two coordinate systems defined by frames R_a, R_{a+da} respectively. Then we have

$$dg_m+\omega^s(a, da)\,Y_s g_m=0.$$

On a Ricci's family $\mathfrak{F}$ of $\mathfrak{H}$, g_m are functins of $y^1,\dots,y^k$ merely, furthermore the rank of the matrix

$$\left\| \frac{\partial g_m}{\partial y^j} \right\|$$

is equal tô k. Hence

$$\frac{\partial g_m}{\partial y^j}dy^j+\omega^s Y_s g_m(y)=0.$$

is nothing else but a Rieci's relation of $\mathfrak{F}$.

Thus we can also obtain a Ricci's relation from knowing Y_s and $g_m(y)$.

4. The case that $\mathfrak{H}$ is intransitive. In particular, let us assume that $\mathfrak{H}$ is intransitive on E^n and has l invariants. For a transformation $T_y\in\mathfrak{G}$, setting $\tilde{\omega}^s=a^s_\nu(y)\,\tilde{\omega}^\nu$, we may see that $\tilde{\omega}^1,\dots,\tilde{\omega}^n$ are generally $(n-l)$-independent. Let T_y be one of the transformations which transform the origin A_0 of the frame R_0 to an arbitrary point $A_y\in E^n$. Then, among the relative components $\omega^{*s}(y, dy)$ of the family $\{T_y\}$ depending upon n parameters $y^1,\dots,y^n$, the first n components $\omega^{*1},\dots,\omega^{*n}$ are independent. Let us represent a relation of the least Ricci's family including $\{T_yR_0\}$ by the formulae

$$\omega^s=a^s_\nu(y^j)\,\tilde{\omega}^\nu+\omega^{*s}(y^j, dy^j),\;\; l^m_\nu\tilde{\omega}^\nu=0\;\; (j=1,\dots,k\leqq n). \tag{2}$$

Then $\omega^1,\dots,\omega^n$ are linearly independent in (2).

Changing parameters y^j conveniently if necessary, we can still make $\omega^1,\dots,\omega^n$ independent even after setting $y^{l+1}\equiv 0,\dots,y^k\equiv 0$ in (2).

Consequently we have:

LEMMA 2. *If $\mathfrak{H}$ is an intransitive subgroup with l invariants on $\boldsymbol{E}^n$, we can gėt a Ricci's relatin* (2) *fulfilling the following conditions*:

(i) $\omega^1,\dots,\omega^n$ *are* (*generally*) *independent,*

(ii) $k=l$.

Interpreting these conditions geometrically, we have:

LEMMA 2′. *There exists a Ricci's family $\mathfrak{F}$ which contains* (at least in the local sense) *a frame whose origin is an arbitrary point of $\boldsymbol{E}^n$, and the invariants $y^1,\dots,y^l$ of $\mathfrak{F}$ are functions of ξ^i and are indeed the invariants of $\mathfrak{H}$ on $\boldsymbol{E}^n$.*

Let S^{n-l} denote a surface defined by $y^j=$ const. $(j=1,\dots,l)$. Then $T_uS^{n-l}=S^{n-l}$ for every $T_u\in\mathfrak{H}$, and $\mathfrak{H}$ operates transitively on the points of S^{n-l}. Hence, in the Klein space $\boldsymbol{E}^n(\mathfrak{H})$, S^{n-l} is an exceptional surface on which all invariants of contact are constant. The Frenet's formulae of S^{n-l} are therefore relations between $\omega^1,\dots,\omega^r$ with coefficients of functions merely of y^j.

Let $\mathfrak{R}$ denote the family of all Frenet's frames of S^{n-l}. We can take $\mathfrak{F}$ such that $\mathfrak{R}\supset\mathfrak{H}T_yR_0$.

It follows that:

LEMMA 3. *We can take the relation* (2) *in Lemma* 2 *fulfilling the further conditions*:

(iii) $a^s_p\bar{\omega}\,(s=1,\dots,r)$ *satisfy the Frenet's formulae of S^{n-l},*

(iv) *some of $y^1,\dots,y^l$ are the invariants of contact of S^{n-l}.*

REMARK. The similar results hold when we consider a neighbourhood U of a point A_0 such that $\mathfrak{H}A_0$ spreads an $(n-l)$-dimensional variety in $\boldsymbol{E}^n$ and $dim\mathfrak{H}A\geqq n-l$ for any $A\in U$.

II. The space with $\mathfrak{G}$-connection.

5. Pfaffian forms of connection. Let us now consider a space $\boldsymbol{R}^n$ of points x^i with $\mathfrak{G}$-connection.

We associate with each point $A(x^i)\in\boldsymbol{R}^n$ a frame R_A in the tangent space $\boldsymbol{E}_A$. R_A may define a coordinate system (ξ^i) to which $\boldsymbol{E}_A$ refers.

If we connect $\boldsymbol{E}_{A'(x+dx)}$ with $\boldsymbol{E}_{A(x)}$, a point $\xi'^i=\xi^i+d\xi^i\in\boldsymbol{E}_{A'}$ coincides with a point $\xi^i\in\boldsymbol{E}_A$. Then we have the formula

$$d\xi^i + \omega^s(x, dx) X_s \xi^i = 0,$$

where $\omega^s(x, dx)$ are Pfaffian forms of connection.

An infinitesimal transformation associated with an infinitesimal cycle of $\boldsymbol{R}^n$ may be given by the formula

$$(d\delta)\xi^i + \Omega^s X_s \xi^i = 0$$

with

$$\Omega^s = \omega^{s\prime} - \frac{1}{2} C^s_{pq} [\omega^p \omega^q] :$$

a point of $\boldsymbol{E}_A$ having coordinates ξ^i with respect to R_A has coordinates $\xi^i + (d\delta)\xi^i$ with respect to the developed frame $\bar{R}_A$ of R_A.

Suppose that $\boldsymbol{R}^n$ admits a holonomy group $\mathfrak{H}$. For a point $A_0 \in \boldsymbol{R}^n$, let us take a Ricci's family $\mathfrak{F}_{A_0}$ of $\mathfrak{H}$ in $\boldsymbol{E}_{A_0}$ and write its Ricci's relation in the formulae

$$\omega^s = a^s_p(y)\bar{\omega}^p + \omega^{*s}(y, dy), \quad l^{m\prime}_p \bar{\omega}^p = 0. \tag{2}$$

Let C be a curve joining A_0 to any other point $A \in \boldsymbol{R}^n$ and let $\mathfrak{F}_A$ denote the developed family of $\mathfrak{F}_{A_0}$ along C into $\boldsymbol{E}_A$. Then $\mathfrak{F}_A$ is also a Ricci's family whose relation is given by (2). Moreover $\mathfrak{F}_A$ does not depend on C, as $\mathfrak{H}$ is the holonomy group of $\boldsymbol{R}^n$.

It follows that, in order that $\boldsymbol{R}^n$ admits a holonomy group $\mathfrak{H}$, it is necessary and sufficient that with each point $A \in \boldsymbol{R}^n$ a Ricci's family $\mathfrak{F}_A$ of $\mathfrak{H}$ in $\boldsymbol{E}_A$ can be associated so that a development along an arbitrary curve $\widehat{A_1 A_2}$ may superpose $\mathfrak{F}_{A_2}$ on $\mathfrak{F}_{A_1}$, in particular a coset $y = y_1 (=\text{const.})$ of $\mathfrak{F}_{A_2}$ on a coset $y = y_1$ of $\mathfrak{F}_{A_1}$.

Therefore, if we take the frame R_A such that $R_A \in \mathfrak{F}$, the Pfaffian forms $\omega^s(x, dt)$ of connection satisfy the relation (2): that is

$$\omega^s(x, dx) = a^s_p(y^j)\bar{\omega}^p(x, dx) + \omega^{*s}(y^j, dy^j),$$

where $\bar{\omega}^p(x, dx)$ are certain Pfaffian forms satisfying the equations

$$l^{m\prime}_p \bar{\omega}^p = 0;$$

and y^j are functions of x^i: for a point $A(x^i)$, the functions $y^j(x)$ represent the values of y^j to which the coset of $\mathfrak{F}_A$ containing R_A corresponds. Hence we have the following:

THEOREM 1. *If $\boldsymbol{R}^n$ admits a holonomy group $\mathfrak{H}$, we can take R_A so that Pfaffian forms of connection may satisfy an arbitrary Ricci's relation of $\mathfrak{H}$. In this case, the invariants y^j are functions* *of x^j, but they depend on the choice of the coset to which R_A belongs.*

Setting

$$\bar{\Omega}^s = \bar{\omega}^{s\prime} - \frac{1}{2} C^s_{pq}[\bar{\omega}^p \bar{\omega}^q],$$

we can obtain formulae

$$l^m_p \bar{\Omega}^p = 0, \quad \Omega^s = a^s_p \bar{\Omega}^p,$$

that is:

THEOREM 2. *Ω^s satisfy the Ricci's relation* (2) *in which we set $dy^j = 0$.*

6. Intransitive holonomy groups. When the holonomy group $\mathfrak{H}$ is intransitive on $\boldsymbol{E}_A$, we can take $\mathfrak{F}_A$ satisfying the conditions (i), (ii) in Lemma 2 and the frame $R_A \in \mathfrak{F}_A$ whose origin is the point $A \in \boldsymbol{R}^n$. Then the coset to which R_A belongs is uniquely determined, and so we have the following result:

THEOREM 3. *If $\boldsymbol{R}^n$ admits an intransitive holonomy group with l invariants, then we can so attach to each point of $\boldsymbol{R}^n$ a frame R_A whose origin is the point of $\boldsymbol{R}^n$ that Pfaffian forms of connection may satisfy a Ricci's relation which has just l invanıants $y',\cdots, y^l$. In this case y^j are functions of x^i and are independent on the choice of the frame R_A.*

Since an arbitrary development separately superposes the cosets of one family on those of the other, we have:

COROLLARY 1. *Let A be a point on a variety defined by $y^j = y^j_1$, $(=const.)$ in $\boldsymbol{R}^n$ and A_0 a point of $\boldsymbol{R}^n$. Develop $\boldsymbol{E}_A$ on $\boldsymbol{E}_{A0}$ along a curve $\widehat{A_0 A}$. Then the developed frame $\bar{R}_A$ of R_A belongs to the coset $y^j = y^j_1$ of $\mathfrak{F}_{A0}$, and its origin $\bar{A}$ is on the surface $y^j = y^j_1$ of $\boldsymbol{E}_{A0}$.*

DEFINITION. Let V^λ be a λ-dimensional variety in $\boldsymbol{R}^n$. Suppose that, in every $\boldsymbol{E}_A$, $A \in V^\lambda$, there exists a surface $S^\lambda_A \ni A$, and that the development along an arbitrary curve $\widehat{AA'} \subset V^\lambda$ superposes $S^\lambda_{A'}$ on S^λ_A. Then we call S^λ_A an *image* of V^λ and V^λ the *inverse image* of S^λ_A.

From the definition we have:

COROLLARY 2. *In Theorem* 3, *a variety V^{n-l} defined by $y^j = y_1{}^j$ in $\boldsymbol{R}^n$ is the inverse image of the surface S^{n-l}_A defined by $y^j = y_1{}^j$ in $\boldsymbol{E}_A$, $A \in V^{n-l}$.*

About the geometric property of the inverse image, taking into account Lemma 3, we have:

COROLLARY 3. *Along the variety V^{n-l}, Pfaffian forms of connection of $\boldsymbol{R}^n$ can satisfy the Frenet's formulae of S^{n-l}.*

REMARK. Suppse that, in $\boldsymbol{E}_{A_0}$, A_0 is such a point as A_0 in § 4, Remark. Then the similar results hold when we consider a neighbourhood of A_0 in $\boldsymbol{R}^n$.

7. Imprimitive subgroups. We have known that, when the holonomy group $\mathfrak{H}$ is intransitive, $\boldsymbol{R}^n$ is generated by ∞^l varieties V^{n-l} which are inverse images of the invariant surfaces S^{n-l} of $\mathfrak{H}$ in $\boldsymbol{E}_A$.

Now we are going to prove that this circumustance occurs in the case that $\mathfrak{H}$ is imprimitive and $\boldsymbol{R}^n$ is without torsion.

Let us assume that $\mathfrak{H}$ is imprimitive on $\boldsymbol{E}^n$: the whole space $\boldsymbol{E}^n$ is filled up by ∞^μ surfaces $M^{n-\mu}$ having (locally) the property that no two of them can have common point without being identical; and $\mathfrak{H}$ can also be regarded as a group of transformations operating (generally intransitively) on the class of objects $M^{n-\mu}$.

We suppose that $\mathfrak{H}$ is intransitive on $\boldsymbol{E}^n$ and has l invariants, $(0 \leqq l \leqq n)$; let $\mathfrak{F}$ be the Ricci's family that we had considered in Lemma 2 and let $b^1, \dots, b^{\rho+l}$ be parameters of $\mathfrak{F}$.

Let $\mathfrak{M}$ denote a family consisting of the frames of $\mathfrak{F}$ whoseo rigins are on a same surface $M^{n-\mu}$. Then two classes of objects $M^{n-\mu}$ and $\mathfrak{M}$, on which $\mathfrak{H}$ operates, are equivalent.

Let $\mathfrak{P}$ denote a subfamily of $\mathfrak{M}$ whose frames have a same point $P \epsilon \boldsymbol{E}^n$ as their origins. Then two classes of $\mathfrak{P}$ and P, on which $\mathfrak{H}$ operates, are equivalent. And so the object $\mathfrak{P}$ are characterized by the equations $\omega^1=0, \dots, \omega^n=0$ on $\mathfrak{F}$.

Since each $\mathfrak{M}$ is generated by $\mathfrak{P}$, the objects $\mathfrak{M}$ are characterized by linear homogemeus equations of $\omega^1, \dots, \omega^n$. And their coefficients may be functions of y^j; because the equations in which we set $y^j = y^j_1 (=\text{const.})$ characterize the class of objects $\mathfrak{M}_{y1}$ which is equivalent to the class of $M^{n-\mu} \cap S^{n-l}_{y1}$ and $\mathfrak{H}$ is transitive on it, therefore the equations which characterize $\mathfrak{M}_{y1}$ may be of constant coefficients.

From the definition of our relations of Lemma 2, the first n

equations of (2) for the unknowns dy^j have a unique solution written as

$$dy^j = \tau^j_k(y)\,\omega^k \quad (j=1,\dots,l;\ k=1,\dots,n). \tag{4}$$

We can now take the forms

$$\pi^p = \lambda^p_q(y)\,\omega^q \quad (p,q=1,\dots,r)$$

with the properties:

(i) $|\lambda^p_q| \neq 0,\ |\lambda^i_k| \neq 0,\ \lambda^i_Q \equiv 0\ (i,k=1,\dots,n;\ Q=n+1,\dots,r)$,

(ii) the relations (2) from which $\bar{\omega}^s$ are eliminated are (4) and

$$\pi^{\rho+l+1}=0,\dots,\pi^r=0, \tag{5}$$

(iii) the equations which characterize the relative components of $\mathfrak{M}$ are expressed by (4), (5) and

$$\pi^1=0,\dots,\pi^\mu=0. \tag{6}$$

If ω^s are the relative components $\omega^s(b,db)$ of $\mathfrak{F}$, then $\pi^1,\dots,\pi^{\rho+l}$ are indepent, $\pi^{\rho+l+1}=0,\dots,\pi^r=0$, and from the imprimitivity of $\mathfrak{H}$ the system of equations (6) is completely integrable on $\mathfrak{F}$. Hence, setting

$$\begin{aligned}\pi^{a\prime} &= \frac{\partial\lambda^a_i}{\partial y^j}\tau^j_k[\omega^k\omega^i] + \frac{1}{2}\lambda^a_k C^k_{pq}[\omega^p\omega^q] \\ &\equiv \frac{1}{2}\gamma^a_{pq}(y)[\pi^p\pi^q] \quad (a=1,\dots,\mu)\end{aligned}$$

with

$$\gamma^a_{pq} + \gamma^a_{pq} = 0,$$

we have

$$\gamma^a_{\sigma\tau} \equiv 0 \quad (a=1,\dots,\mu;\ \sigma,\tau=\mu+1,\dots,\rho+l). \tag{7}$$

8. Imprimitive holonomoy groups. We are now in a position to consider $\boldsymbol{R}^n$ which admits the holonomy group $\mathfrak{H}$. We shall denote by M_A the surface $M^{n-\mu}$ such that $A \in M^{n-\mu} \subset \boldsymbol{E}_A$ and by $\mathfrak{M}_A$ the family $\mathfrak{M}$ such that $R_A \in \mathfrak{M} \subset \mathfrak{F}_A$.

If Γ is a geometric object in a tangent space, then the developed object of Γ will be written as $\bar{\Gamma}$.

The Pfaffian forms $\omega^s(x,dx)$ of connection may be satisfy the relations

$$\pi^{\rho+l+1}=0,\dots,\pi^{r}=0$$

where $y^j=y^j(x)$ such that $dy^j=\tau^j_k(y)\omega^k$, and R_A may be a frame whose origin is the point $A\in\boldsymbol{R}^n$.

Consider a system of differential equations

$$\pi^1=0,\dots,\pi^\mu=0. \qquad (6)$$

On account of the condition that $\boldsymbol{R}^n$ is without torsion, say

$$\Omega^1=0,\dots,\Omega^n=0,$$

we have

$$\pi^{a\prime}=\frac{1}{2}\gamma^a_{\alpha\beta}[\pi^\alpha\pi^\beta] \quad (a=1,\dots,\mu;\ \alpha,\beta=1,\dots,\rho+l),$$

and from

$$\gamma^a_{\sigma\tau}\equiv 0 \quad (a=1,\dots,\mu;\ \sigma,\tau=\mu+1,\dots,\rho+l) \qquad (7)$$

we have

$$\pi^{a\prime}=\frac{1}{2}\gamma^a_{bc}[\pi^b\pi^c]+\gamma^a_{b\sigma}[\pi^b\pi^\sigma] \quad (a,b,c=1,\dots,\mu;\ \sigma=\mu+1,\dots,\rho+l).$$

The system (6) is therefore completely integrable, and $\boldsymbol{R}^n$ is filled up by its ∞^μ integral varieties $L^{n-\mu}$. We can easily see that these varieties $L^{n-\mu}$ depend on neither λ^p_q nor R_A.

Let

$$x^i=x^i(t) \quad (0\leqq t\leqq 1)$$

be a curve C lying on a variety $L^{n-\mu}$, and let $h^s(t)dt$ denote the Pfaffian forms of connection along C. We denote by A, A' the points $x^i(0)$, $x^i(1)$ respectively.

If $\omega^s(b,db)$ denote the relative components of $\mathfrak{F}_A=\{R_b\}$, a system of differential equations

$$\omega^s(b,db)=h^s(t)dt \qquad (8)$$

has a unique solution $b^\alpha(t)\ (\alpha=1,\dots,\rho+l)$ such that $R_{b(0)}=R_A$. Then

$$R_{b(1)}=\bar{R}_{A'} \quad (\text{along } C).$$

Since $\omega^s=h^s(t)dt$ satisfy the relations

$$\pi^1=0,\dots,\pi^\mu=0,$$

we have

$$\bar{R}_{A'} \in \mathfrak{M}_A,$$

and since $\bar{A}'$ is the origin of $\bar{R}_{A'}$, we have

$$\bar{A}' \in M_A$$

and

$$\bar{M}_{A'} = M_A.$$

This proves that M_A is the image of $L^{n-\mu}$ passing through A. Consequently we have the following result:

THEOREM 4. *Suppose that the holonomy group* $\mathfrak{H}$ *of* $\boldsymbol{R}^n$ *without torsion is imprimitive:* $\boldsymbol{E}_A$ *is filled up by* ∞^μ *surfaces* $M^{n-\mu}$ *and* $\mathfrak{H}$ *may also be regarded as the group* $\mathfrak{H}(M^{n-\mu})$ *operating on the class of objects* $M^{n-\mu}$. *Then* $\boldsymbol{R}^n$ *is filled up by* ∞^μ *varieties* $L^{n-\mu}$ *having* (locally) *the property that no two of them can have common point without being identical, and each* $L^{n-\mu}$ *is the inverse image of* $M^{n-\mu}$.

9. Spaces with $\mathfrak{H}(M^{n-\mu})$-connection. Furthermore, let us assume that a transformation associated with an arbitrary closed curve lying on a $L^{n-\mu}$ leaves invariant all $M^{n-\mu}$ in $\boldsymbol{E}_A$. Then we can identify $\boldsymbol{E}_A$ and $\boldsymbol{E}_{A'}$, A, $A' \in L^{n-\mu}$, and we have the tangent space $\boldsymbol{E}_{L^{n-\mu}}$ at a variety $L^{n-\mu}$.

When $\mathfrak{H}(M^{n-\mu})$ is transitive, if we regard $L^{n-\mu}$, $M^{n-\mu}$ as the point in $\boldsymbol{R}^n$, $\boldsymbol{E}^n_A$ respectively, the given connection induces a μ-dimensional space $\boldsymbol{R}^\mu$ of points $L^{n-\mu}$ with $\mathfrak{H}(M^{n-\mu})$-connection.

When $\mathfrak{H}(M^{n-\mu})$ is intransitive, let V^λ denote the inverse image of $S^\lambda_A = \mathfrak{H} M^{n-\mu}_A$, then we have a $(\lambda - n + \mu)$-dimensional space $V^{\lambda-n+\mu}$ with $\mathfrak{H}(M^{n-\mu})$-connection, by attending only to V^λ in $\boldsymbol{R}^n$.

In such cases, the study of spaces with a given connection can be reduced to that of spaces with a different connection.

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