Memoirs OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES, A
Vol. XXVIII, Mathematics No. 1, 1953.

On the Whitney Characteristic classes of the Normal Bundle

By

Toshikazu YAGYU

(Recieved April 20, 1953)

1. It is the aim of this paper to establish a generalization of Chern's formula for the invariant of Whitney ([2], § 4.), that is, to obtain the integral formula of the Whitney characteristic class of the normal bundle. We use the following notations.

R^{n+N}; $(n+N)$-dimensional orientable Riemannian manifold of the class $\geqq 3$.

M^n; n-dimensional closed orientable submanifold of the same class imbedded in R^{n+N}.

$\boldsymbol{N}^{q-1}$; Bundle of the normal $(N-q+1)$-frame to R^{n+N} over M^n.

$\boldsymbol{N}^q$; Bundle of the normal $(N-q)$-frame to R^{n+N} over M^n.

$\boldsymbol{T}^0$; Bundle of the tangent n-frame to M^n over M^n.

$\boldsymbol{B}^0$; Bundle of the tangent $(n+N)$-frame to R^{n+N} over M^n.

The q-th Whitney characteristic class of the normal bundle is the cohomology class of the obstruction $c(F)$ where F is any cross-section to over the $(q-1)$-skeleton in the cellular decomposition of M^n, ([1], $p-190$) The bundle of coefficient of $\boldsymbol{N}^{q-1}$ is the product bundle by the orientability of R^{n+N} and M^n, and the $(q-1)$-th homotopy group of the fibre $V_{N,\,N-q+1}$. of $\boldsymbol{N}^{q-1}$ is ∞ if $q-1$ is even or $N=q$, and 2 if $q-1$ is odd and $N\neq q$. Then our class is regrarded as the ordinary cohmology class with the coefficient of integer or integer mod. 2. Now, we represent $c(F)$ by the integral formula. In the special case, $N=n=q$, our formula is Chern's one.

2. Let Δ be an oriented q-cell in the cellular decomposion of M^n, $\sum$ be its oriented boundary sphere and Δ be contained in a coordinate neighborhood. By the properties of the homotopy group of Stiefel manifold $V_{N,\,N-q}$ which is the fibre of $\boldsymbol{N}^q$ ([1], $p-132$), there exists the expension E_0 of pF over Δ where p is the projection $\boldsymbol{N}^{q-1}\rightarrow\boldsymbol{N}^q$. Now, $\boldsymbol{N}^{q-1}$ being regarded as the bundle over $\boldsymbol{N}^q$, by the covering homotopy theorem ([1], p-54) there exists E_0' in $\boldsymbol{N}^{q-1}$ which is the cross-section to $\boldsymbol{N}^{q-1}$ over $\Delta - x_0$ for any fixed point $x_0 \in \Delta$ and is equal to F on $\sum$. Each element of E_0' over x_0 is the $N-q+1$-frame whose $N-q$ vectors are constantly $E_0|x_0$ and the last vector runs on the oriented unit sphere S in the normtl space at x_0, where the orientation of S is determined uniquely by the orientability of R^{n+N} and M^n for each q-cell. Thus, we obtain the mapping $\sum \to S$ and let D be the degree of this mapping. Then

$$c(F)\cdot\Delta = D, \qquad \text{if } q \text{ is odd or } q=N.$$
$$\equiv D \text{ mod. } 2, \text{ if } q \text{ is even and } q \neq N.$$

3. Let ω_i, ω_{ij} be the coefficients of the connections induced in M^n. We make the following forms.

$$\Phi_k = \sum_{i=n+N-q+2}^{n+N} \epsilon_{i_1 \cdots i_{q-1}} \Omega_{i_1 i_2} \cdots \Omega_{i_{2k-1} i_{2k}} \omega_{i_{2k+1}\, n+N-q+1} \cdots \omega_{i_{q-1}\, n+N-q+1}.$$

$$\Pi = \begin{cases} \dfrac{1}{\pi^p} \displaystyle\sum_{\lambda=0}^{p-1} (-1)^\lambda \dfrac{1}{1\cdot 3 \cdots (2p-2\lambda-1)\, 2^{p+\lambda} \lambda!} \Phi_\lambda, & \text{if } q \text{ is even } 2p. \\ \dfrac{1}{2^{2p+1}\pi^p p!} \displaystyle\sum_{\lambda=0}^{p} (-1)^\lambda \binom{p}{\lambda} \Phi_\lambda, & \text{if } q \text{ is odd } 2p+1. \end{cases}$$

$$\Omega = \begin{cases} (-1)^p \dfrac{1}{2^{2p}\pi^p p!} \displaystyle\sum_{i=n+N-q+1}^{n+N} \epsilon_{i_1 \cdots i_q} \Omega_{i_1 i_2} \cdots \Omega_{i_{q-1} i_q}, & \text{if } q \text{ is even } 2p. \\ 0, & \text{if } q \text{ is odd.} \end{cases}$$

where

$$\Omega_{ij} = \theta_{ij} - \sum_{\alpha=1}^{n+N-q} \omega_{i\alpha}\, \omega_{j\alpha}$$

where θ_{ij} is the curvature form of R^{n+N}.

These forms are in $\boldsymbol{B}^0$ generally but since we use the induced connection, they are forms in the product of bundles, $\boldsymbol{N}^0 \times \boldsymbol{T}_0$. Moreover, it can be proved that Π is the form in $\boldsymbol{N}^{q-1}$ and Ω in $\boldsymbol{N}^q$ by the same methods in Chern's paper, ([2]). And also, $d\Pi = -\Omega$.

4. Therefore, by Stokes' theoreem,

$$\int_{E_0} \Omega = \int_{E_0'} \Omega = -\int_{E_0'} d\Pi = -\int_{\partial E_0'} \Pi = -\int_F \Pi + \int_{E_0'|x_0} \Pi.$$

Now, if elements of pF are equal to frames by vectors $e_1, \cdots e_{n+N-q}$ of "repere" defining ω_i, ω_{ij} on $\sum$, E_0 can be taken so on Δ. Then, Ω_{ij} is zero on $E_0'|x_0$ and Π becomes the following form on $E_0'|x_0$.

$$\begin{cases} \dfrac{1}{\pi^p 1\cdot 3\cdots(2p-1)2^p}\Phi_0 = \dfrac{1}{\pi^p 1\cdot 3\cdots(2p-1)2p}\displaystyle\sum_{i=n+N-q+2}^{n+N}\epsilon_{i_1\dots i_{q-1}}\omega_{i,n+N-q+1}\cdots\omega_{i_{q-1}n+q+1}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } q=2p. \\ \dfrac{1}{2^{2p+1}\pi^p\, p!}\Phi_0 = \dfrac{1}{2^{2p+1}\pi^p p!}\displaystyle\sum_{i=n+N-q+2}^{n+N}\epsilon_{i_1\dots i_{q-1}}\omega_{i_1,n+N-q+1}\cdots\omega_{i_{q-1}n+N-9+1}, \text{ if } q=2p+1. \end{cases}$$

By Kronecker's formula,

$$(-1)^q D(F) = \int_{E_0'|x_0} \Pi$$

Therefore

$$(-1)^q D(F) = \int_{E_0}\Omega + \int_F F$$

5. For the general cross-section F, there exists F' such that $F \sim F'$ and pF' has the property which we assumed in the above section for F. Let E be any extension of pF over Δ.

Now, by the same method in Takizawa's paper ([3], § 6)

$$\int_E \Omega + \int_{F_0}\Pi = \int_{E_0}\Omega + \int_{F'}\Pi, \qquad \text{q ; odd or q} = N.$$

$$\equiv \int_{E_0}\Omega + \int_{F'}\Pi \ \text{mod } 2, \quad \text{q ; even and q} \neq N.$$

and

$$c(F) = c(F')$$

Thus, we obtain the following theorem.

Theorem

$$c(F)\cdot\Delta \begin{cases} -\displaystyle\int_F \Pi, & \text{if q is odd} \\ =\displaystyle\int_E \Omega + \int_F \Pi, & \text{if q} = N \text{ and even.} \\ \equiv\displaystyle\int_E \Omega + \int_F \pi \ \text{mod. } 2, & \text{if q} \neq N \text{ and even.} \end{cases}$$

References

1) N. Steenrod, The topology of fibre bundle. (Princeton Press 1951)

2) S. Chern, On the curvatura integra in a R. M. (Ann. of Math. Vol. 46, 1945 674-684)

4) S. Takizawa, On the Stiefel characteristic classes. (In this memoire)