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Local imbedding of Riemann spaces

By

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C. B. Allendoerfer [13] defined the type number r of Riemann space, which is imbedded in a flat space, and proved that, if $r>3$ and there exist H^{P}_{ij} satisfying the Gauss equation, then we have H^{Q}_{Pj} satisfying the Codazzi and Ricci equations. Hence, in this case, imbedding problem in flat space reduces merely to algebraic one, that is, to solving the Gauss equation. But we were not given by him any intrinsic method to determine the type number of the space.

The second section of the present paper gives a necessary condition that a Riemann n-space be imbedded in an Euclidean $(n+p)$-space. A development of the discussion in this section leads us to the *intrinsic definition of the even type number* of a Riemann space, as will be shown in the third section.

The fourth and subsequent sections concern with the *imbedding of Riemann space in space of constant curvature.* The Riemann curvature K of an enveloping space will be determined by a system of equations of first degree with respect to K. The system of equations is obtained as a consequence of the necessary condition found in the second section. Thus we shall show that the *imbedding problem of Riemann space in space of constant curvature is generally reducible to one in flat space.*

§ 1. Preliminaries and historical notes

Let V_n be a Riemann n-space with the metric form

$$g_{ij}\,dx^i dx^j \qquad (i, j=1,\cdots, n,),$$

imbedded in a Riemann $m(>n)$-space V_m with the metric form

$$g_{\alpha\beta}\,dy^\alpha dy^\beta \qquad (\alpha, \beta=1,\cdots, m,),$$

V_n being defined by equations of the form

$$y^\alpha = \varphi^\alpha(x^1, \cdots, x^n) \qquad (\alpha=1, \cdots, m),$$

where the rank of the functional matrix $\|\partial y^\alpha/\partial x^i\|$ is n. In this place, we suppose that these metric forms are not necessarily positive-definite. For displacement in V_n we have

$$g_{\alpha\beta} dy^\alpha dy^\beta = g_{ij} dx^i dx^j,$$

and it follows that

$$(1\cdot 1) \qquad g_{\alpha\beta} B_i^\alpha B_j^\beta = g_{ij},$$

where we put $B_i^\alpha = \partial y^\alpha/\partial x^i$. Let B_P^α $(P=n+1, \cdots, m)$ be orthogonal unit vectors normal to V_n, so that we have

$$(1\cdot 2) \qquad \begin{aligned} &g_{\alpha\beta} B_P^\alpha B_Q^\beta = 0 \quad (P \neq Q), \qquad = e_P \quad (P=Q), \\ &g_{\alpha\beta} B_i^\alpha B_P^\beta = 0, \end{aligned}$$

where the quantities $e_P = \pm 1$. Differentiations (1·1) and (1·2) give the following equations:

$$(1\cdot 3) \qquad B_{i,j}^\alpha + \Gamma_{\beta\gamma}^\alpha B_i^\beta B_j^\gamma = \sum_P H_{ij}^P B_P^\alpha,$$

$$(1\cdot 4) \qquad B_{P,j}^\alpha + \Gamma_{\beta\gamma}^\alpha B_P^\beta B_j^\gamma = H_{Pj}^k B_k^\alpha + \sum_Q e_Q H_{Pj}^Q B_Q^\alpha,$$

where commas denote the covariant differentiations with respect to g_{ij} and $\Gamma_{\beta\gamma}^\alpha$ is the Christoffel's symbol formed with respect to $g_{\alpha\beta}$. Three systems of functions H_{ij}^P, H_{Pj}^k and H_{Pj}^Q in (1·3) and (1·4) satisfy the equations

$$H_{ij}^P = H_{ji}^P, \qquad H_{Pj}^i = -g^{ik} H_{kj}^P, \qquad H_{Pj}^Q = -H_{Qj}^P.$$

We call usually H_{ij}^P the second fundamental tensors of V_n. As the conditions of integrabilities of (1·3) and (1·4), we get the Gauss equation

$$(1\cdot 5) \qquad B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta R_{\alpha\beta\gamma\delta} = R_{ijkl} - \sum_P e_P (H_{ik}^P H_{jl}^P - H_{il}^P H_{jk}^P),$$

the Codazzi equation

$$(1\cdot 6) \qquad \begin{aligned} B_P^\alpha B_i^\beta B_j^\gamma B_k^\delta R_{\alpha\beta\gamma\delta} = &-H_{ij,k}^P + H_{ik,j}^P \\ &+ \sum_Q e_Q (H_{ij}^Q H_{Qk}^P - H_{ik}^Q H_{Qj}^P), \end{aligned}$$

and finally the Ricci equation

$$(1\cdot 7) \qquad \begin{aligned} B_P^\alpha B_Q^\beta B_j^\gamma B_k^\delta R_{\alpha\beta\gamma\delta} = &H_{Pj,k}^Q - H_{Pk,j}^Q + H_{Pj}^i H_{ik}^Q - H_{Pk}^i H_{ij}^Q \\ &+ \sum_R e_R (H_{Pj}^R H_{Rk}^Q - H_{Pk}^R H_{Rj}^Q). \end{aligned}$$

When enveloping space V_m is flat, above equations become respectively

$$(1\cdot8)\qquad R_{ijkl}=\sum_P e_P(H_{ik}^P H_{jl}^P - H_{il}^P H_{jk}^P),$$

$$(1\cdot9)\qquad H_{ij,k}^P - H_{ik,j}^P = \sum_Q e_Q(H_{ij}^Q H_{Qk}^P - H_{ik}^Q H_{Qj}^P),$$

$$(1\cdot10)\qquad H_{Pj,k}^Q - H_{Pk,j}^Q = g^{il}(H_{ij}^P H_{lk}^Q - H_{ik}^P H_{lj}^Q) - \sum_R e_R(H_{Pj}^R H_{Rk}^Q - H_{Pk}^R H_{Rj}^Q).$$

It is well known that V_n is imbedded in a flat m-space, if and only if there exist two systems of functions H_{ij}^P $(=H_{ji}^P)$ and H_{Pj}^Q $(=-H_{Qj}^P)$, $(P, Q=n+1, \cdots, m)$, satisfying the equations (1·8), (1·9) and (1·10).

If we take another set of normal $\bar{B}_P^\alpha$ defined by

$$B_P^\alpha = \sum_Q l_P^Q \bar{B}_Q^\alpha,$$

where the matrix $\|l_P^Q\|$ of coefficients is to be orthogonal, it follows easily that the functions H_{ij}^P and H_{Pj}^Q subject to the transformations

$$(1\cdot11)\qquad \bar{H}_{ij}^P = \sum_Q l_P^Q H_{ij}^Q,$$

$$(1\cdot12)\qquad \bar{H}_{Pj}^Q = \sum_{R,S} l_P^R l_Q^S H_{Rj}^S + \sum_R l_{P,j}^R l_Q^R.$$

Now, any Riemann n-space can be imbedded locally and isometrically in a flat space of dimension $n(n+1)/2$. This result was enunciated by L. Schlaefli [1] and was first proved by M. Janet [7]. E. Cartan also proved this fact by means of theorems on Pfaff's form [8]. If V_n has some particular properties, V_n may be imbedded in a flat space of a lower dimension. When the lowest dimension is equal to $n+p$, we say that V_n is of *class* p. This term "class" originated with G. Ricci [2]. The imbedding problem, so called, is the intrinsic characterization of this particular properties.

The imbedding of V_n of non-vanishing constant curvature is satisfactorily studied. Such a space V_n is a fundamental hyperquadrics in flat $(n+1)$-space [16, p. 203] and hence V_n is of class one. But it is impossible that V_n of negative constant curvature is really imbedded im Euclidean $(n+1)$-space [3, p. 485]*.

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* We call a *flat* space a space, the metric form being written in the form $\sum_i e_i(dx^i)^2$ $(e_i=\pm1)$. If all e_i are positive, we call it an *Fuclidean* space.

On the other hand, only partial results have been obtained as to imbedding of Einstein spaces, but we have many interesting theories on this. If V_n is an Einstein n-space of vanishing scalar curvature, it is impossible that V_n is imbedded in a flat $(n+1)$-space. This theorem was proved by E. Kasner for dimension four [4] and his method is easily generalized to the case of higher dimension [16, p. 199]. But, for the proof of this theorem, we use a supposition that the elementary divisors of the matrix $\|\rho H_{ij}-g_{ij}\|$ are all simple, and hence, if the supposition on the matrix is omitted, we have the problem to find the condition that an Einstein V_n of vanishing scalar curvature be imbedded in a flat $(n+1)$-space. On this problem we will note in the end of the fifth section. Besides, C. B. Allendoerfer gave the condition that an Einstein space of non-vanishing scalar curvature be of class one [10].

For the case of V_n being conformally flat, all of circumstances of imbedding have become clear. Such a space is a fundamental hypercone in a flat $(n+2)$-space, and hence V_n is of class at most two, which was proved by H. W. Brinkmann [5]. In addition we have already the condition that V_n be of class one [19].

Now, in 1936, we were given by T. Y. Thomas the general theory on V_n being of class one [9]. In his paper, the problem on space of class one was perfectly discussed, except when V_n is of type two. His paper [9] threw a fresh light on the problem, and the algebraic characterization in a true sense has arised from him. Allendoerfer's paper concerning with an Einstein space of class one [10] as well as the paper on a conformally flat space of class one by the present author [19] are residual products of [9]. But Thomas omitted the case of type two, because the general theory on V_n of type greater than two can not be applied to the case of type two. The latter case was studied afterwards by the present author [21], though satisfactory result did not be obtained. Further, A. Kawaguchi got the simple expression of the condition $(8\cdot 4)$ in [9].

The work of Thomas was immediately followed by C. B. Allendoerfer. He got the generalized Frenet equations for V_n in a flat space and discussed the imbedding of an open simply connected domain of V_n [11]. Further, in his paper [13], the notion of type number defined by Thomas in the case of class one was generalized to the case of class greater than one and, making use of this notion, many beautiful theorems were obtained as to the rigidity of sub-space and the independences of the Gauss, Codazzi and Ricci equations. But the type number of these general cases was not defined by the intrinsic property and also he gave not a condition for the Gauss equation to have a solution.

The present author gave the condition for V_n being of class two [18]. It is an natural development of theories by Thomas and Allendoerfer. The type number, which is not the same one as defined by Allendoerfer, is determined by the intrinsic property. In this case also, V_n of type one and two are exceptional cases. As an example of this special case, he offered such a simple space [20].

§ 2. An necessary condition for V_n of class p

We shall limit our investigations in this section to the case when an enveloping space V_{n+p} is Euclidean. This restriction will abbreviate following equations. However, by a little modification, most of the results are perhaps satisfied in the case of V_{n+p} being flat but not Euclidean.

The Gauss equation (1·8) is written in the form

$$R_{ijk_1k_2} = \delta^P_Q H^P_{ia} H^Q_{jb} \delta^{a\ b}_{k_1k_2}, \tag{2·1}$$

where δ^P_Q are the Kronecker's deltas and we use hereafter the summation convention for indices $P, Q=1, \cdots, p$, and further $\delta^{a\ b}_{k_1k_2}$ are their generalizations. In order to generalize (2·1) we put in the first time

$$\underset{(2)}{R}_{i_1i_2|j_1j_2|k_1\cdots k_4} = \delta^{P_1P_2}_{Q_1Q_2} H^{P_1}_{i_1a_1} H^{P_2}_{i_2a_2} H^{Q_1}_{j_1b_1} H^{Q_2}_{j_2b_2} \delta^{a_1a_2b_1b_2}_{k_1\cdots\cdots k_4}. \tag{2·2}$$

This tensor $\underset{(2)}{R}$ is expressible in terms of the components of the curvature tensor. In fact, the right-hand number of (2·2) is written as follows:

$$\frac{1}{2^2}(\delta^{P_1}_{Q_1} H^{P_1}_{i_1a_1} H^{Q_1}_{j_1b_1} \delta^{a_1b_1}_{c_1d_1} \cdot \delta^{P_2}_{Q_2} H^{P_2}_{i_2a_2} H^{Q_2}_{j_2b_2} \delta^{a_2b_2}_{c_2d_2}$$
$$- \delta^{P_1}_{Q_2} H^{P_1}_{i_1a_1} H^{Q_2}_{j_2b_2} \delta^{a_1b_2}_{c_1d_2} \cdot \delta^{P_2}_{Q_1} H^{P_2}_{i_2a_2} H^{Q_1}_{j_1b_1} \delta^{a_2b_1}_{c_2d_1}) \delta^{c_1c_2d_1d_2}_{k_1\cdots\cdot k_4}.$$

Substituting from (2·1) we obtain

$$\underset{(2)}{R}_{i_1i_2|j_1j_2|k_1\cdots k_4} \tag{2·3}$$

$$= \frac{1}{2^2} \underset{(2)}{\varepsilon}^{uv} R_{i_1 j_u a_1 b_u} R_{i_2 j_v a_2 b_v} \delta^{a_1 a_2 b_1 b_2}_{k_1 \cdots k_4}$$

$$= \frac{1}{2^2} (R_{i_1 j_1 a_1 b_1} R_{i_2 j_2 a_2 b_2} - R_{i_1 j_2 a_1 b_2} R_{i_2 j_1 a_2 b_1}) \delta^{a_1 a_2 b_1 b_2}_{k_1 \cdots k_4} .$$

Further, if we put

$$(2\cdot4) \qquad \underset{(3)}{R}_{i_1 i_2 i_3 | j_1 j_2 j_3 | k_1 \cdots k_6}$$
$$= \delta^{P_1 P_2 P_3}_{Q_1 Q_2 Q_3} H^{P_1}_{i_1 a_1} H^{P_2}_{i_2 a_2} H^{P_3}_{i_3 a_3} H^{Q_1}_{j_1 b_1} H^{Q_2}_{j_2 b_2} H^{Q_3}_{j_3 b_3} \delta^{a_1 a_2 a_3 b_1 b_2 b_3}_{k_1 \cdots \cdots \cdots k_6} ,$$

and proceed in similar manner as above shown, we establish then

$$(2\cdot5) \qquad \underset{(3)}{R}_{i_1 i_2 i_3 | j_1 j_2 j_3 | k_1 \cdots k_6}$$
$$= \frac{1}{2^2 \cdot 4!} \underset{(3)}{\varepsilon}^{uvw} R_{i_1 j_u a_1 b_u} \underset{(2)}{R}_{i_2 i_3 | j_v j_w | a_2 a_3 b_v b_w} \delta^{a_1 a_2 a_3 b_1 b_2 b_3}_{k_1 \cdots \cdots \cdots k_6}$$
$$= \frac{1}{2 \cdot 4!} (R_{i_1 j_1 a_1 b_1} \underset{(2)}{R}_{i_2 i_3 | j_2 j_3 | a_2 a_3 b_2 b_3} + R_{i_1 j_2 a_1 b_2} \underset{(2)}{R}_{i_2 i_3 | j_3 j_1 | a_2 a_3 b_3 b_1}$$
$$+ R_{i_1 j_3 a_1 b_3} \underset{(2)}{R}_{i_2 i_3 | j_1 j_2 | a_2 a_3 b_1 b_2}) \times \delta^{a_1 a_2 a_3 b_1 b_2 b_3}_{k_1 \cdots \cdots \cdots k_6}$$
$$= \frac{1}{2^3} \Big[R_{i_1 j_1 a_1 b_1} (R_{i_2 j_2 a_2 b_2} R_{i_3 j_3 a_3 b_3} - R_{i_2 j_3 a_2 b_3} R_{i_3 j_2 a_3 b_2})$$
$$+ R_{i_1 j_2 a_1 b_2} (R_{i_2 j_3 a_2 b_3} R_{i_3 j_1 a_3 b_1} - R_{i_2 j_1 a_2 b_1} R_{i_3 j_3 a_3 b_3})$$
$$+ R_{i_1 j_3 a_1 b_3} (R_{i_2 j_1 a_2 b_1} R_{i_3 j_2 a_3 b_2} - R_{i_2 j_2 a_2 b_2} R_{i_3 j_1 a_3 b_1}) \Big]$$
$$\times \delta^{a_1 a_2 a_3 b_1 b_2 b_3}_{k_1 \cdots \cdots \cdots k_6} .$$

If we generalize above processes and put

$$(2\cdot6) \qquad \underset{(r)}{R}_{i_1 \cdots i_r | j_1 \cdots j_r | k_1 \cdots k_{2r}}$$
$$= \delta^{P_1 \cdots P_r}_{Q_1 \cdots Q_r} H^{P_1}_{i_1 a_1} \cdots H^{P_r}_{i_r a_r} H^{Q_1}_{j_1 b_1} \cdots H^{Q_r}_{j_r b_r} \delta^{a_1 \cdots a_r b_1 \cdots b_r}_{k_1 \cdots \cdots \cdots k_{2r}} ,$$

then it follows that $\underset{(r)}{R}$ is written in the intrinsic form

$$(2\cdot7) \qquad \underset{(r)}{R}_{i_1 \cdots i_r | j_1 \cdots j_r | k_1 \cdots k_{2r}}$$
$$= \frac{1}{2^r} \underset{(r)}{\varepsilon}^{uv \cdots w} R_{i_1 j_u a_1 b_u} R_{i_2 j_v a_2 b_v} \cdots R_{i_r j_w a_r b_w} \delta^{a_1 \cdots a_r b_1 \cdots b_r}_{k_1 \cdots \cdots \cdots k_{2r}}$$

$$= \frac{1}{2\cdot(2r-2)!}\sum_{s=1}^{r}(-1)^{s-1} R_{i_1 j_s a_1 b_s}$$

$$\times \underset{(r-1)}{R}{}_{i_2\cdots i_r|j_1\cdots\hat{j}_s\cdots j_r|a_2\cdots a_r b_1\cdots\hat{b}_s\cdots b_r} \times \delta^{a_1\cdots a_r b_1\cdots b_r}_{k_1\cdots\cdots\cdots k_{2r}}$$

$$= \frac{1}{4!\cdot(2r-4)!}\sum_{s,t(s<t)}^{1,\dots,r}(-1)^{s+t-1}\underset{(2)}{R}{}_{i_1 i_2|j_s j_t|a_1 a_2 b_s b_t}$$

$$\times \underset{(r-2)}{R}{}_{i_3\cdots i_r|j_1\cdots\hat{j}_s\cdots\hat{j}_t\cdots j_r|a_3\cdots a_r b_1\cdots\hat{b}_s\cdots\hat{b}_t\cdots b_r}\delta^{a_1\cdots a_r b_1\cdots b_r}_{k_1\cdots\cdots\cdots k_{2r}}.$$

In these calculations we made use of the following identities satisfied by generalized Kronecker's deltas*.

$$\delta^{a_1\cdots a_t\cdots a_s}_{b_1\cdots\cdots\cdots b_s}\delta^{c_1\cdots c_t}_{a_1\cdots a_t} = t!\cdot\delta^{c_1\cdots c_t c_{t+1}\cdots a_s}_{b_1\cdots\cdots\cdots\cdots b_s},$$

(2·8)

$$\delta^{P_1\cdots P_s}_{Q_1\cdots Q_s} = \underset{(s)}{\varepsilon}{}^{uv\cdots w}\delta^{P_1}_{Q_u}\delta^{P_2}_{Q_v}\cdots\delta^{P_r}_{Q_w}.$$

Observe that components of $\underset{(r)}{R}$ $(2\leqq 2r\leqq n)$ are expressed intrinsically as homogeneous polynomials of r-th degree in terms of components of the curvature tensor.

We can write the Bianchi's identity as follows:

$$R_{ijk_1k_2,k_3}\delta^{k_1k_2k_3}_{l_1l_2l_3} = 0,$$

and, making use of mathematical induction and the second expression of $\underset{(r)}{R}$ in (2·7), we establish

$$\underset{(r)}{R}{}_{i_1\cdots i_r|j_1\cdots j_r|k_1\cdots k_{2r},k_{2r+1}}\delta^{k_1\cdots k_{2r+1}}_{l_1\cdots l_{2r+1}} = 0. \tag{2·9}$$

If V_n is of class p, the indices P's and Q's in (2·6) take the different p values, so that we have evidently from (2·6) and the definition of the generalized Kronecker's delta

$$\underset{(q)}{R}{}_{i_1\cdots i_q|j_1\cdots j_q|k_1\cdots k_{2q}} = 0 \quad (2p<2q\leqq n) \tag{2·10}$$

Therefore

THEOREM 1. *It is necessary for V_n of class p $(2p<n)$ that the tensor $\underset{(p+1)}{R}$ vanishes.*

We remark that, from (2·7), $\underset{(q)}{R}=0$ $(q>p+1)$, if $\underset{(p+1)}{R}=0$.

Though this theorem seems not to play a röle in the case of class one [9] and two [18], we see in the fourth and seventh sections

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* We use throughout this paper the generalized Kronecker's deltas. See O. Veblen: Invariants of quadratic differential forms, Cambridge, 1927.

sections that this is fundamental for the imbedding problem of V_n in an $(n+p)$-space of constant curvature.

§ 3. Allendoerfer's type numbers

If V_n is of class p, we put

$$(3\cdot1)\qquad H_{i_1\cdots i_p|a_1\cdots a_p}=\underset{(p)}{\varepsilon}_{P_1\cdots P_p}H^{P_1}_{i_1a_1}\cdots H^{P_p}_{i_pa_p}$$
$$=\begin{vmatrix} H^1_{i_1a_1} & H^2_{i_1a_1} & \cdots & H^p_{i_1a_1}\\ H^1_{i_2a_2} & H^2_{i_2a_2} & \cdots & H^p_{i_2a_2}\\ \cdots\cdots & & & \cdots\cdots\\ H^1_{i_pa_p} & H^2_{i_pa_p} & \cdots & H^p_{i_pa_p}\end{vmatrix},$$

and it follows immediately that

$$\delta^{P_1\cdots P_p}_{Q_1\cdots Q_p}H^{P_1}_{i_1a_1}\cdots H^{P_p}_{i_pa_p}H^{Q_1}_{j_1b_1}\cdots H^{Q_p}_{j_pb_p}$$
$$=H_{i_1\cdots i_p|a_1\cdots a_p}H_{j_1\cdots j_p|b_1\cdots b_p}.$$

Combining this and (2·6) we get

$$(3\cdot2)\qquad \underset{(p)}{R}_{i_1\cdots i_p|j_1\cdots j_p|k_1\cdots k_p}$$
$$=H_{i_1\cdots i_p|a_1\cdots a_p}H_{j_1\cdots j_p|b_1\cdots b_p}\delta^{a_1\cdots a_pb_1\cdots b_p}_{k_1\cdots\cdots\cdots k_{2p}}.$$

Further we put

$$(3\cdot3)\qquad \underset{(p,r)}{C}_{i_1\cdots i_{pr}|j_1\cdots j_{pr}|k_1\cdots k_{2pr}}$$
$$=H_{i_1\cdots i_p|a_1\cdots a_p}H_{j_1\cdots j_p|b_1\cdots b_p}\cdots$$
$$\times H_{i_{p(r-1)+1}\cdots i_{pr}|a_{p(r-1)+1}\cdots a_{pr}}H_{j_{p(r-1)+1}\cdots j_{pr}|b_{p(r-1)+1}\cdots b_{pr}}$$
$$\times\delta^{a_1\cdots b_{pr}}_{k_1\cdots k_{2pr}}.$$

Making use of (2·8) and (3·2) we obtain

$$(3\cdot4)\qquad \underset{(p,r)}{C}_{i_1\cdots i_{pr}|j_1\cdots j_{pr}|k_1\cdots k_{2pr}}$$
$$=\frac{1}{\{(2p)!\}^r}\underset{(p)}{R}_{i_1\cdots i_p|j_1\cdots j_p|a_1\cdots a_pb_1\cdots b_p}\cdots$$
$$\times\underset{(p)}{R}_{i_{p(r-1)+1}\cdots i_{pr}|j_{p(r-1)+1}\cdots j_{pr}|a_{p(r-1)+1}\cdots a_{pr}b_{p(r-1)+1}\cdots b_{pr}}$$
$$\times\delta^{a_1\cdots b_{pr}}_{k_1\cdots k_{2pr}},$$

and further we have

$$(3\cdot5)\qquad \underset{(p,r)}{C}{}_{i_1\cdots i_{pr}|j_1\cdots j_{pr}|k_1\cdots k_{2pr}}$$
$$=\frac{1}{(2p)!\cdot\{2p(r-1)\}!}\underset{(p,1)}{C}{}_{i_1\cdots i_p|j_1\cdots j_p|l_1\cdots l_{2p}}$$
$$\times\underset{(p,r-1)}{C}{}_{i_{p+1}\cdots i_{pr}|j_{p+1}\cdots j_{pr}|l_{2p+1}\cdots l_{2pr}}\,\delta^{l_1\cdots l_{2pr}}_{k_1\cdots k_{2p}}\,.$$

We observe that components of $\underset{(p,r)}{C}$ are expressible as the homogeneous polynomials of pr-th degree in terms of components of the curvature tensor.

C. B. Allendoerfer defined type number of V_n imbedded in a flat space [13]. The quantities $C_{a_1\cdots a_q|\beta_1\cdots\beta_q}$ defined by (2·3) of [13] are equal to $H_{i_1\cdots i_q|a_1\cdots a_q}$ of (3·1) in the present paper, so that $\underset{(p,r)}{C}$ of (3·3) is equivalent to C_{2r} in (2·4) of [1·3]. We observe from (3·5) that, if $\underset{(p,r+1)}{C}=0$, then $\underset{(p,s)}{C}=0$ $(s>r+1)$. Now, Allendoerfer's type numbers are defined as follows. If $C_r\neq0$ and $C_{r+1}=0$ in a point P of V_n, we say that V_n is of type r at P. Therefore we can define (even) type number by means of the intrinsic properties of V_n as follows.

DEFINITION. *Let V_n be a Riemann n-space. If $\underset{(p,r)}{C}\neq0$ and $\underset{(p,r+1)}{C}=0$ $(2pr\leq n)$ at a point P, we say that V_n is of type $2r$ at P, where $\underset{(p,r)}{C}$ are defined by* (3·4).

If V_n is of type $2r$, there exists such a coordinate system that $\underset{(p,r)}{C}{}_{1\cdots1\,3\cdots3\cdots2r-1\cdots2r-1\,|\,2\cdots2\,4\cdots4\cdots2r\cdots2r\,|\,k_1\cdots k_{2pr}}\neq0$. This quantity is a determinant of $2pr$-th order and hence we construct the normalized cofactors H^{ia}_P of H^P_{ia}, satisfying the equations

$$H^P_{ia}H^{ja}_Q=\delta^j_i\delta^P_Q \qquad \left(\begin{array}{l}a,b=k_1\cdots k_{2pr}\\ i,j=1,\cdots,2r\\ P,Q=1,\cdots,p\end{array}\right)$$
$$H^P_{ia}H^{ib}_P=\delta^b_a$$

Making use of these quantities, Allendoerfer proved remarkable theorems (see [13] in details). If V_n is of type $2r$, V_n is of type $2r$ or $2r+1$ in the sense of Allendoerfer, so that those theorems are stated as follows.

THEOREM 2. (1) *If V_n is of type $\geqq4$ at a point P, the solution H^P_{ij} of the Gauss equation (1·8) at P, if exists, is uniquely determined to within orthogonal transformations (1·11).*

(2) *If V_n is of type $\geqq 4$ in an neighborhood and there exist functions H^P_{ij} satisfying the Gauss equation (1·8), we have the functions H^Q_{Pj} satisfying the Codazzi and Ricci equations (1·9) and (1·10).*

(3) *If V_n is of type $\geqq 2$ at P and there exist functions H^P_{ij} satisfying the Gauss equation (1·8), the solution H^Q_{Pj} of the Codazzi equation (1·9), if exists, is unique.*

§ 4. Imbedding a Riemann n-space in an $(n+1)$-space of constant curvature

The imbedding problem of Riemann space of dimension n in a flat $(n+p)$-space may be generalized to the case when the enveloping space V_{n+p} is not necessarily flat. But those are very hard to study in general, because quantities B^α_i arise in the Gauss equation (1·5). J. E. Campbell seems to the first to have tried this kind of problem. He proved the interesting theorem that any Riemann n-space can be imbedded in an Einstein $(n+1)$-space of vanishing scalar curvature [6, pp. 212–219], the method being very complicated. Also, it is worthy of our notice that K. Yano and Y. Muto considered the imbedding in conformally flat space [15].

In the following we concern with the case when the enveloping space V_{n+p} is of *constant curvature*$\neq 0$. It is to be accentuated in this place that *we do not think of enveloping space as previously given, but it is our purpose to find an enveloping space of the given space,* and hence the constant Riemann curvature K of enveloping space is to be found. The necessary and sufficient condition that a Riemann n-space be imbedded in an $(n+p)$-space S_{n+p} of constant curvature K, whose fundamental metric form is positive definite, is that there exist two systems of functions H^P_{ij} $(=H^P_{ji})$ and H^Q_{Pj} $(=-H^P_{Qj})$ satisfying the Gauss, Codazzi and Ricci equations as follows [16, p. 211]:

$$(4\cdot1)\qquad R_{ijkl}=K(g_{ik}g_{jl}-g_{il}g_{jk})+\sum_P(H^P_{ik}H^P_{jl}-H^P_{il}H^P_{jk}),$$

$$(4\cdot2)\qquad H^P_{ij,k}-H^P_{ik,j}=\sum_Q(H^Q_{ij}H^P_{Qk}-H^Q_{ik}H^P_{Qj}),$$

$$(4\cdot3)\qquad H^Q_{Pj,k}-H^Q_{Pk,j}=g^{il}(H^P_{ij}H^Q_{lk}-H^P_{ik}H^Q_{lj})\\ -\sum_R(H^R_{Pj}H^Q_{Rk}-H^R_{Pk}H^Q_{Rj}).$$

On putting

$$(4\cdot4)\qquad S_{ijkl}=R_{ijkl}-K(g_{ik}g_{jl}-g_{il}g_{jk}),$$

we have from (4·1)

$$(4\cdot5)\qquad S_{ijkl}=\sum(H^p_{ik}H^p_{jl}-H^p_{il}H^p_{jk}).$$

It is clear that S_{ijkl} possesses similar properties as the curvature tensor for interchange of indices. Hence the process, by means of which from (2·1) we obtained Theorem 1, is applied equally well when R_{ijkl} is replaced by S_{ijkl}. Thus from the theorem we have

$$(4\cdot6)\qquad \underset{(p+1)}{S}_{i_1\cdots i_{p+1}|j_1\cdots j_{p+1}|k_1\cdots k_{2p+2}}=0\qquad (2p+2\leqq n),$$

for V_n being imbedded in an $(n+p)$-space of constant cuavature.

In this section we treat the simplest case of $p=1$. The case is typical and we have interesting special type. But the general theory of the case $n\geqq4$ can not be applicable to the case of $n=3$, because of $2p+2\leqq n$ in (4·6), and hence we consider first the former.

I. *The case of dimension $n\geqq4$*

In this case we have from (4·6) $\underset{(2)}{S}=0$, so that (2·3) gives

$$(4\cdot7)\qquad (S_{i_1j_1a_1b_1}S_{i_2j_2a_2b_2}-S_{i_1j_2a_1b_2}S_{i_2j_1a_2b_1})\delta^{a_1a_2b_1b_2}_{k_1\cdots k_2}=0.$$

Substituting from (4·4) we have a system of equations of second degree in terms of K. But it is easily verified that coefficients of K^2 in these equations are identically zero and resulting equations become then

$$(4\cdot8)\qquad \underset{(2)}{A}_{i_1i_2|j_1j_2|k_1\cdots k_4}\cdot K-2\underset{(2)}{R}_{i_1i_2|j_1j_2|k_1\cdots k_4}=0,$$

where we put

$$(4\cdot9)\qquad \underset{(2)}{A}_{i_1i_1|j_1j_2|k_1\cdots k_4}=(R_{i_1j_1a_1b_1}g_{i_2a_2}g_{j_2b_2}-R_{i_1j_2a_1b_2}g_{i_2a_2}g_{j_1b_1}$$
$$+R_{i_2j_2a_2b_2}g_{i_1a_1}g_{j_1b_1}-R_{i_2j_1a_2b_1}g_{i_1a_1}g_{j_2b_2})\delta^{a_1a_2b_1b_2}_{k_1\cdots k_4},$$

which satisfies the identities

$$(4\cdot10)\qquad \underset{(2)}{A}_{i_1i_1|j_1j_2|k_1\cdots k_4,k_5}\delta^{k_1\cdots k_5}_{l_1\cdots l_5}=0,$$

as easily shown. Therefore if V_n can be imbedded in S_{n+1} of constant curvature $K\neq0$, this K must satisfy (4·8). Elimination of K from (4·8) gives

$$(4\cdot11)\qquad \begin{vmatrix} \underset{(2)}{A}_{a_1a_2|b_1b_2|c_1\cdots c_4} & \underset{(2)}{R}_{a_1a_2|b_1b_2|c_1\cdots c_4} \\ \underset{(2)}{A}_{i_1i_2|j_1j_2|k_1\cdots k_4} & \underset{(2)}{R}_{i_1i_2|j_1j_2|k_1\cdots k_4}\end{vmatrix}=0.$$

$(a, b, c, i, j, k=1,\cdots,n)$

If we suppose that $\underset{(2)}{A}$, the coefficients of K in (4·8), is zero tensor, then we obtain, contracting (4·9) by $g^{i_2k_3}g^{j_2k_4}$,

$$C_{i_1j_1k_1k_2} = R_{i_1j_1k_1k_2} - \frac{1}{n-2}(g_{i_1k_1}R_{j_1k_2} - g_{i_1k_2}R_{j_1k_1} + R_{i_1k_1}g_{j_1k_2} - R_{i_1k_2}g_{j_1k_1}) + \frac{R}{(n-1)(n-2)}(g_{i_1k_1}g_{j_1k_2} - g_{i_1k_2}g_{j_1k_1}) = 0.$$

This implies that the conformal curvature tensor of V_n vanishes and hence, if V_n $(n\geqq 4)$ does not be conformally flat, there exists at least one components of $\underset{(2)}{A}$ not to vanish. Conversely we can easily show that, if $C_{ijkl}=0$, $\underset{(2)}{A}$ vanishes. Hereafter we suppose that V_n $(n\geqq 4)$ is not conformally flat. Then the equation (4·8) is thought of as one, from which the constant curvature of enveloping space S_{n+1} and hence S_{n+1} itself is to be determined. The necessary and sufficient condition that (4·8) has a common solution K is clearly (4·11) and then K is uniquely determined.

It is easily seen that K vanishes, if and only if $\underset{(2)}{R}$ is a zero tensor, so that we have the

THEOREM 3. *Let V_n $(n\geqq 4)$ be a Riemann n-space not to be conformally flat. If there exists an $(n+1)$-space S_{n+1} of constant curvature K, in which V_n is imbedded, then K is equal to zero, if and only if the tensor $\underset{(2)}{R}$ of V_n vanishes.*

The solution K of (4·8), under the condition (4·11), will not necessarily be constant, and hence we must find further condition that as thus determined K be constant. Differentiating (4·8) covariantly with respect to x^l, we get in virture of $K_{,l}=0$

$$(4\cdot 12)\qquad \underset{(2)}{A}_{i_1i_2|j_1j_2|k_1\cdots k_4,l}\cdot K - 2\underset{(2)}{R}_{i_1i_2|j_1j_2|k_1\cdots k_4,l} = 0.$$

Equations (4·8) and (4·12) must be consistent and the condition arising from this is clearly given by

$$(4\cdot 13)\qquad \begin{vmatrix} \underset{(2)}{A}_{a_1a_2|b_1b_2|c_1\cdots c_4} & \underset{(2)}{R}_{a_1a_2|b_1b_2|c_1\cdots c_4} \\ \underset{(2)}{A}_{i_1i_2|j_1j_2|k_1\cdots k_4,l} & \underset{(2)}{R}_{i_1i_2|j_1j_2|k_1\cdots k_4,l} \end{vmatrix} = 0.$$

$(a, b, c, i, j, k, l=1, \cdots, n)$

Consequently the constant curvature K of enveloping space is determined from (4·8) and the necessary and sufficient condition for possibility of determination is the equations (4·11) and (4·13). Therefore

THEOREM 4. *Let V_n be a Riemann n-space not to be conformally flat. If there exists an $(n+1)$-space S_{n+1} of constant curvature enveloping V_n, the constant curvature K is given by the equation (4·8) under the condition (4·11) and (4·13).*

It is possible that the solution K of (4·8), if exists, is unconditionally constant, similar to the case of Theorem given by F. Schur [16, p. 83]. Differentiating (4·8) and making use of (2·9) and (4·10) we have

$$\underset{(2)}{A}_{i_1 i_2 | j_1 j_2 | k_1 \cdots k_4} \cdot K_{,k_5} \delta^{k_1 \cdots k_5}_{l_1 \cdots l_5} = 0 .$$

But the author has no hope to deduce from above equation $K_{,j} = 0$, and so the condition (4·13) is unavoidable.

Now we define S_{ijkl} by (4·4), where the intrinsic expression of K as above found is substituted and then our problem reduce to finding the condition that there exists H_{ij} satisfying the following equation:

$$S_{ijkl} = H_{ik} H_{jl} - H_{il} H_{jk} , \tag{4·14}$$

$$H_{ij,k} - H_{ik,j} = 0. \tag{4·15}$$

We remark that (4·14) is formally same as the Gauss equation in the case of space being of class one and (4·15) is the Codazzi equation; and hence, from now on, the similar process in [9] can be applicable to (4·14) and (4·15). Namely, in the first time, we define the type number τ of V_n. If the matrix

$$\left\| \begin{matrix} S_{abc1} & S_{abc2} & \cdots & S_{abcn} \\ \cdots & & & \cdots \\ S_{ijk1} & S_{ijk2} & \cdots & S_{ijkn} \\ \cdots & & & \cdots \\ S_{pqr1} & S_{pqr2} & \cdots & S_{pqrn} \end{matrix} \right\|$$

is of rank one or zero, we say that V_n is of type one. If the rank is τ ($\geqq 2$), we say that V_n is of type τ. Then the rank of matrix $\|H_{ij}\|$ is equal to the type number of V_n. If $\tau \geqq 3$, the solution H_{ij} of (4·14) is uniquely determined to within algebraic sign. If $\tau \geqq 4$, the Codazzi equation (4·15) is a consequence of (4·14). Further, the condition that (4·14) has a real solution is that

$$\begin{vmatrix} S_{bcjk} & S_{bcki} & S_{bcij} \\ S_{cajk} & S_{caki} & S_{caij} \\ S_{abjk} & S_{abki} & S_{abij} \end{vmatrix} \equiv S_{abcijk} \geqq 0 .$$

Finally, if V_n is of type more than two, there exists H_{ij} satisfying (4·14), if and only if $S_{abcijk} \geqq 0$, $\sum S_{abcijk} > 0$ and the system of equations $R_n(S)=0$ be satisfied, where $R_n(S)$ is the resultant system of equation (4·14) and

$$H_{ab}S_{ijkl}+H_{al}S_{ijbk}+H_{ik}S_{jabl}+H_{jk}S_{ialb}=0\,.$$

However, if V_n is of type three, the further condition $H_n(S)=0$ must be subjoined, which is obtained by substituting H_{ij}, as above determined, in the Codazzi equation. Consequently we establish the

THEOREM 5. *Let V_n ($n \geqq 4$) be a Riemann n-space not to be conformally flat. If there exists an $(n+1)$-space S_{n+1} of constant curvature, the curvature is determined by the equations (4·8) under the condition (4·11) and (4·13). If V_n is of type more than three, farther condition that there exists an enveloping space S_{n+1}, is that $S_{abcijk} \geqq 0$, $\sum S_{abcijk} > 0$ and $R_n(S)=0$. If V_n is of type three, the condition $H_n(S)=0$ is subjoined.*

On the other hand, if V_n is of type two, the problem to find H_{ij} satisfying not only (4·14) but also (4·15) does not yet be solved, so far as the author knows. However, it is shown as in [21] that in this case S_{hijk} satisfies the following equation

$$\begin{vmatrix} S_{abij} & S_{abkl} \\ S_{cdij} & S_{cdkl} \end{vmatrix} = 0 \qquad (a, b, c, d, i, j, k, l=1, \cdots, n)$$

and these are necessary and sufficient condition that (4·14) has a solution, which is not be unique. Substituting (4·4), the above equation is written in the form

$$(4\cdot16) \qquad (g_{abij}g_{cdkl}-g_{abkl}g_{cdij})K^2-(R_{abij}g_{cdkl}-R_{abkl}g_{cdij} +g_{abij}R_{cdkl}-g_{abkl}R_{cdij})K+(R_{abij}R_{cdkl} -R_{abkl}R_{cdij})=0,$$

where we put

$$g_{abij}=g_{ai}g_{bj}-g_{aj}g_{bi}\,.$$

Hence there is not any possibility for V_n being of type two, if (4·16) is not consistent to (4·8).

In any case, the problem reduce finally to the consideration of V_n to be of class one.

II. *The case of dimension three*

If the dimension of the space is three, we are in special circumstances; there exists always H_{ij} satisfying the Gauss equation

$$S_{ijkl} = H_{ik}H_{jl} - H_{il}H_{jk}.$$

From the theorems of determinants we have

$$\begin{vmatrix} H_{ap} & H_{aq} & H_{ar} \\ H_{bp} & H_{bq} & H_{br} \\ H_{cp} & H_{cq} & H_{cr} \end{vmatrix}^2 = \begin{vmatrix} S_{bcqr} & S_{bcrp} & S_{bcpq} \\ S_{caqr} & S_{carp} & S_{capq} \\ S_{abqr} & S_{abrp} & S_{abpq} \end{vmatrix} \equiv \sigma$$

where (a, b, c), (p, q, r) are even permutations of $(1, 2, 3)$, and then we get

$$H_{ap} = \sigma^{-1/2} \cdot \begin{vmatrix} S_{carp} & S_{capq} \\ S_{abrp} & S_{abpq} \end{vmatrix}, \tag{4·17}$$

as a unique solution of the Gauss equation. Thus the Riemann curvature K of the enveloping space S_4 is not determined only by the Gauss equation, and hence we must consider the Codazzi equation, by which K will be determined.

In the first time, we substitute (4·17) from (4·4) and obtain

$$H_{ap} = \frac{X_{ap}K^2 + Y_{ap}K + Z_{ap}}{\sqrt{AK^3 + BK^2 + CK + D}}, \tag{4·18}$$

where we put for brevity $\rho^{ap} = R_{bcqr}$ and

$$\begin{aligned} X_{ap} &= g \cdot g_{ap} \qquad (g = |g_{ij}|), \\ Y_{ap} &= -g \cdot R_{ap}, \qquad Z_{ap} = \rho^{bq}\rho^{cr} - \rho^{br}\rho^{cq}, \\ A &= -g^2, \qquad B = \frac{1}{2} g^2 \cdot R, \\ C &= -gR_{ij}\rho^{ij}, \qquad D = |\rho^{ij}|. \end{aligned} \tag{4·19}$$

Observe that X_{ap} and A are covariant constants. From (4·18) the Codazzi equation (4·15) is written in the form

$$L_{ijk} \cdot K^4 + M_{ijk} \cdot K^3 + N_{ijk} \cdot K^2 + P_{ijk} \cdot K + Q_{ijk} = 0, \tag{4·20}$$

where we made use of K being constant and put

$$\begin{aligned} L_{ijk} &= 2AY_{i[j,k]} - X_{i[j}B_{,k]}, \\ M_{ijk} &= 2BY_{i[j,k]} + 2AZ_{i[j,k]} - X_{i[j}C_{,k]} - Y_{i[j}B_{,k]}, \\ N_{ijk} &= 2CY_{i[j,k]} + 2BZ_{i[j,k]} - X_{i[j}D_{,k]} - Y_{i[j}C_{,k]} - Z_{i[j}B_{,k]}, \\ P_{ijk} &= 2DY_{i[j,k]} + 2CZ_{i[j,k]} - Y_{i[j}D_{,k]} - Z_{i[j}C_{,k]}, \\ Q_{ijk} &= 2DZ_{i[j,k]} - Z_{i[j}D_{,k]}. \end{aligned}$$

Therefore, if V_3 can be imbedded in a 4-space of constant curvature, nine equations (4·20) must have a common real solution K, which is constant and does not satify

$$(4\cdot21)\qquad AK^3+BK^2+CK+D=0.$$

Then the Riemann curvature K of enveloping space S_4 is given by a solution as above mentioned and further the second fundamental tensor H_{ap} of V_3 is given uniquely by (4·18). As a result we have the

THEOREM 6. *A space V_3 can be imbedded in a space S_4 of constant curvature, if and only if (4·20) have a common real solution, which is constant and does not satisfy (4·21). Then the curvature K of S_3 is given by the above solution and the second fundamental tensor H_{ap} of V_3 is given by (4·18).*

If (4·20) has many solution $K_1, K_2, \cdots$, as above mentioned, every $K_1, K_2, \cdots$, defines a enveloping space of constant curvature and thus there exist at most four spaces enveloping a given V_3, if exists.

As in general cases of V_n $(n>3)$ being conformally flat, for the case of conformally flat V_3, we have also special circumstances. For such a V_3, we have

$$R_{ij,k}-R_{ij,k}-\frac{1}{4}(g_{ij}R_{,k}-g_{ik}R_{,j})=0,$$

and it is easily verified that L_{ijk} is identically zero and converse. Thus (4·20) is of three degree in terms of K and there exist at most three spaces enveloping V_3.

§ 5. Imbedding an Einstein n-space in an $(n+1)$-space of constant curvature

We can evidently apply the general discussion in the last section to an Einstein n-space, which is not conformally flat. An Einstein space, which is conformally flat, is of constant curvature [16, p. 93] and hence such a space may be excepted from our discussion. However, following the Allendoerfer's treatment on an Einstein space of class one [10], we give the simpler discussion for such a space. The condition for this case is more briefly expressed and so we are interesting about it. On the other hand, A. Fialkow already investigated the similar problem [12]. But he thought of as an enveloping space S_{n+1} being previously given and so his discussion is exactly similar to Allendoerfer's one, while our purpose is to find S_{n+1}, in which a given space V_n is imbedded.

In the general case of the last section, we paid attention to a necessary condition $(2\cdot 10)$ $(q=2)$ for V_n being of class one, and replaced the curvature tensor by S_{hijk} defined by $(4\cdot 4)$. Also, in this case we are going to use the similar process. Allendoerfer deduce the equation

$$(5\cdot 1)\qquad H_{hi}H_{jk} = D_{hi|jk} \\ = \frac{eR}{n(n-2)}g_{hi}g_{jk}+\frac{en}{2R(n-2)}(R^{\cdot b}_{a\cdot hj}R^{\cdot a}_{b\cdot ik}-2R^{\cdot a}_{h\cdot ib}R^{\cdot b}_{j\cdot ka}),$$

from the Gauss equation and hence the matrix $\|D_{hi|jk}\|$ being necessarily of rank one and semi-definite. Further, from $(5\cdot 1)$ and the Gauss equation we must have the equation

$$(5\cdot 2)\qquad \frac{(n-2)R}{n}\left\{R_{hijk}-\frac{R}{n(n-2)}g_{hijk}\right\}-R^{\cdot b}_{a\cdot hi}R^{\cdot a}_{b\cdot jk} \\ +R^{\cdot a}_{h\cdot jb}R^{\cdot b}_{i\cdot ka}-R^{\cdot a}_{h\cdot kb}R^{\cdot b}_{i\cdot ja}=0,$$

Thus the above matrix condition and $(5\cdot 2)$ is the necessary and sufficient condition that an Einstein space V_n be of class one [10]. In order to obtain $(5\cdot 1)$ and $(5\cdot 2)$, it is not necessary but the fact that R_{hijk} is written in the form $(1\cdot 8)$ $(P=1)$ and the Ricci tensor satisfies the characteristic equation $R_{ij}=(R/n)g_{ij}$ of Einstein space. In our case S_{hijk} is also written in the form $(4\cdot 5)$ $(P=1)$ and that we have

$$S_{ij}=\frac{S}{n}g_{ij}\ (=g^{ab}S_{iajb}),\quad S=R-n(n-1)K\ (=g^{ab}S_{ab}),$$

by means of $(4\cdot 4)$. Hence, from $(4\cdot 5)$ we have in like manner the equation

$$(5\cdot 2')\qquad \frac{(n-2)S}{n}\left\{S_{hijk}-\frac{S}{n(n-2)}g_{hijk}\right\}-S^{\cdot b}_{a\cdot hi}S^{\cdot a}_{b\cdot jk} \\ +S^{\cdot a}_{h\cdot jb}S^{\cdot b}_{i\cdot ka}-S^{\cdot a}_{h\cdot kb}S^{\cdot b}_{i\cdot ja}=0.$$

Substitution for S_{hijk} and S the form $(4\cdot 4)$ and $R-n(n-1)K$ respectively gives the following equation of first degree in terms of K:

$$(5\cdot 3)\qquad A_{hijk}\cdot K-B_{hijk}=0,$$

where we put

$$(5\cdot4)\quad \begin{aligned} A_{hijk} &= (n-1)(n-2)\left\{R_{hijk}-\frac{R}{n(n-1)}g_{hijk}\right\}, \\ B_{hijk} &= \frac{(n-2)R}{n}\left\{R_{hijk}-\frac{R}{n(n-2)}g_{hijk}\right\} \\ &\quad -R^{b}_{a\cdot hi}R^{a}_{b\cdot jk}+R^{a}_{h\cdot jb}R^{b}_{i\cdot ka}-R^{a}_{h\cdot kb}R^{b}_{i\cdot ja}. \end{aligned}$$

We may naturally assume that V_n itself is not of constant curvature and hence the tensor A_{hijk}, the coefficient of K in $(5\cdot3)$, is not zero. Accordingly we have clearly unique solution K of $(5\cdot3)$ if and only if the equation

$$(5\cdot5)\quad \begin{vmatrix} A_{abcd} & B_{abcd} \\ A_{hijk} & B_{hijk} \end{vmatrix} = 0 \quad (a, b, c, d, h, i, j, k=1, \cdots, n),$$

be satified. Further it will be unavoidable that the equation

$$(5\cdot6)\quad \begin{vmatrix} A_{abcd} & B_{abcd} \\ A_{hijk,l} & B_{hijk,l} \end{vmatrix} = 0 \quad (a, \cdots, k, l=1, \cdots, n),$$

must satisfy as the condition that K determined as the solution of $(5\cdot3)$ is constant, similar to $(4\cdot13)$.

Now S_{hijk} has been intrinsically determined and the second fundamental tensor H_{ij} satisfying $(4\cdot5)$ is found from the equation

$$(5\cdot7)\quad H_{hi}H_{jk} = e\,D'_{hi|jk},$$

where $D'_{hi|jk}$ is obtained from $D_{hi|jk}$ by replacing R_{hijk} and R by S_{hijk} and S respectively. The fact, that matrix $\|D'_{hi|jk}\|$ is of rank one and semi-definite, is the condition that there exists H_{ij} satisfying $(5\cdot7)$. But we must exclude the special case of $S=0$, this case be characterized by

$$(5\cdot8)\quad T_{hijk} \equiv A_{hijk}\cdot R-n(n-1)B_{hijk} = 0,$$

which follows from $(4\cdot4)$ and $(5\cdot3)$. Thus defined H_{ij} satisfies the Gauss equation, because the condition for this is given by $(5\cdot2')$, which is same as $(5\cdot3)$. It is to be noted that the Codazzi equation is a consequence of the Gauss equation, if S does not vanish. In fact, contracting the Gauss equation, namely

$$R_{hijk}-K\cdot g_{hijk} = e(H_{hj}H_{ik}-H_{hk}H_{ij}),$$

by g^{ik}, we have

$$\frac{S}{n} g_{hj} = e(g^{ik} H_{hj} H_{ik} - g^{ik} H_{hk} H_{ij}).$$

From this it follows by the same process as in [10] that the determinant $|H_{ij}|$ does not vanish for $S \neq 0$. Consequently

THEOREM 7. *Let V_n $(n>3)$ be an Einstein n-space, such that it is not of constant curvature and the tensor T_{hijk} does not be zero. In order that V_n is imbedded in an $(n+1)$-space of constant curvature, it is necessary and sufficient that the equation (5·5) and (5·6) are satisfied and the matrix $\|D'_{hi|jk}\|$ is of rank one and semi-definite. The constant curvature K of the enveloping space is determined by the equation (5·3).*

Next we consider the special case of $S=0$. Then the constant curvature of the enveloping space must be equal to $R/n(n-1)$. It should be remarked that the scalar curvature of any Einstein space is constant [16, p. 93], and accordingly $K=R/n(n-1)$ is constant. We have from the Gauss equation

$$H_{ab} S_{ijkl} + H_{al} S_{ijbk} + H_{ik} S_{jabl} + H_{jk} S_{ialb} = 0^*.$$

Contracting by g^{ab} we have by means of $S_{ij}=0$

$$H S_{ijkl} = H_{al} S^{a}_{k\cdot ij} \quad (H = g^{ab} H_{ab}).$$

If we multiply this by H_{bh} and subtract from it the equation obtained by interchanging h and l, we have in virture of the Gauss equation

$$H(H_{bh} S_{ijkl} - H_{bl} S_{ijkh}) = e S_{ablh} S^{a}_{k\cdot ij}\,.$$

From this and similar expressions for the other terms in the right-hand member of the following equation it follows that

$$(5\cdot 9) \qquad H H_{bk} S_{ijhl} = \frac{e}{2} (S_{ablh} S^{a}_{k\cdot ij} + S_{abkh} S^{a}_{l\cdot ij} + S_{ablk} S^{a}_{h\cdot ij}).$$

Eliminating HH_{bk} we have

$$(5\cdot 10) \qquad \begin{vmatrix} S_{ijhl} & P_{bk|ijhl} \\ S_{acdm} & P_{bk|acdm} \end{vmatrix} = 0 \qquad (a, \cdots, m=1, \cdots, n)\,,$$

where $P_{bk|ijhl}$ is the right-hand member in (5·9) divided by e. From

* Cf. (8·2) in [9].

(5·10) we have the equation of second degree in terms of K and sustitution for K the expression $R/n(n-1)$ gives

$$(5\cdot 11)\qquad \begin{aligned} & R^2(g_{ijhl}Q_{bk\mid acdm}-g_{acdm}Q_{bk\mid ijhl}) \\ & +n(n-1)R(R_{ijhl}Q_{bk\mid acdm}-R_{acdm}Q_{bk\mid ijhl} \\ & -g_{ijhl}P'_{bk\mid acdm}+g_{acdm}P'_{bk\mid ijhl}) \\ & +n^2(n-1)^2(R_{ijhl}P'_{bk\mid acdm}-R_{acdm}P'_{bk\mid ijhl}) = 0, \end{aligned}$$

where we put

$$\begin{aligned} Q_{bk\mid acdm} &= g_{ak}R_{bcdm}-g_{ck}R_{badm}+g_{am}R_{bcdk} \\ &\qquad -g_{cm}R_{badk}+g_{ad}R_{bckm}-g_{cd}R_{bakm}\,, \\ P'_{bk\mid acdm} &= R_{ibmd}R^{i}_{k\cdot ac}+R_{ibkd}R^{i}_{m\cdot ac}+R_{ibmk}R^{i}_{d\cdot ac}\,. \end{aligned}$$

Contracting (5·9) by g^{bk} we have

$$(5\cdot 12)\qquad H^2\cdot S_{ijhl} = eP_{ijhl} \qquad (=eg^{bk}P_{bk\mid ijhl}),$$

where K in S_{ijhl} and P_{ijhl} is replaced by $R/n(n-1)$. Elimination H^2 from (5·12) gives

$$\begin{vmatrix} S_{abcd} & P_{abcd} \\ S_{ijhl} & P_{ijhl} \end{vmatrix} = 0\,,$$

which is a consequence of (5·10). Hence, if (5·11) is satisfied, then (5·10) is satisfied and so we obtain from (5·12) H^2, because S_{ijhl} does not be zero for V_n, which is assumed not to be of constant curvature. In this case e must be chosen so that H is real. Then from (5·9) we have H_{bk}, because the condition that (5·9) has solution H_{bk} is given by (5·10), which is equivalent to (5·11). But we must suppose $H\neq 0$, that is to say, $P_{ijhl}\neq 0$ from (5·12). Therefore H_{ij} is thus determined under the condition (5·11) and $P_{ijhl}\neq 0$.

Further we must get the condition that as above determined H_{ij} satisfy the Gauss equation. From (5·9) and (5·12) we obtain

$$H^2 S_{rstu}\cdot H_{bk}S_{ijhl}\cdot H_{cm}S_{adpq} = \frac{1}{4}P_{bk\mid ijhl}P_{cm\mid adpq}S_{rstu}\,.$$

Interchanging k and m we have from the Gauss equation

$$(5\cdot 13)\qquad \begin{aligned} & P_{rstu}S_{bckm}S_{ijhl}S_{adpq} \\ & = \frac{1}{4}S_{rstu}(P_{bk\mid ijhl}P_{cm\mid adpq}-P_{bm\mid ijhl}P_{ck\mid adpq})\,. \end{aligned}$$

Conversely if (5·13) is satisfied, as above determined H_{ij} satisfies the Gauss equation, as easily seen.

Finally we give the condition that these H_{ij} satisfy the Codazzi equation. In this case, the Codazzi equation is perhaps independent from the Gauss equation. Since H_{ij} is expressed in terms of curvature tensor, the Codazzi equation itself is also expressible in terms of the curvature tensor and its derivatives. But we can explicitly write this condition. In fact, multiplying (5·9) by H and substituting from (5·12), we get

$$H_{bk}P_{ijhl} = HP_{bk|ijhl}\,.$$

Covariant differentiation of this equation with respect to x^m and multiplication by $HS_{pqrs}S_{acdt}$ gives

$$H\cdot H_{bk,m}P_{ijhl}S_{pqrs}S_{acdt} = -HH_{bk}S_{pqrs}S_{acdt}P_{ijhl,m}$$
$$+HH_{,m}S_{pqrs}S_{acdt}P_{bk|ijhl}+H^2S_{pqrs}S_{acdt}P_{bk|ijhl,m}\,.$$

Substituting from (5·9), (5·12) and the equation obtained from (5·12) by covariant differentiation with respect to x^m, we get

$$HH_{bk,m}S_{pqrs}S_{acdt}P_{ijhl} = Q_{pqrs|acdt|ijhl|bkm}\,, \tag{5·14}$$

where we put

$$Q_{pqrs|acdt|ijhl|bkm} = \frac{1}{2}(P_{pqrs,m}S_{acdt}-S_{pqrs,m}P_{acdt})P_{bk|ijhl}$$
$$+(P_{bk|ijhl,m}P_{pqrs}-P_{bk|pqrs}P_{ijhl,m})S_{acdt}\,.$$

The equation (5·14) is satisfied by H_{ij} above determined, so that the equation

$$Q_{pqrs|acdt|ijhl|bkm}-Q_{pqrs|acdt|ijhl|bmk} = 0\,, \tag{5·15}$$

is equivalent to the Codazzi equation. Consequently

THEOREM 8. *Let V_n ($n>3$) be an Einstein n-space such that it is not of constant curvature and the tensor T_{hijk} vanishes, but not P_{hijk}. In order that V_n is imbedded in S_{n+1} of constant curvature, it is necessary and sufficient that the equations (5·11), (5·13) and (5·15) are satisfied. The curvature K of S_{n+1} is equal to $R/n(n-1)$, where R is scalar curvature of V_n.*

This theorem can be applied well when we discuss the problem for an Einstein n-space of vanishing scalar curvature to be of class one. Namely, we replace merely S_{hijk} by R_{hijk} in the above discussion. But, in this case, it must be required that elementary divisors of matrix $\|\rho H_{ij}-g_{ij}\|$ are not all simple [16, p. 199].

§ 6. Imbedding a conformally flat $n(>3)$-space in an $(n+1)$-space of constant curvature

In this section we consider an n-space V_n, whose conformal curvature tensor vanishes. Such a space is the only case excepted from the general discussion of the fourth section, because all of coefficients of K in (4·8) are equal to zero, and hence a particular circumstance will be anticipated. If $n>3$, V_n is conformally flat and we have already the condition that V_n be of class one [18] and so we go along the similar process as shown in the last two sections.

The conformally flat n-space V_n $(n>3)$ is characterized by the equation

$$R_{hijk} = g_{hj}l_{ik}-g_{hk}l_{ij}+l_{hj}g_{ik}-l_{hk}g_{ij}\,, \tag{6·1}$$

where we put

$$l_{ij} = \frac{1}{n-2}\Big(R_{ij}-\frac{R}{2(n-1)}g_{ij}\Big).$$

Making use of (6·1) and the equation

$$H_{ab}R_{ijkl}+H_{al}R_{ijbk}+H_{ik}R_{jabl}+H_{jk}R_{ialb} = 0\,,$$

which is deduced from the Gauss equation

$$R_{ijkl} = e(H_{ik}H_{jl}-H_{il}H_{jk}),$$

we obtain in [18] the equation

$$H_{ij} = a g_{ij}+b l_{ij}\,.$$

Now S_{hijk} in (4·4) is also written in the form

$$S_{hijk} = g_{hj}l'_{ik}-g_{hk}l'_{ij}+l'_{jh}g_{ik}-l'_{hk}g_{ij}\,, \tag{6·2}$$

where l'_{ij} is defined by

$$l'_{ij} = l_{ij}-\frac{K}{2}g_{ij}\,.$$

Thus the similar process used in [18] leads us to

$$H_{ij} = a g_{ij}+b l'_{ij}\,, \tag{6·3}$$

where a and b are scalar.

We remark here that the rank of the matrix $\|H_{ij}\|$ is n. Because the Ricci's directions coincide with principal directions [12] and so it is easily proved that at least $n-1$ principal curvature are equal, so that the process used in [18] is not limited to our case for the proof of $|H_{ij}|\neq 0$. Accordingly the Codazzi equation is a consequence of the Gauss equation and then we consider only the Gauss equation in the following.

Substitution from (6·3) in the Gauss equation

$$R_{hijk}-Kg_{hijk}=e(H_{hj}H_{ik}-H_{hk}H_{ij}), \tag{6·4}$$

gives

$$\left(e-ab+\frac{K}{2}b^2\right)R_{hijk}=\left(a^2+\frac{K^2}{4}b^2+(e-ab)K\right)g_{hijk} +b^2(l_{hj}l_{ik}-l_{hk}l_{ij}). \tag{6·5}$$

In the first place we consider the particular case when the matrix $\|l_{ij}\|$ is of rank less than two. Since V_n may be assumed not to be of constant curvature, it follows that the rank is one, and from (6·5) we have

$$e-ab+\frac{K}{2}b^2=0\,,\quad a^2+\frac{K^2}{4}b^2+(e-ab)K=0\,, \tag{6·6}$$

from which we get immediately

$$a^2=-\frac{e}{4}K\,,\quad b^2=-\frac{e}{K},\quad a=-\frac{K}{2}b\,. \tag{6·7}$$

Conversely if $\|l_{ij}\|$ is of rank one, we take an arbitrary constant $K\neq 0$ and choose e plus or minus one, according as K is negative or positive, and then we define a, b by (6·7). Moreover H_{ij} is defined by (6·3), then we obtain (6·4) by substitution. Consequently V_n can be imbedded in any $(n+1)$-space of constant curvature $\neq 0$. On the other hand, V_n can not be imbedded in flat space; since otherwise we show easily $l_{ij}=0$.

Thus, from our stand-point, we obtain the special type of conformally flat space, in which $\|l_{ij}\|$ is of rank one. In the following, we treat such spaces for a while. We get in the first place

$$l_{ij}=l_il_j\quad (i,j=1,\cdots,n),$$

since l_{ij} is symmetric and $\|l_{ij}\|$ is of rank one. We see that l_i, defined by the above equations, is unique to within algebraic sign.

Multiplying (6·1) by l_m and subtracting from this the equation obtained by interchanging k and m, we have

$$(6\cdot8)\qquad R_{jkhi}l_m+R_{jkim}l_h+R_{jkmh}l_i=0\,,$$

by means of (6·1). Contracting (6·8) by $g^{hj}g^{mk}$ we have

$$(6\cdot9)\qquad \left(R_{ia}-\frac{R}{2}g_{ia}\right)l^a=0\qquad (l^a=g^{ai}l_i),$$

from which we get

$$\frac{R_{ij}l^il^j}{g_{ij}l^il^j}=\frac{R}{2},$$

which is called the mean curvature of the space for the direction l^i and, from (6·9), l^i is the Ricci principal direction [16, p. 113]. Therefore *l^i is the Ricci principal direction and the mean curvature for this direction is $R/2$.*

We return to the general case when $\|l_{ij}\|$ is of rank greater than one. Then we can easily show that (6·5) must have non-trivial solutions $e-ab+(K/2)b^2$, $a^2+(K^2/4)b+(e-ab)K$ and b^2; so that we have as the condition

$$(6\cdot10)\qquad \begin{vmatrix} R_{abcd} & g_{abcd} & l_{abcd}\\ R_{hijk} & g_{hijk} & l_{hijk}\\ R_{pqrs} & g_{pqrs} & l_{pqrs}\end{vmatrix}=0$$

$$(a,\cdots,s=1,\cdots,n)$$

where we put

$$l_{hijk}=l_{hj}l_{ik}-l_{hk}l_{ij}\,.$$

However we have in [18] that a conformally flat n-space $(n>3)$ is of class one, if and only if the matrix $\|l_{ij}\|$ is of rank greater than one and (6·10) is satisfied. Thus such a space can be imbedded in flat space. Namely, the equation

$$tR_{hijk}=Ag_{hijk}+Bl_{hijk}\,,$$

has a non-trivial solution A, B, and t $(\neq0)$ and hence we have

$$(6\cdot11)\qquad R_{hijk}=Cg_{hijk}+Dl_{hijk}\,.$$

In the case of class one, we define a and b as follows:

$$a=\sqrt{\frac{eC}{2}},\qquad b=\sqrt{\frac{eD}{2}}.$$

On the other hand, compairing this with (6·5) we put

$$e-ab+\frac{K}{2}b^2=\rho \quad (\neq 0),$$

$$(6\cdot 12) \qquad a^2+\frac{K^2}{4}b^2+(e-ab)K=\rho C,$$

$$b^2=\rho D,$$

and from this we get

$$a=e_a\sqrt{\rho\left(C-K+\frac{K^2}{4}D\right)}, \quad b=e_b\sqrt{\rho D}, \quad \rho=\frac{e'}{2-KD},$$

where e_a, e_b, and e' are plus or minus one and satisfy the condition

$$\frac{De'}{2-KD}>0, \quad e_a e_b e'>0.$$

Conversely, if we choose a, b as above mentioned, where K is arbitrary constant such that $KD\neq 2$, then (6·12) is satisfied and hence, if we define H_{ij} by (6·3), these H_{ij} satisfy the Gauss equation. Therefore our space V_n can be imbedded in any $(n+1)$-space of constant curvature $K=0$ or $\neq 0$. But we must except the special case when K is taken $=2/D$. If D is constant and $K=2/D$, we see easily $e=0$, and hence V_n can not be imbedded in an $(n+1)$-space of constant curvature $2/D$. Therefore

THEOREM 9. *Let V_n $(n>3)$ be a conformally flat n-space not of constant curvature.*

(1) *If the matrix $\|l_{ij}\|$ is of rank one, V_n is imbedded in any $(n+1)$-space of constant curvature but not in flat space.*

(2) *If the rank is greater than one, V_n is imbedded in an $(n+1)$-space of constant curvature, if and anly if (6·10) is satisfied. Such a space V_n can be imbedded in any $(n+1)$-space of constant curvature $\neq 2/D$, where D is defined by (6·11).*

In addition, we consider n-space V_n, which is imbedded in a conformally flat $(n+1)$-space C_{n+1}. The conformal curvature tensor of V_n is expressed in the form [15]

$$C_{ijkl}=M_{ik}M_{jl}-M_{il}M_{jk}+\frac{1}{n-2}(M_{ia}M_k^a g_{jl}-M_{ia}M_l^a g_{jk}$$
$$+g_{ik}M_{ja}M_l^a-g_{il}M_{ja}M_k^a)-\frac{M_b^a M_a^b}{(n-1)(n-2)}g_{ijkl},$$

where we put

$$M_{ij} = H_{ij} - \frac{1}{n} g_{ij} g^{ab} H_{ab} :$$

from which we have

$$(6\cdot 13) \qquad R_{ijkl} = M_{ik} M_{jl} - M_{il} M_{jk} + g_{ik} A_{jl} - g_{il} A_{jk} + A_{ik} g_{jl} - A_{il} g_{jk} ,$$

where A_{ij} is defined by the following form

$$A_{ij} = \frac{1}{(n-2)} (R_{ij} + M_{ia} M^a_j) - \frac{M^a_b M^b_a + R}{2(n-1)(n-2)} g_{ij} .$$

Now we shall generalize (6·13) and consider V_n, the curvature tensor of which is expressed in the form

$$(6\cdot 14) \qquad R_{ijkl} = N_{ik} N_{jl} - N_{il} N_{jk} + G_{ik} a_{jl} - G_{il} a_{jk} + a_{ik} G_{jl} - a_{il} G_{jk} .$$

It is easily verified that the tensor $\underset{(2)}{R}$ defined by (2·3) vanishes. Therefore if this space V_n can be imbedded in an $(n+1)$-space of constant curvature K, we see from (4·8) that $K=0$ or the tensor $\underset{(2)}{A}$ vanishes, so that V_n is of class one or conformally flat. Hence we have the

THEOREM 10. *If V_n can be imbedded in an $(n+1)$-space of constant curvature and the curvature tensor is expressed in the form (6·14), then V_n is conformally flat or of class one.*

From (6·13) we have the following corollary:

COROLLARY. *Let V_n be such an n-space, that is not conformally flat and not of class one, but is imbedded in a conformally flat $(n+1)$-space. Then V_n can not be imbedded in any $(n+1)$-space of constant curvature.*

§ 7. Imbedding a Riemann n-space in an $(n+p)$-space of constant curvature

In this section we show a remarkable theorem that the problem of imbedding in an $(n+p)$-space of constant curvature $\neq 0$ for $2p=2, \cdots, n-2$ is reducible generally to one of imbedding in a flat $(n+p)$-space, as seen in the fourth section for $p=1$.

From (4·6) we have

$$(7\cdot 1) \qquad \underset{(p+1)}{S}{}_{i_1 \cdots i_{p+1} | j_1 \cdots j_{p+1} k_1 \cdots k_{2p+2}} = 0 ,$$

as a condition for V_n to be imbedded in an $(n+p)$-space of constant curvature. We substitute from (4·4) and then have equations of $(p+1)$-th degree in terms of K, the constant curvature of enveloping space. But coefficients of $K^2, \cdots, K^{p+1}$ in these equations are all equal to zero. In fact we see from (2·7)

$$(7\cdot 2)\quad \begin{aligned} \underset{(p+1)}{R}{}_{i_1\cdots i_{p+1}|j_1\cdots j_{p+1}|k_1\cdots k_{2p+2}} &= \frac{1}{4!(2p-2)!}\sum_{s,t\,(s<t)}^{1\cdots p+1}(-1)^{s+t-1} \\ &\times \underset{(2)}{R}{}_{i_1 i_2|j_s j_t|a_1 a_2 b_s b_t}\,\underset{(p-1)}{R}{}_{i_3\cdots i_{p+1}|j_1\cdots \check{j}_s\cdots \check{j}_t\cdots j_{p+1}|a_3\cdots a_{p+1}b_1\cdots \check{b}_s\cdots \check{b}_t\cdots b_{p+1}} \\ &\times \delta^{a_1\cdots a_{p+1}b_1\cdots b_{p+1}}_{k_1\cdots\cdots\cdots\cdots k_{2p+2}}. \end{aligned}$$

By means of (7·2) coefficients of K^2 are equal to the sum of terms, one of which, for example, is as follows:

$$\begin{aligned} &\frac{1}{4!(2p-2)!}(-1)^{s+t-1} \\ &\times \overline{\underset{(2)}{R}}{}_{i_1 i_2|j_s j_t|a_1 a_2 b_s b_t}\,\underset{(p-1)}{R}{}_{i_3\cdots i_{p+1}|j_1\cdots \check{j}_s\cdots \check{j}_t\cdots j_{p+1}|a_3\cdots a_{p+1}b_1\cdots \check{b}_s\cdots \check{b}_t\cdots b_{p+1}} \\ &\times \delta^{a_1\cdots a_{p+1}b_1\cdots b_{p+1}}_{k_1\cdots\cdots\cdots\cdots k_{2p+2}} \end{aligned}$$

where $\overline{\underset{(2)}{R}}{}_{i_1 i_2|j_s j_t|a_1 a_2 b_s b_t}$ is obtained from

$$\underset{(2)}{R}{}_{i_1 i_2|j_s j_t|a_1 a_2 b_s b_t} = \frac{1}{2^2}\underset{(2)}{\varepsilon}{}^{uv} R_{i_1 j_u c_1 d_u} R_{i_2 j_v c_2 d_v}\,\delta^{c_1 c_2 d_1 d_2}_{a_1 a_2 b_s b_t}$$

by replacing $R_{i_1 j_u c_1 d_u}, \cdots$ by $g_{i_1 c_1} g_{j_u d_u} - g_{i_1 d_u} g_{j_u c_1}, \cdots$, and this is clearly equal to zero. Thus the coefficients of K^2 vanish and the similar proof is applicable to the case when we show that coefficients of $K^3, \cdots, K^{p+1}$ are all vanishing. Consequently from (7·1) we obtain the equations of first degree in terms of K as follows:

$$(7\cdot 3)\qquad \underset{(p+1)}{A}{}_{i_1\cdots i_{p+1}|j_1\cdots j_{p+1}|k_1\cdots k_{2p+2}}\cdot K - 2^p\cdot \underset{(p+1)}{R}{}_{i_1\cdots i_{p+1}|j_1\cdots j_{p+1}|k_1\cdots k_{2p+2}} = 0.$$

The component of $\underset{(p+1)}{A}$, coefficient of K in (7·3), is easily calculated by (2·7) as follows:

$$(7\cdot 4)\quad \begin{aligned} &\underset{(p+1)}{A}{}_{i_1\cdots i_{p+1}|j_1\cdots j_{p+1}|k_1\cdots k_{2p+2}} \\ &= \frac{1}{2}\sum_{s=1}^{p+1}\underset{(p+1)}{\varepsilon}{}^{u\cdots vxy\cdots z}(g_{i_s a_s} g_{j_x b_x} - g_{i_s b_x} g_{j_x a_s}) R_{i_1 j_u a_1 b_u}\cdots R_{i_{s-1} j_v a_{s-1} b_v} \\ &\quad \times R_{i_{s+1} j_y a_{s+1} b_y}\cdots R_{i_{p+1} j_z a_{p+1} b_z}\,\delta^{a_1\cdots b_{p+1}}_{k_1\cdots k_{2p+2}}, \\ &= \sum_{s=1}^{p+1}\underset{(p+1)}{\varepsilon}{}^{u\cdots vxy\cdots z} g_{i_s a_s} g_{j_x b_x} R_{i_1 j_u a_1 b_u}\cdots R_{i_{s-1} j_v a_{s-1} b_v} \\ &\quad \times R_{i_{s+1} j_y a_{s+1} b_y}\cdots R_{i_{p+1} j_z a_{p+1} b_z}\,\delta^{a_1\cdots b_{p+1}}_{k_1\cdots k_{2p+2}}. \end{aligned}$$

The latter is generalization of (4·9). If $\underset{(p+1)}{A}$ does not be zero, (7·3) will uniquely determine the constant curvature of enveloping space S_{n+p}, and accordingly the enveloping space S_{n+p} itself, under the condition

$$(7\cdot5)\quad \begin{vmatrix} \underset{(p+1)}{A}{}_{a_1\cdots a_{p+1}|b_1\cdots b_{p+1}|c_1\cdots c_{2p+2}} & \underset{(p+1)}{R}{}_{a_1\cdots a_{p+1}|b_1\cdots b_{p+1}|c_1\cdots c_{2p+2}} \\ \underset{(p+1)}{A}{}_{i_1\cdots i_{p+1}|j_1\cdots j_{p+1}|k_1\cdots k_{2p+2}} & \underset{(p+1)}{R}{}_{i_1\cdots i_{p+1}|j_1\cdots j_{p+1}|k_1\cdots k_{2p+2}} \end{vmatrix} = 0\,.$$

And further condition that as thus determined K be constant is clearly given by the condition

$$(7\cdot6)\quad \begin{vmatrix} \underset{(p+1)}{A}{}_{a_1\cdots a_{p+1}|b_1\cdots b_{p+1}|c_1\cdots c_{2p+2}} & \underset{(p+1)}{R}{}_{a_1\cdots a_{p+1}|b_1\cdots b_{p+1}|c_1\cdots c_{2p+2}} \\ \underset{(p+1)}{A}{}_{i_1\cdots i_{p+1}|j_1\cdots j_{p+1}|k_1\cdots k_{2p+2},l} & \underset{(p+1)}{R}{}_{i_1\cdots i_{p+1}|j_1\cdots j_{p+1}|k_1\cdots k_{2p+2},l} \end{vmatrix} = 0\,.$$

Now we have the intrinsic form of constant curvature and then ${}_{'}S_{hijk}$ is intrinsically define. Hence our problem reduces to the consideration of the equations (4·5), (4·2) and (4·3), which are formally equivalent to the Gauss, Codazzi and Ricci equations respectively in the case of V_n being of class p. For example, we can give the condition that V_n be imbedded in an $(n+2)$-space of constant curvature, if $\underset{(3)}{A}$ does not vanish; namely, we make merely use of the discussions in [18].

If $\underset{(p+1)}{A}$ does not vanish, the enveloping space of constant curvature, if exists, is unique. While, if V_n can be imbedded in $(n+p)$-space S_{n+p} and S'_{n+p}, both of which are of constant curvature K, K' $(\neq)$, we have from (7·3) $\underset{(p+1)}{A}=0$. Therefore

THEOREM 10. *If the tensor* $\underset{(p+1)}{A}$ *of* V_n *does not vanish, the enveloping space of constant curvature, if exists, is unique. If there exist more than one* $(n+p)$*-space of constant curvature such that these curvatures are different, the tensor* $\underset{(p+1)}{A}$ *of* V_n *is necessarily equal to zero.*

It is very compricated to study such a space that the tensor $\underset{(p+1)}{A}$ vanishes. For instance, we contract $\underset{(3)}{A}{}_{i_1i_2i_3|j_1j_2j_3|k_1\cdots k_6}$ by $g^{i_3k_3}g^{j_3k_6}$ and, if we moreover contract, we have a tensor, which is identically zero. This fact is similar to the case for the conformal curvature tensor. We can easily give an example of such a space that $\underset{(3)}{A}$ vanishes, but the problem of studying the geometrical properties of all the space, in which $\overset{(3)}{A}$ vanishes, is probably very hard. As in the case of $p=1$, it is possible that there exists such a space that can be imbedded in more than one $(n+p)$-spaces of constant curvature.

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