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On the holonomy groups of the group-spaces.

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Let S be the group-space of a continuous group of transformations G_r, in which the $(+)$ or $(-)$-connection is induced. As we shall show in this paper, the holonomy group of S is a group of affine translations. It is a question, therefore, that how many essential parameters are there in the holonomy group. We shall reply to this question by giving a necessary and sufficient condition that the holonomy group has $p(\leqq r)$ essential parameters, and study the relations between the holonomy group and G_r.

We shall make use here principally of the notations of L. P. Eisenhart in his work " Continuous Group of Transformations [1]".

1. Let

$$x'^i = f^i(x, a) \qquad (i=1,\cdots,n) \tag{1·1}$$

be the equations of a continuous transformation group G_r with n independent variables x^i and r essential parameters a^α, and

$$a_3^\alpha = \varphi^\alpha(a_1, a_2) \qquad (\alpha=1,\cdots,r) \tag{1·2}$$

be the equations of the parameter-group of G_r. That is, the groups defined by (1·2) as a_2^α and a_1^α are considered as the parameters are called respectively the first and second parameter-groups of G_r. Let us denote them by $\mathfrak{G}_r^{(+)}$ and $\mathfrak{G}_r^{(-)}$ respectively.

Let

$$\frac{\partial x'^i}{\partial a^\alpha} = \xi_b^i(x') A_\alpha^b(a) \qquad \begin{pmatrix} i=1,\cdots,n; \\ b, \alpha=1,\cdots,r \end{pmatrix},$$

$$\frac{\partial a_3^\alpha}{\partial a_2^\beta} = A_b^\alpha(a_3) A_\beta^b(a_2) \qquad (b, \alpha, \beta=1,\cdots,r),$$

and

$$\frac{\partial a_3^\alpha}{\partial a_1^\beta} = \bar{A}_b^\alpha(a_3) \bar{A}_\beta^b(a_1) \qquad (b, \alpha, \beta=1,\cdots,r)$$

be the fundamental equations of G_r, $\mathfrak{G}_r^{(+)}$ and $\mathfrak{G}_r^{(-)}$ respectively, where $\|A^a_b\|$ is the inverse matrix of $\|A^b_\alpha\|$, and $\bar{A}^\alpha_b(a)$ and $\bar{A}^b_\alpha(a)$ are the corresponding ones to $A^\alpha_b(a)$ and $A^b_\alpha(a)$ respectively when (1·1) are replaced by their inverse transformations. Then we have the next relations:

$$(1\cdot 3)\qquad \frac{\partial A^c_\alpha}{\partial a^\beta}-\frac{\partial A^c_\beta}{\partial a^\alpha}=c_{ab}^{\ c}A^a_\alpha A^b_\beta \qquad (a, b, c, \alpha, \beta=1,\cdots, r),$$

$$(1\cdot 4)\qquad \frac{\partial \bar{A}^c_\alpha}{\partial a^\beta}-\frac{\partial \bar{A}^c_\beta}{\partial a^\alpha}=\bar{c}_{ab}^{\ c}\bar{A}^a_\alpha \bar{A}^b_\beta \qquad (a, b, c, \alpha, \beta=1,\cdots, r),$$

where $c_{ab}^{\ c}$ are the constants of structure, and

$$(1\cdot 5)\qquad \bar{c}_{ab}^{\ c}=-c_{ab}^{\ c}.$$

Let S be the group-space. The two kinds of connections are able to be induced in S, and their coefficients of connections are given by

$$(1\cdot 6)\qquad L_{\beta\gamma}^{\alpha}=-A^b_\beta\frac{\partial A^\alpha_b}{\partial a^\gamma} \qquad (b, \alpha, \beta, \gamma=1,\cdots, r),$$

$$(1\cdot 7)\qquad \bar{L}_{\beta\gamma}^{\alpha}=-\bar{A}^b_\beta\frac{\partial \bar{A}^\alpha_b}{\partial a^\gamma} \qquad (b, \alpha, \beta, \gamma=1,\cdots, r)$$

respectively. These spaces are called the space of (+)-connection and that of (−)-connection. We shall denote them by $S^{(+)}$ and $S^{(-)}$ respectively.

The curvature tensors of $S^{(+)}$ and $S^{(-)}$ thus defined are zero. Therefore both of $S^{(+)}$ and $S^{(-)}$ are spaces of affine connections without curvatures. We shall say simply that they are flat following Eisenhart (although the torsions do not vanish in general). Moreover, from (1·6),

$$\frac{\partial A^\alpha_b}{\partial a^\gamma}+L_{\beta\gamma}^{\alpha}A^\beta_b=0,$$

hence the vectors $\vec{A}_b$ whose components are $A_b^1,\cdots, A_b^r$ are absolutely parallel in $S^{(+)}$. Similarly from (1·7) the vectors $\vec{\bar{A}}_b$ whose components are $\bar{A}_b^1,\cdots, \bar{A}_b^r$ are absolutely parallel in $S^{(-)}$.

At each point a^α in $S^{(+)}$ we may assigne a "repère" $R_a(a-\vec{A}_1\cdots\vec{A}_r)$. Let C be a closed curve upon $S^{(+)}$. Develop it on the tangential space at a_1^α. We obtain the image $\mathfrak{R}_{a_1}$ of the repère associated to a_1^α regarded as the terminal point of C. This image does not coincide with the image repère associated to initial point

a_1^α wihch is R_{a_1} itself. We can over-lap them, however, by translating the one to the other, since $S^{(+)}$ is flat and the vectors $\vec{A}_b$ are absolutely parallel. Therefore we have:

Theorem 1. *On $S^{(+)}$, the holonomy group is a group of affine translations. So is it on $S^{(-)}$.*

2. Let us represent by $a^\alpha(t)$ $(t_0 \leqq t \leqq t_1)$ a curve C which, we suppose, is situated on $S^{(+)}$. Develop C on the tangential space at $a^\alpha(t_0)$, then the image vectors $\vec{\mathfrak{A}}_b(a(t))$ of $\vec{A}_b(a(t))$ are always parallel to each other. We shall denote them, therefore, simply by $\vec{\mathfrak{A}}_b$. Take a point $P(a^\alpha(t))$ on C. The image $\overrightarrow{\delta P}$ of an infinitesimal vector $\overrightarrow{dP}$ is given by

$$\overrightarrow{\delta P} = \left\{ A_\alpha^b(a(t)) \frac{da^\alpha}{dt} dt \right\} \vec{\mathfrak{A}}_b \qquad (b, \alpha = 1, \cdots r). \tag{2·1}$$

Hence the position of the image of any point $a^\alpha(t)$ on C refered to the initial repère can be written

$$\left\{ \int_{t_0}^{t_1} A_\alpha^b(a(t)) \frac{da^\alpha}{dt} dt \right\} \vec{\mathfrak{A}}_b \qquad (b, \alpha = 1, \cdots, r).$$

Let us denote by $\vec{\mathfrak{T}}(C)$ the translation which brings the image of the terminal point of C to the initial point, then

$$\vec{\mathfrak{T}}(C) = -\left\{ \int_{t_0}^{t_1} A_\alpha^b(a(t)) \frac{da^\alpha}{dt} dt \right\} \vec{\mathfrak{A}}_b \qquad (b, \alpha = 1, \cdots, r). \tag{2·2}$$

When C is a closed curve, we may represent it by

$$\vec{\mathfrak{T}}(C) = -\left\{ \int_C A_\alpha^b(a) da^\alpha \right\} \vec{\mathfrak{A}}_b \qquad (b, \alpha = 1, \cdots, r). \tag{2·3}$$

Each development is taken, hitherto, on the tangential space at a point on C. It is clear, however, that the equations (2·1), (2·2) and (2·3) are available even when that developments are taken on the tangential space at any point on $S^{(+)}$ (not situated on C), so long as the latter can be connected to the point on C. Let us call $\vec{\mathfrak{T}}(C)$ thus defined "the transformation attached to C". Then we have:

Theorem 2. *Concerning $S^{(+)}$, the transformation attached to a closed curve C is given by*

$$-\left\{ \int_C A_\alpha^b(a) da^\alpha \right\} \vec{\mathfrak{A}}_b \qquad (b, \alpha = 1, \cdots, r).$$

The similar therem will be obtained concerning $S^{(-)}$.

3. Let us now research for the condition that the component of a certain direction, say $\vec{A}_e$, of the transformation attached to any closed curve on C always vanishes. By the above theorem it is necessary and sufficient that the quantity $\int_C A^e_\alpha(a)da^\alpha$ is equal to zero for any closed curve C. Applying Stoke's theorem, we can represent this condition by

$$(3\cdot1)\qquad \frac{\partial A^e_\alpha}{\partial a^\beta}-\frac{\partial A^e_\beta}{\partial a^\alpha}=0 \qquad (\alpha,\beta=1,\cdots,r),$$

and from (1·5), we have the equivalent relations

$$c^e_{ab}A^a_\alpha A^b_\beta=0 \qquad (a,b,\alpha,\beta=1,\cdots,r),$$

which gives

$$c^e_{ab}=0 \qquad (a,b=1,\cdots,r)$$

as the determinant $|A^a_\alpha|\fallingdotseq 0$.

On the other hand, the rank of the matrix

$$M=\|c^c_{ab}\|$$

where c indicates the columns and a and b the rows, is equal to the order of the derived group of $\mathfrak{G}^{(+)}_r$. When this order is p, we can choose $A_1f,\cdots,A_rf$ as symbols of $\mathfrak{G}^{(+)}_r$ so that the first p of them generate the derived group. Concerning this set of symbols we have

$$(3\cdot2)\qquad c^e_{ab}=0 \qquad \begin{pmatrix} e=p+1,\cdots,r; \\ a,b=1,\cdots,r \end{pmatrix}$$

Moreover, for every d from 1 to p there exists at least one set of (a,b)'s for which $c^d_{ab}\fallingdotseq 0$.

Consequently, when the rank of the matrix M is $p(\leqq r)$, the symbols $A_bf(b=1,\cdots,r)$ of $\mathfrak{G}^{(+)}_r$ can be chosen so that we have the next two properties:

(1) When b takes any one of $1,\cdots,p$, there exists at least one closed curve C for which the component in the direction $\vec{\mathfrak{A}}_b$ of $\vec{\mathfrak{T}}(C)$ does not vanish.

(2) When b takes any one of $p+1,\cdots,r$, for every closed curve C, the component in the direction $\vec{\mathfrak{A}}_b$ of $\vec{\mathfrak{T}}(C)$ always vanishes.

Therefore we have:

Theorem 3. *A necessary and sufficient condition that the holo-*

nomy group of $S^{(+)}$ (or $S^{(-)}$) has $p(\leqq r)$ essential parameters is that the matrix $\|c_{ab}^{c}\|$ where c indicates the columns and a and b the rows is of rank p.

$S^{(+)}$ can be replaced by $S^{(-)}$ in the above theorem, because we can take $\bar{A}_{\alpha}^{b}$, $\bar{c}_{ab}^{c}$ instead of A_{α}^{b}, c_{ab}^{c} respectively in (3·1) and (3·2) by (1·4), (1·5) and the analogous theorem to Theorem 2.

Since the G_r and $\mathfrak{G}_r^{(+)}$ have the same structure, we have:

Theorem 4. *The order of the derived group of G_r are equal to the number of the essential parameters of the holonomy group of the group-space $S^{(+)}$ (or $S^{(-)}$).*

It is well known that when the matrix M is of rank r the group $\mathfrak{G}_r^{(+)}$, consequently G_r, is simple and vice-versa. Hence we have the following corollary of Theorem 3.

Corollary 1. *When G_r is a simple group, the holonomy group of the group-space $S^{(+)}$ (or $S^{(-)}$) is a group of affine translations with r essential parameters and vice-versa.*

When $\mathfrak{G}_r^{(+)}$, and accordingly G_r, are Abelian, the matrix M is is of rank zero, hence we have:

Corollary 2. *When G_r is Abelian, the holonomy group of the group-space $S^{(+)}$ (or $S^{(-)}$) is an identical transformation, and vice-versa.*

In the proof of Theorem 3 we have used Stoke's theorem, this theorem is available, however, when the curve considered is continuously contractible to a point in the space S. Accordingly the holonomy group of $S^{(+)}$ (or $S^{(-)}$) in the above theorem are considered in the sense of " in the small."

4. We shall now deduce Theorem 3 by a geometrical method. In this section we take $S^{(+)}$ as the group-space.

On the tangential space at each point $P(a^{\alpha})$ on a closed curve C, associate a new repère $\mathfrak{R}(\mathfrak{O}, \vec{\mathfrak{J}}_b)$ which is determined by

$$(4\cdot1)\qquad \begin{cases} \overrightarrow{P\mathfrak{O}} = u^{b}(a)\vec{\mathfrak{A}}_{b} & (b=1,\cdots,r), \\ \vec{\mathfrak{J}}_{b} = \pi_{b}^{a}(a)\vec{\mathfrak{A}}_{a} & (a,b=1,\cdots,r). \end{cases}$$

Bringing the tangential space at $P+dP(a^{\alpha}+da^{\alpha})$ to that at P, by the $(+)$-connection, we have the image repère $\mathfrak{R}'_{a+da}$ of $\mathfrak{R}_{a+da}$ which does not coincide with $\mathfrak{R}_{a}$ in general. The infinitesimal displacement from $\mathfrak{R}_{a}$ to $\mathfrak{R}'_{a+da}$ is determined by $\overrightarrow{\delta\mathfrak{O}}$ and $\overrightarrow{\delta\mathfrak{J}_b}$. By a theorem due to Cartan [2] the functions u^{b} and π_{b}^{a} in (4·1) can be chosen so that this infinitesimal displacement agrees with that induced by the holonomy group of the space. Now let $\mathfrak{G}_{(h)}$ be the holonomy group. We have shown in Theorem 1 that it is a group of affine translations. Suppose that it has $p(\leqq r)$ essential parameters. Then we can choose r vectors $\vec{A}_b$ of a repère R_a at a point $P(a^\alpha)$, so that there is no translation of $\mathfrak{G}_{(h)}$ whose component in the direction $\vec{\mathfrak{A}}_e (e=p+1,\cdots,r)$ does not vanish. Accordingly we must have

$$(4\cdot2)\qquad \begin{cases} \overrightarrow{\delta\mathfrak{Q}}=\sum_{d=1}^{p}\omega^d(a,da)\vec{\mathfrak{A}}_d & (d=1,\cdots,p),\\ \overrightarrow{\delta\mathfrak{J}_b}=0 & (b=1,\cdots,r),\end{cases}$$

where $\omega^d(a,da)$ are Pfafflan forms.

On the other hand,

$$(4\cdot3)\qquad \overrightarrow{\delta\mathfrak{Q}}=\overrightarrow{\delta P}+\overrightarrow{\delta P\mathfrak{Q}}.$$

As $\vec{\mathfrak{A}}_b$ are always constant, from the first of (4·1), we have

$$(4\cdot4)\qquad \overrightarrow{\delta P\mathfrak{Q}}=du^b\vec{\mathfrak{A}}_b \qquad (b=1,\cdots,r),$$

$$(4\cdot5)\qquad \overrightarrow{\delta\mathfrak{J}_b}=d\pi_b^a\vec{\mathfrak{A}}_a \qquad (a,b=,\cdots,r).$$

In virtue of (2·1) and (4·4), (4·3) can be written

$$(4\cdot6)\qquad \overrightarrow{\delta\mathfrak{Q}}=(A_\alpha^b(a)da^\alpha+du^b)\vec{\mathfrak{A}}_b \qquad (a,b=1,\cdots,r).$$

Substituting the first and second of (4·2) in (4·6) and (4·5) respectively, we have the following two systems of differential equations.

$$(4\cdot7)\qquad \begin{cases} du^d=-A_\alpha^d(a)da^\alpha+\omega^d(a,da)\\ du^e=-A_\alpha^e(a)da\end{cases} \quad \begin{pmatrix} d=1,\cdots,p;\ e=p+1,\\ \cdots,r;\ \alpha=1,\cdots,r\end{pmatrix},$$

$$(4\cdot8)\qquad d\pi_b^a=0 \qquad (a,b=1,\cdots,r).$$

So far as $\mathfrak{G}_{(h)}$ is the holonomy group, it is necessary and sufficient that the functions u^a and π_b^a can be solved from (4·7) and (4·8) respectively. As r vectors $\vec{\mathfrak{J}}_b$ in (4·1) are independent, we may suppose, by (4·8), that $\pi_b^a=\delta_b^a$ without loss of generality. The second of (4·7) shows that the right-hand members must be total differential forms. The condition of integrability are (3·1), where $e=p+1,\cdots,r$. Finally, we must show that $u^d(d=1,\cdots,p)$ can be solved from the first of (4·7). Although $\omega^d(a,da)$ are chosen as arbitrary Pfaffian forms, they must not be identically zero. For, if one of them is taken zero, then it is shown from the first of (4·2) that the number of the essential parameters $\mathfrak{G}_{(h)}$ becomes less than p.

Therefore it is necessary and sufficient that there is at least one set of (α, β)'s for which $\frac{\partial A^d_\alpha}{\partial a^\beta} - \frac{\partial A^d_\beta}{\partial a^\alpha}$ does not vanish whenever $d(=1, \cdots, p)$ is determined. Since every $A^d_\alpha(a)da^\alpha$ is not a total difierential under this condition, $u^d (d=1, \cdots, p)$ can be solved by choosing $\omega^d(a, da)$ suitably which are not identically zero.

Thus the necessary and sufficient conditions that there exist just p essential parameters in $\mathfrak{G}_{(h)}$ is that we have the equations (3·1) for $\alpha, \beta = 1, \cdots, r$; and only for $e = p+1, \cdots, r$.

BIBLIOGRAPHY

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[2] E. Cartan: Leçon sur la théorie des espaces à connexion projective: Paris, 1937.