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On the convergence of solutions of the non-linear differential equation

By

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In the foregoing papar* we have researched sufficient conditions for the *ultimate boundedness* of solutions of the system of differential equations,

$$(1) \qquad \frac{dx}{dt}=f(t, x, y), \qquad \frac{dy}{dt}=g(t, x, y),$$

and we have obtained an existence theorem of a periodic solution by aid of the boundedness theorem. Namely under some conditions, it is proved that there exist two positive numbers A and B independent of particular solutions such that

$$|x(t)|<A, \qquad |y(t)|<B$$

for $t\geqq t_0$ (t_0 depending upon each particular solution), where $(x(t), y(t))$ is any solution of (1).

Let $f(t, x, y)$ and $g(t, x, y)$ be two *continuous functions* of (t, x, y) in the domain

$$\Delta_1: \quad 0\leqq t<\infty, \quad -\infty<x<+\infty, \quad -\infty<y<+\infty.$$

Now we will show that under some conditions every solution of (1) converges to the periodic solution as $t\rightarrow\infty$ provided the solutions of (1) are ultimately bounded. At first, we shall prove two following lemmas.

Lemma 1. *Let Δ_2 be the 5-dimensional domain of (t, x, u, y, v) such as*

$$t_0\leqq t<\infty, \ |x|\leqq A, \ |u|\leqq A, \ |y|\leqq B, \ |v|\leqq B,$$

where t_0 may be arbitrarily great, but it is a constant. Now suppose

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that there exists a continuous function $\Phi(x, u, y, v)$ *satisfying the following conditions in* Δ_2; *namely*

1° $\Phi(x, u, y, v)>0$, *provided* $|x-u|+|y-v|>0$,

2° $\Phi(x, u, y, v)=0$, *provided* $|x-u|+|y-v|=0$,

3° $\Phi(x, u, y, v)$ *satisfies the Lipschitz condition with regard to* (x, u, y, v) *and for every point in the interior of this domain* Δ_2, *we have*

$$\varlimsup_{h\to 0} \frac{1}{h}\Big\{\Phi(x+hf(t, x, y), u+hf(t, u, v), y+hg(t, x, y), v+hg(t, u, v)) - \Phi(x, u, y, v)\Big\} \leqq 0,$$

where for every $\lambda>0$ (*small* λ'*s alone being worth to consider*), *if* $|x-u|+|y-v|\geqq\lambda$, *the left hand side of this inequality* $\leqq\kappa(\lambda)<0$ ($\kappa(\lambda)$ *may be arbitrarily small, but it is a fixed constant for fixed* λ).

Then choosing $\delta(>0)$ *suitably for any* $\varepsilon>0$ (ε *however small*), *if any two solutions of* (1), $(x(t), y(t))$ *and* $(\bar{x}(t), \bar{y}(t))$, *which satisfy* $|x|<A$, $|y|<B$ *for* $t\geqq t_0$, *satisfy the following inequality at* $t=T$ $(\geqq t_0)$ (T *being arbitrary*),

(2) $$|x(T)-\bar{x}(T)|+|y(T)-\bar{y}(T)|<\delta,$$

then for $t>T$ *we have always*

(3) $$|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|<\varepsilon.$$

Proof. For a given ε, let δ' be the minimum of $\Phi(x, u, y, v)$ when $|x-u|+|y-v|=\varepsilon$. Then since $\Phi(x, u, y, v)$ is positive for $|x-u|+|y-v|>0$, it is clear that $\delta'>0$, and δ' is independent of t. Moreover since $\Phi(x, u, y, v)$ satisfies the Lipschitz condition, we have a positive constant K such as

$$|\Phi(x, u, y, v)-\Phi(x', u', y', v')|\leqq K(|x-x'|+|u-u'|+|y-y'|+|v-v'|).$$

Now we put

$$\delta=\min(\delta'/K, \varepsilon).$$

Then it is proved as follows that for any two solutions $(x(t), y(t))$ and $(\bar{x}(t), \bar{y}(t))$ satisfying (2) at an arbitrary $t=T$, the inequality (3) holds good: Namely if otherwise suppose that we have at some $t=T'$

$$|x(T')-\bar{x}(T')|+|y(T')-\bar{y}(T')|=\varepsilon.$$

Now consider the function $\Phi(x(t), \bar{x}(t), y(t), \bar{y}(t))$ for $T \leqq t \leqq T'$ and then $\Phi \geqq \delta'$ at $t=T'$, while we have

$$\Phi(x(T), \bar{x}(T), y(T), \bar{y}(T)) \geqq \delta', \tag{4}$$

since this function is a non-increasing function of t by the condition 3°. Moreover we have

$$\begin{aligned}\Phi(x(T), \bar{x}(T), y(T), \bar{y}(T)) - \Phi(\bar{x}(T), \bar{x}(T), \bar{y}(T), \bar{y}(T))\\ \leqq\ & K(|x(T)-\bar{x}(T)|+|y(T)-\bar{y}(T)|)\\ <\ & K\delta\\ \leqq\ & K\cdot\delta'/K=\delta'.\end{aligned}$$

Hence we have

$$\Phi(x(T), \bar{x}(T), y(T), \bar{y}(T)) < \delta',$$

for by the condition 2°

$$\Phi(\bar{x}(T), \bar{x}(T), \bar{y}(T), \bar{y}(T)) = 0.$$

This contradicts (4) and hence (3) holds good. Thus the proof is completed.

Lemma 2. *Suppose that the same assumptions as those in Lemma* 1 *hold good. Then given any positive number* δ (δ *may be sufficiently small), it for any two solutions of* (1) $(x(t), y(t))$ *and* $(\bar{x}(t), \bar{y}(t))$ *which satisfy* $|x|<A$, $|y|<B$ *for* $t \geqq t_0$, *we have at some* $t=T$ ($\geqq t_0$) (*T being arbitrary, but fixed*)

$$|x(T)-\bar{x}(T)|+|y(T)-\bar{y}(T)| \geqq \delta, \tag{5}$$

then we have at some T' ($>T$)

$$|x(T')-\bar{x}(T')|+|y(T')-\bar{y}(T')| < \delta. \tag{6}$$

Proof. Let Δ_3 and Δ_4 be two domains such as

$$T \leqq t < \infty,\ |x| \leqq A,\ |u| \leqq A,\ |y| \leqq B,\ |v| \leqq B$$

and

$$T \leqq t < \infty,\quad |x-u|+|y-v| < \delta'$$

respectively, where $\delta' < \delta$. Now consider a function

$$\Psi(t, x, u, y, v) = e^{Nt}\,\Phi(x, u, y, v) \qquad (N>0)$$

in $\Delta_3 - \Delta_4$ and then we have

1° $\Psi(t, x, u, y, v) > 0$, since $|x-u|+|y-v|>0$,

2° $\Psi(t, x, u, y, v)$ tends to infinity uniformly for (x, u, y, v) as $t\to\infty$.

And it is clear that $\Psi(t, x, u, y, v)$ satisfies locally for t the Lipschitz condition with regard to (x, u, y, v).

Moreover we have

$$\overline{\lim_{h\to 0}}\frac{1}{h}\Big\{\Psi(t+h, x+hf(t,x,y), u+hf(t,u,v), y+hg(t,x,y), v+hg(t,u,v)) - \Psi(t,x,u,y,v)\Big\}$$

$$=\overline{\lim_{h\to 0}}\frac{1}{h}\Big\{e^{N(t+h)}\Phi(x+hf, u+hf, y+hg, v+hg) - e^{Nt}\Phi(x,u,y,v)\Big\}$$

$$=\overline{\lim_{h\to 0}}\frac{1}{h}\Big\{e^{N(t+h)}[\Phi(x+hf, u+hf, y+hg, v+hg) - \Phi(x,u,y,v)] + (e^{N(t+h)}-e^{Nt})\Phi(x,u,y,v)\Big\}$$

$$=e^{Nt}\,\overline{\lim_{h\to 0}}\frac{1}{h}[\Phi(x+hf, u+hf, y+hg, v+hg) - \Phi(x,u,y,v,)] + N\,e^{Nt}\,\Phi(x,u,y,v)$$

$$\leqq e^{Nt}\Big\{\kappa(\delta') + N\Phi(x,u,y,v)\Big\}.$$

Now for δ', we can choose $N(\delta')$ so small that this expression becomes always non-positive in the interior of $\Delta_3-\Delta_4$. Therefore

3° $\Psi(t, x, u, y, v)$ satisfies locally the Lipschitz condition with regard to (x, u, y, v) and for all points in the interior of $\Delta_3-\Delta_4$ we have

$$\overline{\lim_{h\to 0}}\frac{1}{h}\Big\{\Psi(t+h, x+hf(t,x,y), u+hf(t,u,v), y+hg(t,x,y), v+hg(t,u,v)) - \Psi(t,x,u,y,v)\Big\}\leqq 0.$$

Now suppose that the assertion (6) is not true for δ. Let Δ_5 and Δ_6 be two domains such as

$$|x|\leqq A,\quad |u|\leqq A,\quad |y|\leqq B,\quad |v|\leqq B$$

and

$$|x-u|+|y-v|<\delta$$

respectively. Then we can choose T'' by 2° such that

$$(7)\quad \min_{\Delta_5-\Delta_6} e^{N(\delta')T''}\Phi(x,u,y,v) > \max_{\Delta_5-\Delta_6} e^{N(\delta')T}\ \Phi(x,u,y,v),$$

while by 3°

$$\Psi(T'',x(T''),\bar{x}(T''),y(T''),\bar{y}(T''))\leqq\Psi(T,x(T),\bar{x}(T),y(T),\bar{y}(T)).$$

This contradicts (7). Therefore the assertion is true. This T'' depends upon T and $N(\delta')$.

Now we can prove the following *convergence theorem* by aid of these lemmas.

Theorem 1. *Suppose that the solutions of* (1) *are ultimately bounded for A and B and that the same assumptions as those in Lemma* 1 *hold good. Then for any two solutions of* (1) $(x(t), y(t))$ *and* $(\bar{x}(t), \bar{y}(t))$, *we have*

$$(8)\quad \begin{cases} \lim\limits_{t\to\infty}(x(t)-\bar{x}(t))=0 \\ \lim\limits_{t\to\infty}(y(t)-\bar{y}(t))=0. \end{cases}$$

Proof Let $(x(t),y(t))$ and $(\bar{x}(t),\bar{y}(t))$ be any two solutions of (1). Then since these are ultimately bounded with the bounds A and B, there exist T_1 and T_2 such that

$$|x(t)|<A,\ |y(t)|<B \text{ for } t\geqq T_1$$

and

$$|\bar{x}(t)|<A,\ |\bar{y}(t)|<B \text{ for } t\geqq T_2$$

respectively. Now we put $T=\max(T_1,T_2,t_0)$. By Lemma 1, δ is chosen for an $\varepsilon>0$ (however small) and if, for this δ, we do not have

$$|x(T)-\bar{x}(T)|+|y(T)-\bar{y}(T)|<\delta,$$

then we can choose T' such as

$$|x(T')-\bar{x}(T')|+|y(T')-\bar{y}(T')|<\delta$$

by Lemma 2, where $T'>T$. Then by Lemma 1 we have

$$|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|<\varepsilon,$$

for $t>T'$. Namely (8) holds good.

From this fact, it is easy to prove the following theorem.

Theorem 2. *If the same assumptions as those in Theorem 1* *hold good and the system of differential equations* (1) *has a periodic solution of period* ω, *it is unique and the other solutions of* (1) *converge to that periodic solution as* $t\to\infty$.

Remark. If $\Phi(x, u, y, v)$ in condition 3° be *totally differentiable*, then the inequality under 3° reduces to

$$\frac{\partial\Phi(x,u,y,v)}{\partial x}f(t,x,y)+\frac{\partial\Phi(x,u,y,v)}{\partial u}f(t,u,v)$$
$$+\frac{\partial\Phi(x,u,y,v)}{\partial y}g(t,x,y)+\frac{\partial\Phi(x,u,y,v)}{\partial v}g(t,u,v)\leqq 0.$$

Instead of sufficient conditions under which the results of Theorems 1 and 2 are concluded, we can modify the conditions in Lemma 1 and those in Lemma 2 *independently* as follows. Namely

Lemma 3. *Suppose that there exists a continuous function of* (t, x, u, y, v) $\Phi(t, x, u, y, v)$ *in* Δ_2 *satisfying the following conditions; namely*

1° $\Phi(t,x,u,y,v)=0$, *provided* $|x-u|+|y-v|=0$,

2° *there exists a positive number* $\delta(\varepsilon)$ *such that* $\Phi(t,x,u,y,v)>\delta(\varepsilon)>0$ *when* $|x-u|+|y-v|\geqq\varepsilon$, *where* ε *is an arbitrary positive number and* δ *depends on* ε,

3° $\Phi(t,x,u,y,v)$ *satisfies the Lipschitz condition with regard to* (x,u,y,v) *and for a positive constant* K, *and in the interior of* Δ_2 *we have*

$$\overline{\lim_{h\to 0}}\frac{1}{h}\Big\{\Phi(t+h,\ x+hf(t,x,y),\ u+hf(t,u,v),\ y+hg(t,x,y),$$
$$v+hg(t,u,v))-\Phi(t,x,u,y,v)\Big\}\leqq 0.$$

Then for any two solutions of (1), $(x(t),y(t))$ *and* $(\bar{x}(t),\bar{y}(t))$ *satisfying* $|x|<A$, $|y|<B$ *for* $t\geqq t_0$, *being given an arbitrary positive number* ε *(however small), there exists a positive number* $\lambda(<\varepsilon)$ *independent of* T *such that, if we have for an arbitrary* $T(\geqq t_0)$

(9) $$|x(T)-\bar{x}(T)|+|y(T)-\bar{y}(T)|\leqq\lambda(\varepsilon),$$

then

(10) $$|x(t)-\bar{x}(t)|+|y(t)-\bar{y}(t)|<\varepsilon$$

holds for $t\geqq T$.

Remark. Since the case where ε is small alone is worth to consider, it is sufficient that the condition 3° is satisfied in the domain in Δ_2 such as

$$t_0 \leqq t < \infty, \quad |x-u|+|y-v| \leqq \varkappa,$$

where $\varkappa(>0)$ may be sufficiently small. Of course it is sufficient that Φ exists in the domain where $|x-u|+|y-v|$ is sufficiently small.

Proof. For an ε, choosing λ such as

$$\lambda(\varepsilon) < \min(\delta/K, \varepsilon),$$

this λ depends only on ε. Now suppose that, for any two solutions of (1) now considering $(x(t), y(t))$ and $(\bar{x}(t), \bar{y}(t))$ satisfying (9) at $t=T$, we have at some $t(>T)$, say T',

$$|x(T')-\bar{x}(T')|+|y(T')-\bar{y}(T')|=\varepsilon. \tag{11}$$

Then we can consider this T' as the first t where (11) holds by the continuity of the solutions. Hence by considering the function $\Phi(t, x(t), \bar{x}(t), y(t), \bar{y}(t))$ for $T \leqq t \leqq T'$, the conclusion of this lemma follows in the same way as in Lemma 1. This first T' is taken according to the fact mentioned in the above remark.

Lemma 4. *For every $\delta>0$ (δ may be sufficiently small), let Δ_7 be the domain such as*

$$t_0 \leqq t < \infty, \quad |x-u|+|y-v| < \delta.$$

Suppose that there exists a continuous function of (t, x, u, y, v), $\Psi_\delta(t, x, u, y, v) = \Psi(t, x, u, y, v)$, in $\Delta_2 - \Delta_7$ which satisfies the following conditions; namely

1° *$\Psi(t, x, u, y, v)$ is positive in $\Delta_2 - \Delta_7$,*

2° *$\Psi(t, x, u, y, v)$ tends to zero uniformly for (x, u, y, v) when $t \to \infty$ (or tends to infinity uniformly as $t \to \infty$),*

3° *$\Psi(t, x, u, y, v)$ satisfies locally the Lipschitz condition with regard to (x, u, y, v) and in the interior of this domain $\Delta_2 - \Delta_7$, we have*

$$\lim_{h \to 0} \frac{1}{h} \Big\{ \Psi(t+h, x+hf(t, x, y), u+hf(t, u, v), y+hg(t, x, y),$$
$$v+hg(t, u, v)) - \Psi(t, x, u, y, v) \Big\} \geqq 0$$

$$(\text{or } \overline{\lim_{h\to 0}} \frac{1}{h}\Big\{\Psi(t+h, x+hf, u+hf, y+hg, v+hg) - \Psi(t, x, u, y, v)\Big\} \leqq 0).$$

Then for any two solutions of (1) $(x(t), y(t))$ *and* $(\bar{x}(t), \bar{y}(t))$ *satisfying* $|x| < A$, $|y| < B$ *for* $t \geqq t_0$, *if we have at some* $t = T(\geqq t_0)$

$$|x(T) - \bar{x}(T)| + |y(T) - \bar{y}(T)| > \delta,$$

then at some T' *such as* $T' > T$, *we have*

$$|x(T') - \bar{x}(T')| + |y(T') - \bar{y}(T')| \leqq \delta.$$

The proof is omitted, for it is the same with Lemma 2.

Theorem 3. *If the solutions of* (1) *are ultimately bounded for* A *and* B *and the assumptions in Lemmas 3 and 4 hold good, then we have* (8) *for any two solutions of* (1), $(x(t), y(t))$ *and* $(\bar{x}(t), \bar{y}(t))$.

Remark. Theorems 1, 2 and 3 can be generalized for the more general system of differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, x_2, \cdots\cdots, x_n) \qquad (i = 1, 2, \cdots\cdots, n).$$

Example. Reuter has obtained a convergence theorem for the solutions of the differential equation of the second order

$$\ddot{x} + kf(x)\dot{x} + g(x) = kp(t) \qquad (k > 0) \tag{12}$$

in *the Journal of the London Mathematical Society, Vol. 26 (1951).* Together with conditions for the ultimate boundedness, he has supposed that $g'(x) > 0$ and that $g''(x)$ exists and is bounded for $|x| \leqq x_0$. Here x_0 corresponds to A in our theorems. And using his notations, we have $|x(t)| \leqq x_0$ and $|\dot{x}(t)| \leqq v_0$. Thus there exist positive constants a_1, a_2, a_3, a_4 and $\gamma(x_0)$, independent of k, such that for $|x| \leqq x_0$

$$\begin{cases} a_1 \leqq f(x) \leqq a_2 \\ a_3 \leqq g'(x) \leqq a_4 \\ |g''(x)| \leqq \gamma(x_0), \end{cases}$$

by the assumptions for $f(x)$, $g'(x)$ and $g''(x)$. And he concludes that, if $k > k_0 = v_0\gamma(x_0)/a_1 a_3$, then for any two solutions of (12), $(x_1(t), \dot{x}_1(t))$ and $(x_2(t), \dot{x}_2(t))$, we have

$$x_2(t)-x_1(t)\to 0 \text{ and } \dot{x}_2(t)-\dot{x}_1(t)\to 0 \text{ as } t\to\infty.$$

For our part, instead of (12), we consider the system

$$\dot{x}=y-kF(x),\ \dot{y}=-g(x)+kp(t), \tag{13}$$

where $F(x)=\int_0^x f(x)\,dx$.

Then for $\Phi(x, u, y, v)$ in Theorem 1, we may take the expression

$$(g(x)-g(u))(x-u)+(y-v)^2-2c(x-u)(y-v) \tag{14}$$

which Reuter has denoted by Q and used it in his research, where c is a positive suitable constant and is chosen so small that (14) is positive definite with regard to $(x-u)$ and $(y-v)$.