MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SESIRS A
Vol. XXVIII, Mathematics No. 2, 1953.

Note on integral closures of Noetherian domains

By

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(Received November 3, 1953)

Previously Prof. Akizuki[1] proved that if $\mathfrak{o}$ is a Noetherian local integrity domain[2] of dimension 1 and if $\hat{\mathfrak{o}}$ is its integral closure[3], then any ring $\mathfrak{s}$ such that $\mathfrak{o}\subseteq\mathfrak{s}\subseteq\hat{\mathfrak{o}}$ is Notherian[4].

As for the case of higher dimension, there arise the following problems:

Let $\mathfrak{o}$ be a Noetherian local integrity domain of dimension n and let $\hat{\mathfrak{o}}$ be its integral closure. Then

Problem I. Does it holds in general that any ring $\mathfrak{s}$ such that $\mathfrak{o}\subseteq\mathfrak{s}\subseteq\hat{\mathfrak{o}}$ is Noetherian?

Problem II. Does it holds in general that $\hat{\mathfrak{o}}$ is Noetherian?

In the present note, we show a counter example against the problem I when $n=2$ in § 2 and then a counter example against the problem II when $n=3$ in § 3[5].

§ 1. A preliminary.

Let $\mathfrak{k}_0$ be a perfect field of characteristic p ($\neq 0$) and let $u_1 \cdots$, $u_n, \cdots$ (infinitely many) be algebraically independent elements over $\mathfrak{k}_0$. Set $\mathfrak{k}=\mathfrak{k}_0(u_1,\cdots, u_n,\cdots)$. Further let $x_1,\cdots, x_n$ be indeterminates and denote by $\mathfrak{o}_n$ and $\mathfrak{r}_n$ the rings $\mathfrak{k}^p\{x_1,\cdots, x_n\}[\mathfrak{k}]$ and $\mathfrak{k}\{x_1,\cdots, x_n\}$[6] respectively.

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1) Y. Akizuki, Einige Bemerkunge über primäre Integritätsbereiche mit Teilerkettensatz, Proc. Phys.-Math. Soc. Japan, 3rd Ser., 17 (1935), pp. 327-336.

2) We say in the present note that a ring $\mathfrak{o}$ is a local ring if it has only one maximal ideal $\mathfrak{m}$ and if the intersection of all powers of $\mathfrak{m}$ is zero, where we consider the $\mathfrak{m}$-adic topology for $\mathfrak{o}$.

3) This means the integral closure in its quotient field.

4) This result shows also the similar result for "einartig" Noetherian integrity domains.

5) It was communicated to the writer that this problem II was proved affirmatively by Mr. Mori, when $n=2$.

6) $\mathfrak{k}\{x_1,\cdots, x_n\}$ denotes the ring of formal power series in $x_1,\cdots x_n$ with coefficients in $\mathfrak{k}$.

Then we have

Lemma. $\mathfrak{o}_n$ is a regular local ring and $\mathfrak{r}_n$ is the completion of $\mathfrak{o}_n$.

Proof. When we see that $\mathfrak{o}_n$ is Noetherian, our assertion follows easily. Therefore we prove that $\mathfrak{o}_n$ is Noetherian. When $n=1$, our assertion is evident. Therefore we prove our assertion by induction on n. Since $\mathfrak{k}^p\{x_1,\cdots, x_n\}$ is a complete regular local ring, $\mathfrak{k}^p\{x_1,\cdots, x_n\}[a_1,\cdots, a_t]$ is a complete regular local ring and therefore it is an integrally closed integrity domain, provided that $a_i \in \mathfrak{k}$. This shows that $\mathfrak{o}_n$ is integrally closed. Therefore if y is an element of $\mathfrak{o}_n$, $y\,\mathfrak{r}_n \cap \mathfrak{o}_n = y\,\mathfrak{o}_n$[7]. Now let $\mathfrak{q}$ be an arbitrary prime ideal of $\mathfrak{o}_n$; we have only to show that $\mathfrak{q}$ has a finite basis[8]. Let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{o}_n$ contained in $\mathfrak{q}$ and let $\bar{\mathfrak{p}}$ be the prime ideal of $\mathfrak{r}_n$ such that $\bar{\mathfrak{p}} \cap \mathfrak{o}_n = \mathfrak{p}$. Set $\mathfrak{o}_0 = \mathfrak{k}^p\{x_1,\cdots,x_n\}$ and $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{o}_0$. Then since $\mathfrak{o}_0$ is regular, we see that $\mathfrak{p}_0$ is a principal ideal: $\mathfrak{p}_0 = f\,\mathfrak{o}_0$. Since $\mathfrak{o}_0$ is complete, we may assume that $f = a_0 + a_1 x_n + \cdots a_{s-1} x_n^{s-1} + x_n^s$ with $a_i \in \mathfrak{o}' = \mathfrak{k}^p\{x_1,\cdots, x_{n-1}\}$ by Weierstrass preparation theorem.

Case 1). When f is irreducible over $\mathfrak{o}_{n-1}$, we see that $f\,\mathfrak{o}_n$ is prime because $\mathfrak{o}_{n-1}$ is integrally closed. Hence $\mathfrak{p} = f\,\mathfrak{o}_n$.

Case 2). When f is not irreducible over $\mathfrak{o}_{n-1}$, we take an irreducible monic factor f' of f in the polynomial ring $\mathfrak{o}_{n-1}[x_n]$. Since $\mathfrak{o}_{n-1}$ is a purely inseparable integral extension of $\mathfrak{o}'$ with exponent p, $f = f'^p$. Therefore $f'\mathfrak{r}_n$ must be a prime ideal, because $\mathfrak{r}_n$ is a purely inseparable integral extenson of $\mathfrak{o}_0$ with exponent p. Hence $f'\mathfrak{o}_n = f'\mathfrak{r}_n \cap \mathfrak{o}_n$ is a prime ideal.

Thus, in either case, we see that $\mathfrak{p}$ is principal and that $\mathfrak{o}_n/\mathfrak{p}$ is a finite module over $\mathfrak{o}_{n-1}$ (and therefore that $\mathfrak{o}_n/\mathfrak{p}$ is Noetherian). This shows that $\mathfrak{q}$ has a finite basis.

§ 2. A counter example against the problem I.

We denote, in this paragraph, by x and y instead of x_1 and x_2 respectively. We take elements $c = y \sum_{i=1}^{\infty} u_i x^i$, $c_n = (c - \sum_{i<n} y u_i x^i)/x^n$ $(n=1, 2,\cdots)$. We consider the ring $\mathfrak{o} = \mathfrak{o}_2[c_1,\cdots, c_n,\cdots]$.

Proposition 1. $\mathfrak{o}$ is a counter example against the problem I.

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7) Observe that $\mathfrak{r}_n$ is integral over $\mathfrak{o}_n$.

8) Cf. I. S. Cohen, Commutative rings with restricted minimum condition, Duke Math. J., 17 (1950), pp. 27-42.

Proof. Since $\mathfrak{o}_2$ is Noetherian, $\mathfrak{o}_2[c]$ is Noetherian. It is evident that $\mathfrak{o}$ contains $\mathfrak{o}_2[c]$ and is contained in the integral closure of $\mathfrak{o}_2[c]$. Therefore we have only to show that $\mathfrak{o}$ is not Noethrian.

We first show that this local ring $\mathfrak{o}$ is a dense subspace of $\mathfrak{r}_2$. Let b be an element of $(x, y)^r\mathfrak{r}_2 \cap \mathfrak{o}$. Since b is in $\mathfrak{o}$, we can write b as a polynomial in $c_1, \cdots, c_n$ (by a suitable n) with coefficients in $\mathfrak{o}_2$. Since $c_i = u_i + x c_{i+1}$, we can write b as a polynomial in $x^r c_{n+r}$ with coefficients in $\mathfrak{o}_2$: $b = b_0 + b_1(x^r c_{n+r}) + \cdots + b_s(x^r c_{n+r})^s$ $(b_i \in \mathfrak{o}_2)$. Since $b \in (x, y)^r\mathfrak{r}_2$, we have $b_0 \in (x, y)^r\mathfrak{r}_2 \cap \mathfrak{o}_2 = (x, y)^r\mathfrak{o}_2 \subseteq (x, y)^r\mathfrak{o}$. Therefore $b \in (x, y)^r\mathfrak{o}$, which shows that $(x, y)^r\mathfrak{r}_2 \cap \mathfrak{o} = (x, y)^r\mathfrak{o}$. Now that $\mathfrak{o}$ is dense in $\mathfrak{r}_2$ is evident. We see that the completion of $\mathfrak{o}$ is regular. Therefore, if $\mathfrak{o}$ is Noetherian, $\mathfrak{o}$ must be regular and therefore $\mathfrak{o}$ must be integrally closed. But $\mathfrak{o}$ cannot be integrally closed because c_1/y is not in $\mathfrak{o}$.

§ 3. A counter example against the problem II.

We denote, in this paragraph, by x, y and z instead of x_1, x_2 and x_3 respectively. Further we denote by $u_1, v_1, u_2, v_2, \cdots$ instead of $u_1, u_2\, u_3\, u_4, \cdots$ respectively. Take an element $c = y\sum_{i=1}^{\infty} u_i x^i + z\sum_{i=1}^{\infty} v_i x^i$ of $\mathfrak{r}_3$ and let $\mathfrak{o}$ be the integral closure of $\mathfrak{o}_3[c]$. For the simplicity of our calculus, we treat the case $p=2$[9].

Proposition 2. $\mathfrak{o}_3[c]$ is a counter example against the problem II.

Proof. That $\mathfrak{o}_3[c]$ is Noetherian is evident. Therefore we have only to show that $\mathfrak{o}$ is not Noetherian. Since $\mathfrak{o}$ is integrally closed, $x\mathfrak{r}_3 \cap \mathfrak{o} = x\mathfrak{o}$ and therefore $x\mathfrak{o}$ is a prime ideal. We consider valuation rings $\mathfrak{o}' = \mathfrak{o}_{3(x\mathfrak{o}_3)}$ and $\mathfrak{o}'' = \mathfrak{o}_{(x\mathfrak{o})}$. Then since c is in the completion of $\mathfrak{o}'$, we see that $\mathfrak{o}'$ is a dense subspace of $\mathfrak{o}''$ and therefore $\mathfrak{o}'/x\mathfrak{o}' = \mathfrak{o}''/x\mathfrak{o}''$. On the other hand, since $\mathfrak{o}_3/x\mathfrak{o}_3$ is regular, $\mathfrak{o}_3/x\mathfrak{o}_3$ is integrally closed. Therefore $\mathfrak{o}_3/x\mathfrak{o}_3 = \mathfrak{o}/x\mathfrak{o}$. Therefore the maximal ideal of $\mathfrak{o}$ can be generated by x, y and z. Therefore, if $\mathfrak{o}$ is Noetherian, $\mathfrak{o}$ must be regular. Now we have only to show that $\mathfrak{o}$ is not regular.

Assume for a moment that $\mathfrak{o}$ is regular. Then $\mathfrak{o}$ must contain

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9) We need not for our calculus that $p=2$. Whenever p is not equal to zero, the same construction yields a counter example against our problem II. Our calculus for the case is similar, but it is somewhat more complicated, because there must appear some more terms in the formulars below.

an element $\sum_{i=1}^{\infty} u_i x^i + zf$ with a suitable $f \in \mathfrak{r}_3$ because $\mathfrak{o}/z\mathfrak{o}$ is regular. Therefore we can write

$$\sum_{i=1}^{\infty} u_i x^i + zf = (a_0 + a_1 c)/d_1 \quad (a_0,\ a_1,\ d_1 \in \mathfrak{o}_3),$$

where we choose a_0, a_1, d_1 so that they have no common factor. Now we have

$$d_1 \sum_{i=1}^{\infty} u_i x^i + d_1 zf = a_0 + a_1 y \sum_{i=1}^{\infty} u_i x^i + a_1 z \sum_{i=1}^{\infty} v_i x^i.$$

Since 1, $\sum u_i x^i$ are linearly independent over $\mathfrak{o}_2(=\mathfrak{o}_3/z\mathfrak{o}_3)$, we have $a_0 \in z\mathfrak{o}_3$, $d_1 - a_1 y \in z\mathfrak{o}_3$. Therefore we can write $a_0 = za_0'$, $d_1 = a_1 y + dz$ $(a_0',\ d \in \mathfrak{o}_3)$. Then

$$(a_1 y + dz)(\sum u_i x^i + fz) = za_0' + a_1 y \sum u_i x^i + a_1 z \sum v_i x^i,$$

and therefore

$$a_1 yzf + dz \sum u_i x^i + dfz^2 = za_0' + a_1 z \sum v_i x^i.$$

we write $a_1 = \sum_{i=0}^{\infty} a_{1i} z^i$, $a_0' = \sum_{i=0}^{\infty} a_{0i} z^i$, $d = \sum_{i=0}^{\infty} d_i z^i$, $f = \sum_{i=0}^{\infty} f_i z^i$ $(a_{ji},\ d_i \in \mathfrak{o}_3,$ $f_i \in \mathfrak{r}_2)$. Then comparing the coefficients of z, we have

$$a_{10} y f_0 + d_0 \sum u_i x^i = a_{00} + a_{10} \sum v_i x^i.$$

Since 1, $\sum u_i x^i$ and $\sum v_i x^i$ are linearly independent over $\mathfrak{o}_1(=\mathfrak{o}_2/y\mathfrak{o}_2)$, we have d_0, a_{00} and a_{10} are in $y\mathfrak{o}_2$. We show next that d_r, a_{0r} and a_{1r} are in $y\mathfrak{o}_2$ for any r, by induction on r. Comparing the coefficients of z^{r+1}, we have

$$y(\sum_{j=0}^{r} a_{1j} f_{r-j}) + d_r \sum u_i x^i + \sum_{j=0}^{r-1} d_j f_{r-1-j} = a_{0r} + a_{1r} \sum v_i x^i.$$

Since $d_0, d_1, \cdots, d_{r-1}$ are in $y\mathfrak{o}_2$, by our induction assumption, we have that d_r, a_{0r} and a_{1r} are in $y\mathfrak{o}_2$.

Thus we see that d_1, a_0 and a_1 are in $y\mathfrak{o}_3$, which is a contradiction to that d_1, a_0 and a_1 have no common factor. Thus our proof is completed.