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Note on intersection multiplicity of proper components of algebraic or algebroid varieties

By

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Let $\mathfrak{o}$ be the ring of polynomials or the ring of formal power series in indeterminates $x_1, \cdots, x_n$ over a field k. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals in $\mathfrak{o}$ and let $\mathfrak{n}$ be a minimal prime divisor of $(\mathfrak{p}, \mathfrak{q})\mathfrak{o}$. It is easy to see that rank $\mathfrak{n} \leq$ rank $\mathfrak{p}+$ rank $\mathfrak{q}$.[1] When rank $\mathfrak{n}=$ rank $\mathfrak{p}+$rank $\mathfrak{q}$, we say that $\mathfrak{n}$ is a proper component of $\mathfrak{p} \cup \mathfrak{q}$. On the other hand, the multiplicity $i(\mathfrak{n}; \mathfrak{p} \cup \mathfrak{q})$ of a minimal prime divisor $\mathfrak{n}$ of $(\mathfrak{p}, \mathfrak{q})\mathfrak{o}$ with respect to $\mathfrak{p} \cup \mathfrak{q}$ is defined as follows: Let $\mathfrak{o}'$ be a copy of $\mathfrak{o}$ and we construct $\mathfrak{o}^*=\mathfrak{o} \times_k \mathfrak{o}'$.[2] We denote by $\mathfrak{d}$ the set $\{x_1-x_1', \cdots, x_n-x_n'\}$, where x_i' is the copy of x_i (in $\mathfrak{o}'$). Let $\mathfrak{q}'$ be the copy of $\mathfrak{q}$. Set $\mathfrak{n}^*=(\mathfrak{n}, \mathfrak{d})\mathfrak{o}$. It is evident that $\mathfrak{n}^*$ is a prime ideal of $\mathfrak{o}^*$. Set $\hat{\mathfrak{o}}=\mathfrak{o}^*_{\mathfrak{n}^*}$. Then we define

$$i(\mathfrak{n}; \mathfrak{p} \cup \mathfrak{q})=e((\mathfrak{d}, \mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}/(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}).$$ [3][4]

The purpose of the present paper is to show the following

Theorem. Assume that $(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}$ is a prime ideal of $\hat{\mathfrak{o}}$ and that $\mathfrak{n}$ is a proper component of $\mathfrak{p} \cup \mathfrak{q}$. Then we have

(1) $i(\mathfrak{n}; \mathfrak{p} \cup \mathfrak{q}) \leqq e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{n}})$, and the equality holds if and only if $\mathfrak{p}\mathfrak{o}_{\mathfrak{n}}$ is generated by elements of number rank $\mathfrak{p}$;

(2) $i(\mathfrak{n}; \mathfrak{p} \cup \mathfrak{q}) \leq e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}})$, and the equality holds if and only

1) It is easy to see that if $\mathfrak{r}$ is a regular local ring and if $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of $\mathfrak{r}$, then for any minimal prime divisor $\mathfrak{n}$ of $(\mathfrak{p}, \mathfrak{q})\mathfrak{r}$ we have rank $\mathfrak{n} \leqq$ rank $\mathfrak{p}+$rank $\mathfrak{q}$.

2) When $\mathfrak{o}$ is the ring of polynomials, we mean under this notation $\mathfrak{o} \times_k \mathfrak{o}'$ the tensor product of $\mathfrak{o}$ and $\mathfrak{o}'$ over k (therefore $\mathfrak{o} \times_k \mathfrak{o}'=k[x_1, \cdots, x_n, x_1', \cdots, x_n']$); when $\mathfrak{o}$ is the ring of formal power series, we mean under the same notation the Kroneckerian product of $\mathfrak{o}$ and $\mathfrak{o}'$ over k in the sense of C. Chevelley, Intersections of algebraic and algebroid varieties, Trans. Amer. Math. Soc. 57 (1945), pp. 1-85 (in this case, $\mathfrak{o} \times_k \mathfrak{o}'=k\{x_1, \cdots, x_n, x_1', \cdots, x_n'\}$).

3) Cf. C. Chevalley, 1. c. note 2).

if $\mathfrak{p}\mathfrak{o}_{\mathfrak{n}}$ and $\mathfrak{q}\mathfrak{o}_{\mathfrak{n}}$ are generated by elements of numbers rank $\mathfrak{p}$ and rank $\mathfrak{q}$ respectively.

In §2 we translate this theorem into geometric language.

§1. Proof of the theorem.

1) Proof of (1). We choose a subset $\mathfrak{b}=(b_1, \cdots, b_r)$ ($r=\text{rank}\ \mathfrak{p}$) of $\mathfrak{p}$ so that $\mathfrak{b}$ mod $\mathfrak{q}\mathfrak{o}_{\mathfrak{n}}$ is a system of parameters in $\mathfrak{o}_{\mathfrak{n}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{n}}$ and that $e((\mathfrak{b}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{n}})=e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{n}})$.[4] Then $e((\mathfrak{d}, \mathfrak{b}, \mathfrak{q}')\hat{\mathfrak{o}}/\mathfrak{q}'\hat{\mathfrak{o}})= e((\mathfrak{d}, \mathfrak{q}')\hat{\mathfrak{o}}_{(\mathfrak{d}, \mathfrak{q}')\hat{\mathfrak{o}}}/\mathfrak{q}'\hat{\mathfrak{o}}_{(\mathfrak{d}, \mathfrak{q}')\hat{\mathfrak{o}}})e((\mathfrak{d}, \mathfrak{b}, \mathfrak{q}')\hat{\mathfrak{o}}/(\mathfrak{d}, \mathfrak{q}')\hat{\mathfrak{o}})$[5] $=e((\mathfrak{b}, \mathfrak{d}, \mathfrak{q}')\hat{\mathfrak{o}}/(\mathfrak{d}, \mathfrak{q}')\hat{\mathfrak{o}}) =e((\mathfrak{b}, \mathfrak{q}')\mathfrak{o}_{\mathfrak{n}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{n}}) =e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{n}})$. On the other hand, $e((\mathfrak{d}, \mathfrak{b}, \mathfrak{q}')\mathfrak{o}/\mathfrak{q}'\hat{\mathfrak{o}})=\sum e((\mathfrak{b}, \mathfrak{q}')\hat{\mathfrak{o}}_{\hat{\mathfrak{p}}_i}/\mathfrak{q}_{\hat{\mathfrak{p}}_i})e((\mathfrak{d}, \hat{\mathfrak{q}})\hat{\mathfrak{o}}/\hat{\mathfrak{q}})$,[5] where $\hat{\mathfrak{p}}_i$ runs over all minimal prime divisors of $(\mathfrak{q}', \mathfrak{b})\hat{\mathfrak{o}}$. Since $\mathfrak{b}\subseteq\mathfrak{p}$ and since rank $\mathfrak{b}\hat{\mathfrak{o}}=\text{rank}\ \mathfrak{p}$, we see that $(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}$ must appear among $\hat{\mathfrak{p}}_i$. Therefore we have $e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{n}})\geq e((\mathfrak{d}, \mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}/(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}})=i(\mathfrak{n}\,;\ \mathfrak{p}\cup\mathfrak{q})$. This proves the inequality in (1). Now that $e((\mathfrak{b}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{n}})=i(\mathfrak{n}\,;\ \mathfrak{p}\cup\mathfrak{q})$ is equivalent to the following two conditions: a) $e((\mathfrak{b}, \mathfrak{q}')\hat{\mathfrak{o}}_{(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}}/\mathfrak{q}'\hat{\mathfrak{o}}_{(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}})=1$ and b) $(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}$ is the unique minimal prime divisor of $(\mathfrak{b}, \mathfrak{q}')\hat{\mathfrak{o}}$. a) is equivalent to that the primary component of $(\mathfrak{b}, \mathfrak{q}')\hat{\mathfrak{o}}$ belonging to $(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}$ coincides with $(\mathfrak{b}, \mathfrak{q}')\hat{\mathfrak{o}}$. Therefore, together with b), they are equivalent to the condition that $(\mathfrak{b}, \mathfrak{q}')\hat{\mathfrak{o}}=(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}$, that is, $\mathfrak{b}\mathfrak{o}_{\mathfrak{n}}=\mathfrak{p}\mathfrak{o}_{\mathfrak{n}}$. Thus (1) is proved completely.

2) Proof of (2). We choose a subset $\bar{\mathfrak{a}}=(\bar{a}_1, \cdots, \bar{a}_s)$ ($s=\text{rank}\ \mathfrak{p}+\text{rank}\ \mathfrak{q}$) of $(\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}}$ so that $\bar{\mathfrak{a}}$ is a system of parameters in $\mathfrak{o}_{\mathfrak{n}}$ and that $e(\bar{\mathfrak{a}}\mathfrak{o}_{\mathfrak{n}})=e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}})$. Further we choose elements $a_1, \cdots, a_s$ of $(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}$ so that $a_i\equiv\bar{a}_i$ mod $\mathfrak{d}\hat{\mathfrak{o}}$. Set $\mathfrak{a}=(a_1, \cdots, a_s)$. Then $e((\mathfrak{a}, \mathfrak{d})\hat{\mathfrak{o}})=e(\mathfrak{d}\hat{\mathfrak{o}}_{\mathfrak{d}\hat{\mathfrak{o}}})e((\mathfrak{d}, \mathfrak{a})\hat{\mathfrak{o}}/\mathfrak{d}\hat{\mathfrak{o}})$[5] $=e(\bar{\mathfrak{a}}\mathfrak{o}_{\mathfrak{n}})=e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}})$. On the other hand, $e((\mathfrak{a}, \mathfrak{d})\hat{\mathfrak{o}})=\sum e(\mathfrak{a}\hat{\mathfrak{o}}_{\hat{\mathfrak{p}}_i})e((\mathfrak{d}, \hat{\mathfrak{p}}_i)\hat{\mathfrak{o}}/\hat{\mathfrak{p}}_i)$,[5] where $\hat{\mathfrak{p}}_i$ runs over all minimal prime divisors of $\mathfrak{a}\hat{\mathfrak{o}}$. Since $\mathfrak{a}\subseteq(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}$, we see that $(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}$ appears among $\hat{\mathfrak{p}}_i$. Therefore we have $e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}})\geq e((\mathfrak{d}, \mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}/(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}})=i(\mathfrak{n}\,;\ \mathfrak{p}\cup\mathfrak{q})$. Now that $e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}})=i(\mathfrak{n}\,;\ \mathfrak{p}\cup\mathfrak{q})$ is equivalent to the following two conditions: a) $e(\mathfrak{a}\hat{\mathfrak{o}}_{(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}})=1$ and b) $(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}$ is the unique minimal prime divisor of $\mathfrak{a}\hat{\mathfrak{o}}$. And therefore it is equivalent to the following: c) $\mathfrak{a}\hat{\mathfrak{o}}=(\mathfrak{p}, \mathfrak{q}')\hat{\mathfrak{o}}$. This shows our assertion.

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4) Cf. P. Samuel, La notion de multiplicicité en algèbre et en géométrie algébrique, J. Math. Pures Appl. (9), 30 (1951), pp. 159-274.

5) This equality follows from the theorem of associativity formula due to C. Chevally, 1. c. note 2), which can be proved easily without assumption on basic field; see, Nagata, Local rings, Sûgaku, 5 No. 4 (1954) pp. 229—238 (in Japanese).

§2. Translation into geometric language.

Let U, V be algebraic (or algebroid) varieties in n-space (or local n-space) S^n. Assume that a variety W is a proper component of the intersection of U and V. Take a common field k of definition for U, V and W. Let $\mathfrak{o}$ be the ring of polynomials (or formal power series) in indeterminates $x_1, \cdots, x_n$. Take prime ideals $\mathfrak{n}$, $\mathfrak{p}$ and $\mathfrak{q}$ which corresonds to W, U and V respectively. Then we have

(1) $i(W; U\cdot V) \leq e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{n}})$, and the equality holds if and only if U is locally complete intersection at W;

(2) $i(W; U\cdot V) \leq e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}})$, and the equality holds if and only if both U and V are locally complete intersections respectively at W.

Remark. It is easy to see that the following four conditions are equivalent to each other:

(1) $e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}}/\mathfrak{q}\mathfrak{o}_{\mathfrak{n}}) = 1$,

(2) $e((\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}}) = 1$,

(3) $(\mathfrak{p}, \mathfrak{q})\mathfrak{o}_{\mathfrak{n}} = \mathfrak{n}\mathfrak{o}_{\mathfrak{n}}$,

(4) $i(W; U\cdot V) = 1$ (for the case of our theorem, $i(\mathfrak{n}; \mathfrak{p} \cup \mathfrak{q}) = 1$).