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Derivation and cohomology in simple and other rings. II

(*A remark on the Kronecker product* $A\times_C A$)

By

Tadasi NAKAYAMA

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In our first paper I[1] we proved first that if A is a simple ring (having unit element 1 and satisfying minimum condition) and if C is a weakly normal simple subring of A (which contains 1 and over which A is assumed to be finite for the sake of simplicity), then the Kronecker product (or, direct product, as we called it in I) $A\times_C A$ over C is completely reducible as A-C-double-module, under ordinary operation.[2] This we proved indeed by combining the following two facts, which were proved either in I or in a former paper of the writer: Under the same assumption, 1) the A-C-module A is completely reducible; 2) $A\times_C A$ is A-two-sided completely reducible and is a direct sum of minimal A-double-submodules which are A-left-semilinearly and A-right-linearly isomorphic to A. Thus arises our interest in investigating the relationship between the A-two-sided complete reducibility of $A\times_C A$ and the A-C-complete reducibility of A itself, where A is a ring with unit element 1 and C is a subring of A which contains 1. A typical case, where we have the latter but not the former, is the case of a field C and a non-separable semisimple algebra A over C. It is also clear that the former does not imply the latter in general. For instance, let A be the complete matric ring $\varepsilon_{11}\Omega+\varepsilon_{12}\Omega+\varepsilon_{21}\Omega+\varepsilon_{22}\Omega$ over a field Ω and C be its subring $\Omega+\varepsilon_{21}\Omega$; observe that A has even an (independent) two-sided basis over C,

1) Duke Math. J. 19 (1952), 51-63.

2) We proved the same also under Hochschild's cohomological operation. Further we considered Kronecker products $A\times_C A\times_C\dots\times_C A$ with more factors than 2, and proved their A-B-complete reducibility, where B is any (necessarily weakly normal) simple subring of A which contains C.

for example $\{1, \varepsilon_{12}\}$. Now, in the present short note we want to mention an easy condition under which the A-two-sided complete reducibility of $A \times_C A$ implies the A-C-complete reducibility of A. Thus,

Proposition. *Let A be a ring with unit element 1. Let C be a subring of A which contains 1, and let A satisfy the minimum condition for its A-C-submodules. Suppose that A possesses an independent finite left (and in fact two-sided) C-basis, say $a_1, a_2, \ldots, a_n$, satisfying*[3]

$$(1) \qquad Ca_i = a_iC \qquad (i=1, 2, \ldots, n).$$

If $A \times_C A$ is, under ordinary operation, A-two-sided completely reducible, then A is completely reducible as A-C-double-module.

Proof. For each $i=1, 2, \ldots, n$ and for each element c of C, there exist, because of the assumption (1), elements c^{γ_i}, c^{δ_i} of C such that

$$(2) \qquad ca_i = a_ic^{\gamma_i},$$

$$(3) \qquad a_ic = c^{\delta_i}a_i.$$

Let M be the sum of all minimal A-C-submodules of A. Then for each element $x \neq 0$ of A we have $AxC \cap M \neq 0$. Now, consider the submodule $M \times_C A$ of $A \times_C A$; because of the existence of a left C-basis of A, the Kronecker product $M \times_C A$ itself may be considered as a submodule of $A \times_C A$. Any element u of $A \times_C A$ may be expressed, uniquely, in the form

$$(4) \qquad u = x_1 \times a_1 + x_2 \times a_2 + \ldots + x_n \times a_n \qquad (x_i \in A).$$

Let $u \neq 0$. We wish to prove that

$$AuC \cap (M \times_C A) \neq 0.$$

To do so let t be the youngest index such that $x_t \neq 0$. We take $y_\mu \in A$ and $c_\mu \in C$ so that we have

$$(5) \qquad 0 \neq \textstyle\sum_\mu y_\mu x_t c_\mu \in M.$$

Construct then the element

$$(6) \qquad v = \textstyle\sum_\mu y_\mu u c_\mu^{\gamma_t}$$

of AuC. This element v is equal to

3) It seems, to the writer, to be indicated that some properties of algebras over a commutative ring could be extended to rings of our type.

$$\sum_\mu y_\mu(x_t\times a_t+x_{t+1}\times a_{t+1}+\dots+x_n\times a_n)c_\mu^{\tau_t}$$
$$=\sum_\mu(y_\mu x_t\times a_t c_\mu^{\tau_t})+\sum_\mu(y_\mu x_{t+1}\times a_{t+1}c_\mu^{\tau_t})+\dots$$
$$+\sum_\mu(y_\mu x_n\times a_n c_\mu^{\tau_t})$$
$$=\sum_\mu(y_\mu x_t\times c_\mu a_t)+\sum_\mu(y_\mu x_{t+1}\times c_\mu^{\tau_t\delta_{t+1}}a_{t+1})$$
$$+\dots+\sum_\mu(y_\mu x_n\times c_\mu^{\tau_t\delta_n}a_n)$$
$$=(\sum_\mu y_\mu x_t c_\mu)\times a_t+(\sum_\mu y_\mu x_{t+1}c_\mu^{\tau_t\delta_{t+1}})\times a_{t+1}$$
$$+\dots+(\sum_\mu y_\mu x_n c_\mu^{\tau_t\delta_n})\times a_n\,.$$

Here the first summand $(\sum_\mu y_\mu x_t c_\mu)\times a_t$ is, because of (5), a non-zero element of $M\times_C A$. Thus, if the remaining sum

(7) $$w=(\sum_\mu y_\mu x_{t+1}c_\mu^{\tau_t\delta_{t+1}})\times a_{t+1}+\dots+(\sum_\mu y_\mu x_n c_\mu^{\tau_t\delta_n})\times a_n$$

is 0, then our element v of AuC is a non-zero element of $M\times_C A$, whence $AuC\cap(M\times_C A)\neq 0$. If however w is not 0, suppose that $AwC\cap(M\times_C A)\neq 0$, i.e. that there are $z_\nu\in A$ and $d_\nu\in C$ such that

(8) $$0\neq\sum_\nu z_\nu w d_\nu\in M\times_C A.$$

We have

(9) $$\sum_\nu z_\nu v d_\nu=\sum_\nu(z_\nu(\sum_\mu y_\mu x_t c_\mu)\times a_t d_\nu)$$
$$+\sum_\nu z_\nu w d_\nu\in M\times_C A.$$

Moreover, this element (9), which belongs to AuC, is not equal to 0, since $\sum_\nu(z_\nu(\sum_\mu y_\mu x_t c_\mu)\times a_t d_\nu)$ is in $A\times a_t$ (indeed in $M\times a_t$) while $\sum_\nu z_\nu w d_\nu$ is in $A\times a_{t+1}+\dots+A\times a_n$ and $(A\times a_t)\cap(A\times a_{t+1}+\dots+A\times a_n)=0$.

By an easy induction, on the number of non-zero coefficients x in (4), we see $AuC\cap(M\times_C A)=0$, as is desired. Since this is the case for any non-zero element of $A\times_C A$, it follows that $M\times_C A$ contains all minimal A-C-submodules of $A\times_C A$. If $A\times_C A$ is A-two-sided completely reducible, this implies $M\times_C A=A\times_C A$, whence $M=A$, or, what amounts to the same, A is completely reducible as A-C-module. Our proposition is proved.

Our proposition thus proved is however of mere formal nature. A somewhat, if not much, deeper consideration on the complete reducibility of $A\times_C A$, particularly on its relationship with the complete reducibility of the Kronecker product $A\times_C A\times_C\dots\times_C A$ with more factors than two, will be given in our third note III.

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