MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXIX, Mathematics No. 3, 1955.

Some remarks on invariant forms of a sphere bundle with connexion

By

Seizi TAKIZAWA

(Received September 26, 1955)

Let us consider an $(n-1)$-sphere bundle $\mathfrak{B}^{n-1}(M, S^{n-1}, O_n^+)$ over a differentiable manifold M with the proper orthogonal group O_n^+ of degree n. Let $\mathfrak{B}^q(M, Y^q, O_n^+)$ $(0 \leqq q \leqq n-1)$ denote the associated bundle of $\mathfrak{B}^{n-1}$ with the Stiefel manifold $Y^q = O_n^+/O_q^+$ as fibre. Then, the associated principal bundle $\mathfrak{B}^0(M, O_n^+)$ can be also regared as principal bundles $\mathfrak{B}^0(\mathfrak{B}^q, O_n^+)$ over $\mathfrak{B}^q$ with groups O_q^+. From a connexion defined on $\mathfrak{B}^0(M, O_n^+)$, we can induce naturally connexions on $\mathfrak{B}^0(\mathfrak{B}^q, O_q^+)$. In the present paper we show that by employing these induced connexions, the formulas in our preceding papers[1] and their generalizations can be expressed in a simple manner

§ 1. Let V be an r-dimensional vector space over the real number field R. Its exterior algebra Λ is a graded ring whose homogeneous elements of degree k constitute the space $\Lambda^k (0 \leqq k \leqq n)$ of all exteior k-vectors; in particular $\Lambda^0 = R$ and $\Lambda^1 = V$. Let M be a differentiable manifoid. For the sake of simplicity, we understand that the term "differentiable" means always the differentiability of suitable class. We denote by $T(M)$ and $T_x(M)$ the tangent vector bundle over M and the tangent vecter space of M at $x \in M$ respectively. From any differentiable mapping $\varphi: M \rightarrow M'$, we can induce a linear map $T_x(M) \rightarrow T_{\varphi(x)}(M')$ which we shall denote by φ^*. We consider a p-form θ with values in Λ^k: to any set of vectors $t_1, \cdots, t_p \in T_x(M)$ $x \in M$, is assigned an element $\theta(t_1, \cdots, t_p) \in \Lambda^k$

---

1) S. Takizawa; *On the Stiefel characteristic classes of a Riemannian manifold.*
——————; *On the primary difference of two frame functions in a Riemannian manifold.*
T. Yagyu; *On the Whitney characteristic classes of the normal bundle.* These memoirs, 28, No. 1 (1953)

being multilinear and alternating with respect to the vectors $t_1, \cdots, t_p$. Take a base $(e_1, \cdots, e_n)$ of V. Then the elements

$$e_{i_1}\wedge\cdots\wedge e_{i_k} \ (i_1<\cdots<i_k\,;\, i_1, \cdots, i_k=1, \cdots, r)$$

constitute a base of Λ^k, and the p-form θ can be expressed by

$$\theta=\sum\theta^{i_1\cdots i_k}\otimes e_{i_1}\wedge\cdots\wedge e_{i_k}, \tag{1}$$

where $\theta^{i_1\cdots i_k}$ are p-forms with real values and are skew-symmetric with respect to the indices, and where $\otimes$ denotes the tensor product. In (1), we have made use of the convention, to be used throughout, that when the same indices appear twice in a term the symbol $\sum$ means the sum of the terms obtained by giving the indices each of their values. The exterior derivative of θ is then given by

$$d\theta=\sum d\theta^{i_1\cdots i_k}\otimes e_{i_1}\wedge\cdots\wedge e_{i_k}. \tag{2}$$

DEFINITION. Let θ be a p-form on M with values in Λ^k and φ be a q-form on M with values in Λ^l. The *exterior multiplication* $\theta\wedge\varphi$ being a $(p+q)$-form with values in Λ^{k+l} is defined by

$$\theta\wedge\varphi(t_1, \cdots, t_{p+q}) \tag{3}$$
$$=\sum_{\sigma}\frac{\varepsilon(\sigma)}{(p+q)!}\,\theta(t_{\sigma(1)}, \cdots, t_{\sigma(p)})\wedge\varphi(t_{\sigma(p+1)}, \cdots, t_{\sigma(p+q)})$$

for $t_1, \cdots, t_{p+q}\in T(M)$ $x\in M$, where the summation is extended over all permutation σ of the set $\{1, 2, \cdots, p+q\}$ and where $\varepsilon(\sigma)$ is $+1$ or -1 according as σ is even or odd.

Expressing the forms θ, φ by their components referred to the base $(e_1, \cdots, e_r)$:

$$\theta=\sum\theta^{i_1\cdots i_k}\otimes e_{i_1}\wedge\cdots\wedge e_{i_k},$$
$$\varphi=\sum\varphi^{j_1\cdots j_l}\otimes e_{j_1}\wedge\cdots\wedge e_{j_l},$$

we have

$$\theta\wedge\varphi=\sum\theta^{i_1\cdots i_k}\wedge\varphi^{j_1\cdots j_l}\otimes e_{i_1}\wedge\cdots\wedge e_{i_k}\wedge e_{j_1}\wedge\cdots\wedge e_{j_l}. \tag{4}$$

By the definition we get the following formulas:

$$\varphi\wedge\theta=(-1)^{pq+kl}\theta\wedge\varphi, \tag{5}$$

$$d(\theta\wedge\varphi)=d\theta\wedge\varphi+(-1)^{p}\theta\wedge d\varphi. \tag{6}$$

For any form θ, we shall set

$$(\wedge\theta)^m=\theta\wedge\theta\wedge\cdots\wedge\theta \qquad (m \text{ factors}).$$

Now we assume that V is a Lie algebra, and denote by C^i_{jk} its structure constants for a base $(e_1, \cdots, e_r)$:

$$(7) \qquad [e_i, e_j]=\sum C^k_{ij}e_k\,.$$

DEFINITION. Let θ, φ be forms with values in V of degrees p, q respectively. The *bracket product* $[\theta, \varphi]$ being $(p+q)$-form with values in V is defined by

$$(8) \qquad [\theta, \varphi](t_1, \cdots, t_{p+q})$$
$$=\sum_\sigma\frac{\varepsilon(\sigma)}{(p+q)!}[\theta(t_{\sigma(1)}, \cdots, t_{\sigma(p)}), \varphi(t_{\sigma(p+1)}, \cdots, t_{\sigma(p+q)})]$$

for $t_1, \cdots, t_{p+q}\in T_x(M)$ $x\in M$.

Expressing the forms θ, φ by their components referred to the base $(e_1, \cdots, e_n)$: $\theta=\sum\theta^i\otimes e_i$, $\varphi=\sum\varphi^j\otimes e_j$, we have

$$(9) \qquad [\theta, \varphi]=\sum C^k_{ij}\theta^i\wedge\varphi^j\otimes e_k.$$

By the definition we get the following formulas:

$$(10) \qquad [\varphi, \theta]=(-1)^{pq-1}[\theta, \varphi],$$

$$(11) \qquad d[\theta, \varphi]=[d\theta, \varphi]+(-1)^p[\theta, d\varphi].$$

Moreover if θ, φ, ψ are forms of degrees p, q, s respectively, then

$$(12) \qquad (-1)^{q(p+s)}[\theta, [\varphi, \psi]]+(-1)^{s(p+q)}[\varphi, [\psi, \theta]]$$
$$+(-1)^{p(q+s)}[\psi, [\theta, \varphi]]=0.$$

It follows that, if θ, φ, ψ have same degree,

$$(13) \qquad [\theta, [\varphi, \psi]]+[\varphi, [\psi, \theta]]+[\psi, [\theta, \varphi]]=0\,;$$

and in particular

$$(14) \qquad [\theta, [\theta, \theta]]=0.$$

Let $\mathfrak{B}(M, G, \pi)$ be a differentiable prinicipal fibre bundle over a differentiable manifold M with a Lie group G and with the projection $\pi:\mathfrak{B}\to M$. A connexion on $\mathfrak{B}$ is given by a differentiable 1-form ω satisfying the conditions:

(i) ω is a 1-form on $\mathfrak{B}$ with values in the Lie algebra of G.

(ii) If $t\in T_b(\mathfrak{B})$ $\pi^*t=0$, then $\omega(t)=(\chi^*(b))^{-1}t$ where $\chi(b)$ denotes the admissible map corresponding to $b\in\mathfrak{B}$.

(iii) For any $s\in G$, $\alpha(s^{-1})\circ\omega=\omega\circ\rho^*(s)$, where $\rho(s)$ denotes the right translation of $\mathfrak{B}$ and α denotes the linear adjoint representation of G.

The curvature form Ω of the connexion is given by the equation of structure:

$$d\omega=-\tfrac{1}{2}[\omega, \omega]+\Omega;$$

and taking the exterior derivative of this equation we get immediately Bianchi's identity:

$$d\Omega=[\Omega, \omega].$$

§**2.** Let $\alpha(\sigma)$ denote the similarity by any matrix σ: *i.e.* $\alpha(\sigma)\tau=\sigma\tau{}^t\sigma$, where σ, τ are matrices of degree n and ${}^t\sigma$ is the transpose of σ. Let O_n^+ be the group of all real proper orthogonal matrices of degree n, and V^n be an n-dimensional real vector space with a fixed orthonormal base $(e_1, \cdots, e_n)$. Each vector $x\in V^n$ can be expressed by its components $x={}^t(x_1, \cdots, x_n)$ with respect to the base, and O_n^+ can be regarded as a group of linear automorphisms of $V^n: x\to\sigma x$, $\sigma\in O_n^+$, $x\in V^n$. Denote by $\Lambda^k(V^n)$ the space of all k-vectors generated by V^n. Any element $a\in\Lambda^2(V^n)$ is given by a skew-symmetric matrix: *i.e.* $a=\sum a_{ij}e_i\wedge e_j$, $a_{ij}+a_{ji}=0$. An automorphism $\sigma\in O_n^+$ of V^n induces an automorphism of $\Lambda^2(V^n)$ which is given by $\alpha(\sigma)$: *i.e.* $a\to\alpha(\sigma)a$ for all $a\in\Lambda^2(V^n)$. Let L_n be the Lie algebra of O_n^+. Each element of L_n can be expressed by a skew-symmetric matrix, and an elemet of the linear adjoint group of O_n^+ is given by $\alpha(\sigma)$: *i.e.* $a\to\alpha(\sigma)a$ for $\sigma\in O_n^+$, $a\in L_n$. Accordingly, we may make the identification $L_n=\Lambda^2(V^n)$ which is preserved by any operation of O_n^+. Let O_q^+ be the subgroup of O_n^+ consisting of all matrices of the type

$$\sigma=\begin{pmatrix} \tilde{\sigma} & 0 \\ 0 & \varepsilon \end{pmatrix},$$

where $\tilde{\sigma}$ is a proper orthogonal matrix of degree q and ε is the unit matrix of degree $n-q$. The Lie algebra L_q of O_q^+ is the subalgebra of L_n consisting of all matrices of the type

$$a=\begin{pmatrix} \tilde{a} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\tilde{a}$ is a skew-symmetric matrix of degree q. Let V^q be the subspace of V^n spaned by the vectors $e_1, \cdots, e_q$. Then O_q^+ becomes a group of antomorphisms of V^q, and L_q can be identified with

$\Lambda^2(V^q)$. Let us define an endomorphism $\pi_q: L_n \to L_n$ by

$$\pi_q = \alpha(\iota_q) \quad \text{with} \quad \iota_q = \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix},$$

where ε is the unit matrix of degree q. Then it becomes a projection $\pi_q: L_n \to L_q$. Since $\sigma\iota_q = \iota_q\sigma$ for $\sigma \in O_q^+$, we have

$$\pi_q\alpha(\sigma) = \alpha(\sigma)\pi_q \quad \text{for} \quad \sigma \in O_q^+.$$

To each $a \in L_q$, we assign a vector $pa \in V^{q-1}$ whose components are the q-th column ${}^t(a_{1q}, a_{2q}, \cdots, a_{q-1,q}, 0, \cdots, 0)$ of the matrix a. Thus we have the projection $p: L_q \to V^{q-1}$ for all q. It follows that $p\alpha(\sigma)a = \sigma pa$ for $a \in L_q$, $\sigma \in O_{q-1}^+$. The base $e_1 \wedge \cdots \wedge e_q$ of $\Lambda^q(V^q)$ is invariant under any operation of O_q^+; and so we can identify $\Lambda^q(V^q)$ with the real field R.

§ **3.** Let $\mathfrak{B}^0(M, O_n^+)$ be a differentiable prinicipal bundle over a compact connected differentiable manifold M with group O_n^+, and let $\mathfrak{B}^q(M, Y^q, O_n^+)$ denote the associated bundle of $\mathfrak{B}^0$ having the Stiefel manifold $Y^q = O_n^+/O_q^+$ as fibre. Since the group O_n^+ becomes a prinicipal bundle over Y^q with the group O_q^+, we can regard $\mathfrak{B}^0$ as a principal bundle $\mathfrak{B}^0(\mathfrak{B}^q, O_q^+)$ over $\mathfrak{B}^0$ with the group O_q^+. Let ω be a connexion on $\mathfrak{B}^0(M, O_n^+)$. If we set $\omega^{(q)} = \pi_q\omega$, then $\omega^{(q)}$ becomes a connexion on $\mathfrak{B}^0(\mathfrak{B}^q, O_q^+)$, because the relation $\alpha({}^t\sigma)\circ\omega^{(q)} = \omega^{(q)}\circ\rho^*(\sigma)$ holds for any $\sigma \in O_q^+$. Denoting by $\Omega^{(q)}$ the curvature form of the connexion $\omega^{(q)}$, we define the q-form Θ^q on $\mathfrak{B}^0$ with real values as follows: if q is even, we set

$$\Theta^q = \{(-1)^{q/2}2^{-q}\pi^{-q/2}/(q/2)!\}(\wedge\Omega^{(q)})^{q/2};$$

and if q is odd, $\Theta^q = 0$. Then Θ^q being a form with values in $\Lambda^q(V^q)$ is invariant under any right translation of $\mathfrak{B}^0(\mathfrak{B}^q, O_q^+)$, and is regarded as a form on $\mathfrak{B}^q$ with real values. Obviously $d\Theta^q = 0$; and by Weil's theorem,[2] the cohomology class of Θ^q does not depend on the choice of the connexion on $\mathfrak{B}^0(\mathfrak{B}^q, O_q^+)$. Accordingly it is independent on the choice of the connexion ω on $\mathfrak{B}^0(M, O_q^+)$. Moreover, if we set

$$\Pi^{q-1} = (-1)^q 2^{-q}\pi^{-(q-1)/2}\sum_{k=1}^{[(q-1)/2]}\{(-1)^k/k!\,\Gamma((q-2k+1)/2)\}$$
$$\times(\wedge\Omega^{(q)})^k\wedge(\wedge p\omega^{(q)})^{q-2k-1},$$

2) Cf. S. S. Chern; Topics in differential geometry, Princeton, 1951.

then Π^{q-1} is regarded as a form on $\mathfrak{B}^{q-1}$ with real values. It can be seen that $-d\Pi^{q-1}=\Theta^q$, and that the restriction of Π^{q-1} on a fibre Y_x^{q-1} of $\mathfrak{B}^{q-1}$ reduces to the fundamental cocycle of Y_x^{q-1}. Thus we can consider Θ^q to be the Whitney characteristic class of the bundle $\mathfrak{B}^{n-1}(M, S^{n-1}, O_n^+)$ and Π^{q-1} to represent the difference cochain of cross-sections of $\mathfrak{B}^{q-1}(M, Y^q, O_n^+)$.[1]

In general, by taking an invariant polynomial of the linear adjoint representation of O_q^+, we can obtain, on the associated bundle $\mathfrak{B}^q$ of $\mathfrak{B}^0(M, O_n^+)$, a form whose cohomology class does not depend on the choice of the connexion on $\mathfrak{B}^0(M, O_n^+)$.