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Note on a paper of Lang concerning quasi algebraic closure

By

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As was defined by Lang,[1] a field K is said to be C_i if every homogeneous form in K in n variables and of degree d with $n>d^i$ has a non-trivial zero in K; K is said to be strongly C_i if every polynomial in K in n variables, without constant term and of degree d with $n>d^i$ has a non-trivial zero in K.

Defining further more the notion of normic forms of order i,[2] he proved the following theorems:

(1) Let K be a C_i field admitting at least one normic form of order i. Let $f_1, \cdots, f_r$ be r homogeneous forms in K in n common variables each of degree d. If $n>rd^i$, then the forms have a non-trivial common zero in K.

(2) With the same K as above, a function field L over K is C_{i+r} with $r=\dim_K L$. If K is strongly C_i and if K admits normic forms of any given degree, then L is strongly C_{i+r}.

The purpose of the present note is to prove these theorems without assuming the existence of normic forms. Afterwards, we shall offer some related questions.

§ 1. The main theorems.

We shall prove here the following theorems:

THEOREM 1a. *Let $f_1, \cdots, f_r$ be r homogeneous forms in n common variables each of degree d in a C_i field K. If $n>rd^i$, then the forms have common non-trivial zero in K.*

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1) S. Lang, On quasi algebraic closure, Ann. of Math, vol. 55, pp. 373-390 (1952).

2) Since we shall not make use of the notion of normic forms, we shall not recall the definition of the notion.

THEOREM 1b. *Let $f_1, \cdots, f_r$ be r polynomials in n common variables each of degree at most d in a strongly C_i field K. If $n > rd^i$, then the polynomials have non-trivial common zero in K.*

THEOREM 2a. *Let K be a C_i field and let L be an extension field of K such that $dim_K L = r$. Then L is C_{i+r}.*

THEOREM 2b. *If we assume in Theorem 2a that K is strongly C_i, then L is strongly C_{i+r}.*

In order to prove the C_j case and the strongly C_j case simultaneously, we shall mean under a *form* i) a homogeneous form when we are to prove the C_j case and ii) a polynomial without constant term when we are to prove the strongly C_j case.

According to Lang's technic, we shall introduce the following notation: Let $\phi, f_1, \cdots, f_r$ are forms in a field K such that the number of variables of ϕ is not less than r. In the C_i case we assume furthermore that $f_1, \cdots, f_r$ are of the same degree. Starting from $\phi = \phi^{(0)}(f_1, \cdots, f_r)$, we shall define $\phi^{(t)}(f_1, \cdots, f_r)$ as follows: Assume that $\phi^{(t-1)}(f_1, \cdots, f_r) = g(x_1, \cdots, x_N)$ $(N \geq r)$ and let $x_1, \cdots, x_n$ be variables of $f_1, \cdots, f_r$. Introducing $n[N/r]$ variables $x_{11}, \cdots, x_{1n}, \cdots, x_{[N/r]1}, \cdots, x_{[N/r]n}$, we define $\phi^{(t)}(f_1, \cdots, f_r)$ to be $g(f_1(x_{11}, \cdots, x_{1n}), \cdots, f_r(x_{11}, \cdots, x_{1n}), \cdots, f_1(x_{[N/r]1}, \cdots, x_{[N/r]n}), \cdots, f_r(x_{[N/r]1}, \cdots, x_{[N/r]n}), 0, \cdots, 0)$ (number of zeros is less than r). Then each $\phi^{(t)}(f_1, \cdots, f_r)$ is a form in K. The following two facts are important in our treatment:

(1) *If $f_1, \cdots, f_r$ have no non-trivial common zero in K, and if ϕ has no non-trivial zero in K, then $\phi^{(i)}(f_1, \cdots, f_r)$ has no non-trivial zero in K. (Lang)*

Proof. It will be sufficient to prove the case where $i=1$ and the case will be easy.

(2) *If $n > rd^s$, s being a non-negative real number and d the maximum of degrees of the f_r's, then N_i/D_i^s tends to infinity, N_i and D_i being the number of variables and the degree of $\phi^{(i)}(f_1, \cdots, f_r)$.*

Proof. $N_{i+1} = n[N_i/r]$ and $D_{i+1} \leq dD_i$. Set $a = n - rd^s$. Then $N_{i+1}/D_{i+1}^s \geq (rd^s + a)[N_i/r]/d^s D_i^s$. Therefore, if N_i and D_i^s are sufficiently large, $(N_{i+1}/D_{i+1}^s) - (N_i/D_i^s)$ is nearly equal to $(a/rd^s)(N_i/D_i^s)$. Since a/rd^s is a positive constant real number, we see that the sequence N_i/D_i^s tends to infinity.[3)]

Now we shall prove theorems 1a and 1b. If K is C_0, that is, if K is algebraically closed, then the assertion is obvious. There-

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3) It will be easy to see that D_i^s is bounded if and only if every $D_i^s = 1$. Since N_i is not bounded (by the assumption that $n > rd^s \geq 1$), such extreme case is easy.

fore we shall assume that K is not algebraically closed. Then there exists a form ϕ in N variables $(N>1)$ which has no non-trivial zero in K. We may assume that $N\geq r$, because if $N<r$, then it is sufficient to consider $\phi^{(t)}(\phi)$ for a sufficiently large t. Assume that $f_1, \cdots, f_r$ have no non-trivial common zero in K. Then by (1) and (2) above, for every natural number $t, \phi^{(t)}(f_1,\cdots,f_r)$ has no non-trivial common zero in K and for sufficiently large t its number of variables is greater than the i-th power of its degree, which is a contradiction. Thus Theorems 1a, 1b are proved.

For the proof of Theorems 2a and 2b, Lang's proof works good by virtue of Theorems 1a and 1b. We shall remark here that, though Lang's proof uses Theorem 1a and 1b even in the case where $r=0$, we can prove the case directly.

Indeed, we may assume at first that L is finite over K, as was done in Lang's paper and as is obvious. Set $j=[L:K]$. Let ϕ be a form in L in n variables and of degree d such that $n>d^i$. For a sufficiently large t, $f=\phi^{(t)}(\phi)$ is in N variables and of degree D with $N>j^iD^i$, by (2) above or as is easily seen directly. Let g be the norm of f with resprct to K. Then g is a form in K in N variables and of degree jD. Since K is C_i, g has a non-trivial zero in K, which must be a non-trivial zero of f in K hence in L. Therefore, by virtue of (1), we see that ϕ has a non-trivial zero in L and L is also C_i.

§ 2. Some related questions.

We shall offer here some related questions.

Problem 1. Let $\mathfrak{r}$ be a regular local ring with the field of quotients K. Let $\mathfrak{r}^*$ be the completion of $\mathfrak{r}$ and let K^* be the field of quotients of $\mathfrak{r}^*$. Assume that K is C_i. Is K^* C_i, too? If K is strongly C_i, is then K^* strongly C_i, too? In particular, how is the answer when $\mathfrak{o}$ is a discrete valuation ring?.

Problem 2. We have see that function fields over an algebraically closed field in r variables and function fields over a prime fields of non-zero characteristic in $r-1$ variables are not only C_r but also strongly C_r.[4] Therefore we shall dare to ask whether there exists a C_i field which is not strongly C_i.[5]

4) As was proved by C. Chevalley, (Démonstration d'une hypothèse de M. Artin, Abh. Sem. Hansischen Univ., vol. 11, p. 73 (1935)), every finite field is strongly C_1.

5) As was stated in Lang's paper, it seems us very likely that every C_i field is strongly C_{i+1}.

Problem 3. Let L be a finite algebraic extension of a field K. Assume that L is C_i and K is not C_i. Then, what can we say about these paire? (Very easy example is: Let C and R be the fields of complex numbers and real numbers respectively. Then C is C_0 and R is not C_i for any i. When $x_1, \cdots, x_r$ are variables, $C(x_1, \cdots, x_r)$ is C_r and $R(x_1, \cdots, x_r)$ is not C_i for any i.)

Problem 4. Let L be a C_i field which is a function field of dimension r over a field K. Find some good sufficient conditions for K to be C_{i-r}.

We shall add here an easy sufficient condition as follows: *Assume that there are fields $L_0=K, L_1, \cdots, L_r=L$ such that each $L_j (j=1, 2, \cdots, r)$ is the function field of a normal curve V_j over L_{j-1} which has a rationl point over L_{j-1}. Then K is C_{i-r}.*

Proof. We have only to treat the case where $r=1$. Let $\mathfrak{o}$ be the spot of a rational point of V_1 over K. Let p be a prime element of $\mathfrak{o}$. Assume that K is not C_{i-1}. Then there exists a form ϕ in K in n variables and of degree d with $n > d^{i-1}$ such that ϕ has non-trivial zero in K. Set $f=x_1+px_2+\cdots+p^{d-1}x_d$ and set $g=f^{(1)}(\phi)$. Then g has no non-trivial zero in L and we see that L is not C_i.

REMARK 1. Above proof shows really that if a field L has a discrete valuation ring $\mathfrak{v}$ with residue class field K. If L is C_i, then K is C_{i-1}.

REMARK 2. If we have a good answer of Problem 3, then problem 4 will be easy.

Problem 5. In the definition of C_i fields and strongly C_i fields, we need not assume that i is an integer; i may be any non-negative real number and our theorems holds good in this generalized sense. Now, by Theorem 1 in Lang's paper, we see that if a field K is C_i with $i<1$, then K is C_0 (cf. Problen 6 below). Let C_K be, for a given field K, the lower limit of i such that K is C_i (which may be infinity). Determine the set of C_K, where K runs over all fields. The known properties are 1) the smallest is zero and the second is 1 and 2) if there exists a field K such that $s=C_K$, then for every natural number n there exists a field L such that $C_L=s+n$.

Problem 6. Assume that a field K is C_1. Then for any finite algebraic extension L of K, $N_{L/K}(L)=K$. Indeed, let $c_1, \cdots, c_n$ be a linearly independent base of L over K. Introducing variables $x_1, \cdots, x_n$, set $g=N_{L/K}(\sum x_i c_i)$. Then g has no non-trivial zore in K.

Let a be an arbitrary element of K and let x_0 be a new variable. Set $g^*=g-ax_0^n$. Since g^* is a homogeneous form in K in $n+1$ variables and of degree n, g^* has a non-trivial zero, say $(s_0, s_1, \cdots, s_n)$. Since g has no non-trivial zero in K, s_0 cannot be zero. Therefore we may assume that $s_0=1$. Then $N_{L/K}(\sum s_i c_i)=a$, which proves the assertion.

We want to ask here whether the converse of the above fact is true or not.