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On the necessary and sufficient condition for the uniform boundedness of solutions of $x'=F(t, x)$

By

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The stability of the solution of a system of differential equations has been discussed by Liapounoff [2]* and various authors. Massera has discussed the Liapounoff's function and the relations between various stabilities and has obtained necessary and sufficient conditions for the asymptotic stability of the solution ([3] and [4]). The present author has researched the boundedness of solutions which is the concept corresponding to the stability. And we have obtained sufficient conditions for the boundedness or the ultimate boundedness of solutions for the purpose of using Massera's theorem in the discussion of the existence of a periodic solution of the non-linear differential equation of the second order ([5] and [6]). Moreover it has been shown that those conditions are also necessary conditions for some equations ([9] and [10]). In the papers [7] and [8] we have obtained some necessary and sufficient conditions for the uniform boundedness or the uniform stability, but they are not satisfactory in practical cases, because they are the conditions which are satisfied for general equations. Then we will obtain necessary conditions for the uniform boundedness or the uniform stability.

Now we consider a system of differential equations,

$$\frac{dx}{dt}=F(t, x), \tag{1}$$

where x denotes an n-dimensional vector and $F(t, x)$ is a given

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* Numbers in [ ] refer to the bibliography at the end of the paper.

vector field which is defined and continuous in the domain

$$\Delta: 0 \leqq t < \infty, \quad |x| < \infty.$$

The norm $|a|$ of a vector $a=(a_1, a_2, \cdots, a_n)$ represents $\{\sum_{i=1}^{n} a_i^2\}^{1/2}$.

We will say that $F(t, x)$ belongs to the class C_0 with respect to (t, x) in a region $\mathfrak{R}$ if for any bounded set $\mathfrak{R}' \subset \mathfrak{R}$ a Lipschitz condition is satisfied, i.e., there is a constant $M(\mathfrak{R}')$ such that, for any pair of points $(t, x) \in \mathfrak{R}'$, $(t', x') \in \mathfrak{R}'$, we have

$$|F(t, x) - F(t', x')| \leqq M[|t-t'| + |x-x'|].$$

On the other hand, we will say that $F(t, x)$ belongs to the class $\overline{C}_0$ with respect to (t, x) in a region $\mathfrak{R}$ if M is a Lipschitz constant which may depend on x and is independent of t. Moreover we will say that $F(t, x) \in C_0$ with respect to x if it satisfies a Lipschitz condition with regard to x. We will say that $F(t, x) \in \overline{C}_0$ if the Lipschitz constant is independent of t.

In the former discussions of necessary conditions for the boundedness of solutions or the stability, we have seen the existence of a certain function by utilizing the *supremum* of the norm of the solution. This method is useful in the studies of the ultimate boundedness or the asymptotic stability [4]. But since the supremum of the norm is not necessarily continuous in the case where we will discuss the simple boundedness or the simple stability, we cannot obtain a continuous function. Therefore we had to use the expression as follows; for any solution of (1), $x=x(t)$, the function $\varphi(t, x(t))$ is a non-decreasing function of t or a non-increasing function of t ([7] and [8]). The supremum is useful in the proof of the following theorem which is more general than the theorem in the case where $F(t, x)$ is periodic of t. It will be proved by Massera's method.

Theorem 1. *We assume that $F(t, x)$ belongs to the class C_0 with respect to x. In order that the solutions of (1) are uniformly bounded and uniformly ultimately bounded, it is necessary and sufficient that there exists a positive continuous function $\varphi(t, x)$ satisfying the following conditions in the domain $\overline{\Delta}[0 \leqq t < \infty, |x| \geqq R]$ (R be a positive constant which may be sufficiently great),*

1° *$\varphi(t, x)$ has the property A (cf. p. 279 in [9]),*

2° *$\varphi(t, x)$ tends to infinity uniformly as $|x| \to \infty$,*

3° *$\varphi(t, x)$ belongs to the class C_0 with respect to (t, x) and we have*

$$\overline{\lim_{h\to 0}}\frac{1}{h}\Big\{\varphi(t+h,\ x+hF)-\varphi(t,\ x)\Big\}\leqq -c(|x|),$$

where $c(r)$ is a positive continuous function of r.

Moreover if $F(t, x)\in\overline{C}_0$ with respect to x and $F(t, x)$ is bounded for $|x|$ bounded, $\varphi(t, x)$ belongs to the class $\overline{C}_0$ with respect to (t, x).

In the following two examples, the supremum of the norm of the solution is not continuous. As a simple example for the uniform boundedness, we consider the first order differential equation

$$x'=\begin{cases}(n+1-x)(x-n) & (n\leqq x<n+1)\quad (n=0,\ 1,\ 2,\ \cdots\cdots)\\ 0 & (x<0).\end{cases}$$

The solutions of this equation are uniformly bounded, but the supremum of the norm of the solution is not continuous with respect to the initial value. As the desired function $\varphi(t, x)$ for this equation, we can take

$$\varphi(t,\ x)=|x(0\ ;\ x,\ t)|,$$

namely

$$\varphi(t,\ x)=\begin{cases}\dfrac{n+(n+1)\dfrac{x-n}{n+1-x}e^{-t}}{1+\dfrac{x-n}{n+1-x}e^{-t}} & (n\leqq x<n+1)\\ -x & (x<0).\end{cases}$$

In the case where $x\equiv 0$ is uniformly stable we can also show a similar example. Dividing the interval of x into

$$\left[\frac{1}{2^{n+1}},\ \frac{1}{2^n}\right]\quad (n=0,\ 1,\ 2,\ \cdots\cdots),$$

consider such an equation as one in the example above, where $x=\dfrac{1}{2^n}$ $(n=0,\ 1,\ 2,\ \cdots\cdots)$ is a solution: namely consider the first order differential equation $x'=F(t, x)$ in $0\leqq t<\infty$, $|x|\leqq 1$, where

$$F(t,\ x)=\begin{cases}\left(\dfrac{1}{2^n}-x\right)\left(x-\dfrac{1}{2^{n+1}}\right) & \left(\dfrac{1}{2^{n+1}}<x\leqq\dfrac{1}{2^n}\right)\ (n=0,\ 1,\ 2,\ \cdots)\\ 0 & (x\leqq 0).\end{cases}$$

As the example above, when the solutions are continuable to the left and each solution is unique, if we put $\varphi(t, x)=|x(0\ ;\ x,\ t)|$, $\varphi(t, x)$ is continuous, but this $\varphi(t, x)$ does not necessarily satisfy the other required conditions. For example, we have $\varphi(t, x)=|x(0; x, t)|=|x|e^{t}$ for the equation $x'=-x$ and hence we cannot show that $\varphi(t, x)$ has the property A. Of course, even if the solutions are not continuable to the left, there is the desired function $\varphi(t, x)$. For example, in a equation

$$x'=\begin{cases}-x^{2} & (x\geqq 0)\\ 0 & (x<0),\end{cases}$$

all the solutions are not continuable to $t=0$, but it is sufficient that for this equation, we put

$$\varphi(t, x)=|x|.$$

Adding a slight assumption to $F(t, x)$, we will obtain a more useful condition for the uniform boundedness or the uniform stability.

Theorem 2. *We assume that $F(t, x)$ in the differential equation* (1) *belongs to the class C_0 with respect to x. In order that the solutions of* (1) *are uniformly bounded, it is necessary and sufficient that there exists a positive continuous function $\varphi(t, x)$ satisfying the following conditions in the domain*

$\bar{\Delta}$: $0\leqq t<\infty$, $|x|\geqq R$ (*R: a positive constant which may be sufficiently great*):

1° $\varphi(t, x)$ *has the property A,*

2° $\varphi(t, x)$ *tends to infinity uniformly as* $|x|\to\infty$,

3° $\varphi(t, x)$ *belongs to the class C_0 with respect to* (t, x) *and we have*

$$D_F\varphi(t, x)=\overline{\lim_{h\to 0}}\frac{1}{h}\left\{\varphi(t+h, x+hF)-\varphi(t, x)\right\}\leqq 0.$$

Proof. We have shown that the condition is sufficient (Theorem 3 in [9]). Now we will show that this condition is necessary. Let $x=x(t; x_0, t_0)$ be the solution of (1) through the initial point (t_0, x_0). We observe the solution for t such as $0\leqq t\leqq t_0$. Then this solution is continuable to $t=0$ or for a certain $\sigma\geqq 0$, we have $|x(t; x_0, t_0)|\to\infty$ when $t\to\sigma+0$. Now we put

$$\varphi(t, x)=\min_{\tau}\left[|x(\tau; x, t)|;\ 0\leqq\tau\leqq t\right], \tag{2}$$

where the region of τ is the region for which the solution $x=x(\tau; x, t)$ exists. It is clear that we can define $\varphi(t, x)$ for each point (t, x) in $\bar{\Delta}$. From the definition of $\varphi(t, x)$, we can easily see that we have

$$\varphi(t, x) \leqq |x|,$$

i. e., $\varphi(t, x)$ has the property A and for any solution of (1), $x=x(t)$, the function $\varphi(t, x(t))$ is a non-increasing function of t.

By the uniform boundedness of solutions, for any $G>0$, there exists a positive number $\bar{G}$ such that, if $|x_0| \leqq G$, we have

$$|x(t\,;\, x_0,\, t_0)| < \bar{G}.$$

Hence if $|x| > \bar{G}$, we have

$$|x(\tau\,;\, x,\, t)| > G \quad (\tau \leqq t).$$

From this we have $\varphi(t, x) > G$, that is to say, $\varphi(t, x)$ tends to infinity uniformly as $|x| \to \infty$.

Next we will show that $\varphi(t, x)$ belongs to the class C_0 with respect to (t, x). When the solution $x=x(t; x_0, t_0)$ through (t_0, x_0) is continuable to $t=0$, if we take a suitable neighborhood $U(\delta\,;\, t_0,\, x_0)$ of (t_0, x_0), the solutions issuing from there to the left lie in ε-neighborhood of $x=x(t\,;\, x_0,\, t_0)$, because each solution is unique by the assumption $F(t, x) \in C_0$. Therefore for the solutions issuing from $U(\delta\,;\, t_0,\, x_0)$, we may consider (2) exactly in $0 \leqq \tau \leqq t$. On the other hand, we suppose that for a certain $\sigma \geqq 0$, we have $|x(t; x_0, t_0)| \to \infty$ as $t \to \sigma + 0$. If we put

$$\sup_{\substack{t_0 \leqq t < \infty \\ |x_0| \leqq \eta \\ 0 \leqq t_0 < \omega}} |x(t\,;\, x_0,\, t_0)| = f(\eta),$$

this $f(\eta)$ is uniquely determined, since the solutions are uniformly bounded. Now let $\bar{t}$ be the value of t at which we have

$$|x(t\,;\, x_0,\, t_0)| = 2f(2|x_0|)$$

for the first time. In $\bar{t} \leqq t \leqq t_0$, if we take an ε-neighborhood of the solution $x=x(t\,;\, x_0,\, t_0)$ (ε : be small), the solutions of (1) issuing from a suitable neighborhood $U(\delta\,;\, t_0,\, x_0)$ of (t_0, x_0) to the left lie in ε-neighborhood of $x=x(t\,;\, x_0,\, t_0)$ by the uniqueness of the solution. Therefore they are continuable to $t=\bar{t}$. Now we suppose that a certain solution of them satisfies

$$|x(t'\,;\, x,\, t)| < 2|x_0| \quad ((t,\, x) \in U(\delta\,;\, t_0,\, x_0))$$

at t' such as $0 \leqq t' < \bar{t}$. Then we have

$$|x(\bar{t}\,;\, x,\, t)| > f(2|x_0|) \qquad (|x| < 2|x_0|)$$

for a solution of (1) issuing from the region $|x| < 2|x_0|$ to the right. This contradicts the definition of $f(\eta)$. Therefore they cannot take their minima in $t < \bar{t}$. Namely there exists $\bar{t}$ such that all the solutions issuing from a suitable neighborhood of (t_0, x_0) to the left are continuable to $\bar{t}$ and they take their minima in $\bar{t} \leqq \tau \leqq t$. Hence in all cases, we may consider in $\bar{t} \leqq \tau \leqq t$ $(\bar{t} \geqq 0)$ $\min|x(\tau; x, t)|$ for the point (t, x) of a suitable neighborhood of (t_0, x_0).

Now we consider

$$\begin{aligned}&\varphi(t, x) - \varphi(t, x') \\ &= \min[|x(\tau\,;\, x,\, t)|;\ \bar{t} \leqq \tau \leqq t] - \min[|x(\tau\,;\, x',\, t)|;\ \bar{t} \leqq \tau \leqq t].\end{aligned}$$

If we put

$$\min[|x(\tau\,;\, x',\, t)|;\ \bar{t} \leqq \tau \leqq t] = |x(\tau'\,;\, x',\, t)|,$$

we have

$$\begin{aligned}\varphi(t, x) - \varphi(t, x') &\leqq |x(\tau'\,;\, x,\, t)| - |x(\tau'\,;\, x',\, t)| \\ &\leqq A|x - x'| \qquad \text{(cf. [10])},\end{aligned}$$

where A is a suitable constant in the considered neighborhood. Moreover if we put

$$\min[|x(\tau\,;\, x,\, t)|;\ \bar{t} \leqq \tau \leqq t] = |x(\tau''\,;\, x,\, t)|,$$

we have

$$\begin{aligned}\varphi(t, x) - \varphi(t, x') &\geqq |x(\tau''\,;\, x,\, t)| - |x(\tau''\,;\, x',\, t)| \\ &\geqq -A|x - x'|.\end{aligned}$$

Therefore we have

$$|\varphi(t, x) - \varphi(t, x')| \leqq A|x - x'|. \tag{3}$$

Next we consider $\varphi(t, x) - \varphi(t', x)$, where $t < t'$. If $|x(\tau\,;\, x,\, t')|$ takes the minimum at $\bar{t} \leqq \tau' \leqq t$, we have

$$\begin{aligned}\varphi(t, x) - \varphi(t', x) &\leqq |x(\tau'\,;\, x,\, t)| - |x(\tau'\,;\, x,\, t')| \\ &\leqq |x(\tau'\,;\, x,\, t)| - |x(\tau'\,;\, X,\, t)| \\ &\leqq A|X - x| \\ &\leqq C(t' - t) \qquad \text{(cf. [10])},\end{aligned}$$

where $X=x(t\,;x,t')$ and C is a suitable constant. In the similar way we have

$$\varphi(t,x)-\varphi(t',x)\geqq -C(t'-t).$$

Now if $|x(\tau\,;x,t)|$ takes the minimum at $\bar{t}\leqq\tau''\leqq t$ and $|x(\tau\,;x,t')|$ takes the minimum at $t<\tau'\leqq t'$, we have

$$\begin{aligned}\varphi(t,x)-\varphi(t',x)&=|x(\tau'';x,t)|-|x(\tau';x,t')|\\&\leqq|x(t\,;x,t)|-|x(t\,;x,t')|+|x(t\,;x,t')|-|x(\tau';x,t')|\\&\leqq|x-X|+|x(t\,;x,t')-x(\tau'\,;x,t')|\\&\leqq C(t'-t).\end{aligned}$$

In the similar way we have

$$\varphi(t,x)-\varphi(t',x)\geqq -C(t'-t).$$

Therefore we have

$$(4)\qquad |\varphi(t,x)-\varphi(t',x)|\leqq C(t'-t).$$

From (3) and (4) we have

$$(5)\qquad |\varphi(t,x)-\varphi(t',x')|\leqq M[|t-t'|+|x-x'|],$$

that is to say, $\varphi(t,x)$ belongs to the class C_0 with respect to (t,x). From this and the fact that $\varphi(t,x)$ is non-increasing along any solution, we can see

$$D_F\varphi(t,x)\leqq 0.$$

When we add a slight condition to $F(t,x)$ again, we can obtain a *regular* function $\varphi(t,x)$ by Massera's method [4]: namely

Theorem 3. *We assume that $F(t,x)$ belongs to the class C_0 with respect to (t,x). In order that the solutions of* (1) *are uniformly bounded, it is necessary and sufficient that there exists a regular function $\varphi(t,x)$ satisfying the conditions in Theorem 2. Hence the condition* 3° *is replaced by the inequality*

$$\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial x}\cdot F(t,x)\leqq 0.$$

Proof. By Theorem 2, there exists a function $\varphi(t,x)$ in $\bar{\Delta}$. Since the solutions are uniformly bounded, we have $|x(t\,;x_0,t_0)|<\beta(\alpha)$ when we have $|x_0|\leqq\alpha$. We may assume that $\beta(\alpha)$ is a continuous strictly monotone increasing function of α. Then there exists a function $\alpha(|x|)$ such as

$$0<\alpha(|x|)\leqq\varphi(t, x),$$

where $\alpha(\beta)$ is the inverse function of $\beta(\alpha)$ and $\alpha(\beta)$ is a continuous strictly monotone increasing function of β and $\alpha(\beta)\to\infty$ as $\beta\to\infty$.

If we put

$$U(t, x)=\frac{1+t}{1+2t}\varphi(t, x),$$

we have

$$\frac{1}{2}\alpha(|x|)\leqq U(t, x)\leqq|x|$$

and

$$D_r U(t, x)\leqq-\alpha(|x|)\frac{1}{(1+2t)^2}.$$

Let (τ, x) and (τ', x') be any two points such that $0\leqq\tau\leqq t$, $0\leqq\tau'\leqq t$ and $|x|\leqq r$, $|x'|\leqq r$. We have

$$|U(\tau, x)-U(\tau', x')|\leqq M(t, r)[|\tau-\tau'|+|x-x'|],$$

where we may assume that $M(t, r)$ is monotone with respect to t and r. Moreover let $N(t, r)\geqq 1$ be a Lipschitz constant of $F(\tau, \xi)$ in the region $0\leqq\tau\leqq t$, $|\xi|\leqq r$, where we may assume that $N(t, r)$ is monotone. We have a function $L(t)\geqq 1$ which satisfies

$$|F(t, 0)|\leqq L(t)$$

and is monotone with respect to t. It is possible to find a positive regular function $\theta(t, r)$ in the region $0\leqq t<\infty$, $R\leqq r$, such that

$$\theta(t, r)<\frac{1}{2}$$

$$8r\theta(t, r)L(t+1)M(t+1, r+1)N(t+1, r+1)(1+2(t+1))^2\leqq 1$$

$$8r|\theta_t'|L(t+1)M(t+1, r+1)N(t+1, r+1)(1+2(t+1))^2\leqq 1$$

$$8r|\theta_r'|L(t+1)M(t+1, r+1)N(t+1, r+1)(1+2(t+1))^2\leqq 1.$$

To find $\theta(t, r)$, we put

$$[8rL(t+1)M(t+1, r+1)N(t+1, r+1)(1+2(t+1))^2]^{-1}=P(t, r).$$

Consider a regular non-negative function $K(t, r)$ vanishing outside the set $|t|\leqq 1$, $|r|\leqq 1$, such that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K(t, r)\,dtdr=1$. We define

$$\tilde{\theta}(t, r)=\iint P(\tau, \xi)K(\tau-t-1, \xi-r-1)\,d\tau d\xi.$$

Then it is sufficient to put

$$\theta(t, r)=\int_t^\infty\int_r^\infty \tilde{\theta}(\tau, \xi)e^{-\tau-\xi}d\tau d\xi.$$

Now let $K(t, x)$ be a regular non-negative function vanishing outside the set $0\leqq t\leqq 1$, $|x|\leqq 1$, such that $\int K(t, x)dtdx=1$, extended over the whole space, where x is an n-dimensional vector. When we define

$$V(t, x)$$
$$=\int U(\tau, \xi)K\{(\tau-t)\theta(t, |x|)^{-1}, (\xi-x)\theta(t, |x|)^{-1}\}\theta(t, |x|)^{-n-1}d\tau d\xi,$$

$V(t, x)$ is a regular function defined in $0\leqq t<\infty$, $|x|\geqq R+\frac{1}{2}$. If we follow Massera's proof, we can see that this function is the desired function.

Corresponding to the above-mentioned Theorems 2 and 3, we can obtain the theorems for the uniform stability in the same way.

Theorem 4. *We assume that* $F(t, x)$ *in* (1) *is continuous in the domain*

$$0\leqq t<\infty, \quad |x|\leqq H$$

and $F(t, 0)\equiv 0$. *Moreover we suppose that* $F(t, x)$ *belongs to the class* C_0 *with respect to* x. *Then in order that the solution* $x\equiv 0$ *of* (1) *is uniformly stable, it is necessary and sufficient that there exists a continuous positive definite function* $\varphi(t, x)$ *satisfying the following conditions in the domain,*

$$0\leqq t<\infty, \quad |x|\leqq \bar{H}(\leqq H) \qquad (\bar{H} \text{ be a suitable constant});$$

1° $\varphi(t, x)$ *has the infinitely small upper bound* (*cf.* [3]),

2° $\varphi(t, 0)\equiv 0$,

3° $\varphi(t, x)$ *belongs to the class* C_0 *with respect to* (t, x) *and we have*

$$D_F\varphi(t, x)\leqq 0.$$

Proof. Since the solution $x\equiv 0$ is uniformly stable, for any $\varepsilon>0$ there exists a positive number $\delta(\varepsilon)$ such that, if we have $|x_0|\leqq\delta(\varepsilon)$, we have $|x(t; x_0, t_0)|<\varepsilon$. We may assume that this $\delta(\varepsilon)$ is a continuous strictly monotonic function of ε. Let a be a positive constant such as $H>a$. If we define $\varphi(t, x)$ in the region $0\leqq t<\infty$, $|x|<\delta(\delta(a))$ in the same way as one in the case of the uniform boundedness, we can see that the condition is necessary.

Recently Krasovskii has obtained a necessary and sufficient condition for the uniform stability [1], but by Massera's method, we can obtain a *regular* $\varphi(t, x)$ from Theorem 4 in the following case.

Theorem. 5. *We assume that $F(t, x)$ belongs to the class C_0 with respect to (t, x). In order that the solution $x\equiv 0$ is uniformly stable, it is necessary and sufficient that there exists a regular function $\varphi(t, x)$ satisfying the conditions in Theorem* 4.

In the similar way as one in the proof of Theorem 3, using a function $\theta(t)$ such that

$$\theta(t)M(t+1)N(t+1)(1+2(t+1))^2\leqq 1$$

$$|\theta'(t)|M(t+1)N(t+1)(1+2(t+1))^2\leqq 1,$$

we define

$$V(t, x) = \int \Phi(U(\tau, \xi))K\{(\tau-t)[\rho(|x|)\theta(t)]^{-1}, (\xi-x)[\rho(|x|)\theta(t)]^{-1}\} \times [\rho(|x|)\theta(t)]^{-n-1}d\tau d\xi,$$

where $\Phi(u)$ is a function which Massera has represented by $\phi(u)$ and $\rho(r)$ is a similar function as Massera's one and

$$U(t, x) = \frac{1+t}{1+2t}\varphi(t, x) \quad (\varphi(t, x) \text{ obtained in Theorem 4}).$$

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