MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXXI, Mathematics No. 1, 1958.

An example of a normal local ring which is analytically reducible

By

Masayoshi Nagata

(Received January 27, 1958)

Previously the writer [2] gave an example of a normal local ring which is analytically ramified. In that example, the zero ideal of its completion is a primary ideal. In the present note, we shall show a normal local ring such that the zero ideal of its completion is not primary, which gives a counter example to a problem of the writer [3].

We shall remark here that our example is a finite separable integral extension of a regular local ring of rank 2 which contains a field k whose characteristic may be zero: we shall construct an example under the assumption that the characteristic of k is different from 2. It should be noted that even in the case of characteristic 2, a similar example can be given easily by a slight modification of our example.

(1) Construction of the example.

Let k be a field of characteristic not equal to 2. Let $w=\sum a_i x^i$ ($a_0=0$, $a_i \in k$) be an element of the formal power series ring $k\{x\}$ over k such that w is transcendental over $k(x)$.

Now, let x, y, z be algebraically independent elements over k and set $z_1=z$, $z_{i+1}=[z-(y+\sum_{j<i} a_j x^j)^2]/x^i$. Set $\mathfrak{r}=k[x, y, z_1, z_2, \cdots]_{(x, y, z_1, z_2, \cdots)}$. Then $\mathfrak{o}=\mathfrak{r}[W]/(W^2-z)$ is the required example.

(2) Properties of the ring $\mathfrak{r}$.

Since w is transcendental over $k(x)$, we can identify z with the element $(y+w)^2$ in the power series ring $k\{x, y\}$. Then as is easily seen, every z_i is identified with an element of $k\{x, y\}$ whose leading form is not constant. Thus we may say that $\mathfrak{r}$ is dominated by $k\{x, y\}$. On the other hand, by the definition of z_i, there is a polynomial $f_i(x, y)$ in x and y such that $xz_{i+1}=z_i+f_i(x, y)$ $(f_i(0, 0)=0)$. Therefore every z_i is in the ideal $x\mathfrak{r}+y\mathfrak{r}$. Thus the maximal ideal of $\mathfrak{r}$ is generated by x, y. Furthermore, it is easy to see that for any element a of $\mathfrak{r}$ and for any given natural number n, there exists a polynomial $g(x, y)$ such that $a-g(x, y) \in (x\mathfrak{r}+y\mathfrak{r})^n$, using the relation $xz_{i+1}=z_i+f_i(x, y)$. Thus we know that $k\{x, y\}$ is the completion of $\mathfrak{r}$, in view of the fact that $\mathfrak{r}$ dominates $k[x, y]_{(x, y)}$.

In the next step, we want to prove that $\mathfrak{r}$ is Noetherian. Let $\mathfrak{p}$ be any prime ideal of rank 1 in $\mathfrak{r}$. i) Assume that $x \in \mathfrak{p}$. Since obviously $x\mathfrak{r}$ is a prime ideal, we have $\mathfrak{p}=x\mathfrak{r}$. ii) Assume now that $x \notin \mathfrak{p}$. Since every z_i is contained in $k[x, y, z, 1/x]$, $\mathfrak{r}[1/x]$ is a ring of quotients of $k[x, y, z]$. Since $k[x, y, z]$ is a unique factorization ring, $\mathfrak{r}[1/x]$ is also a unique factorization ring, which shows that $\mathfrak{p}\mathfrak{r}[1/x]$ is a principal ideal. We can choose a generator p of $\mathfrak{p}\mathfrak{r}[1/x]$ such that $p \in \mathfrak{p}$ and such that $p \notin x\mathfrak{r}$ (this is possible because $x\mathfrak{r}$ is a prime ideal and because $\mathfrak{r}$ is dominated by a local ring). Let a be any element of $\mathfrak{p}$. Then there exists an integer r such that $x^r a \in p\mathfrak{r}$. Assume that r is positive. Then $x^r a = pb$ shows $b \in x\mathfrak{r}$ because $p \notin x\mathfrak{r}$ and $x\mathfrak{r}$ is prime. Thus we see that $a \in p\mathfrak{r}$, hence $\mathfrak{p}=p\mathfrak{r}$. Thus we have proved that every prime ideal of rank 1 in $\mathfrak{r}$ is principal.

We shall recall here a theorem of Cohen [1] as follows:

A ring (commutative and having the identity) is Noetherian if (and only if) every prime ideal in the ring has a finite basis.

By virtue of this theorem, it will be sufficient to show that if $\mathfrak{q}$ is a prime ideal of rank 2 in $\mathfrak{r}$, then $\mathfrak{q}$ is maximal. Assume the contrary. Since co-rank $x\mathfrak{r}=1$ as is easily seen, $x \notin \mathfrak{q}$. Therefore $\mathfrak{q}\mathfrak{r}[1/x] \cap k[x, y, z]$ is a prime ideal of rank 2 in $k[x, y, z]$. Therefore the transcendence degree of $\bar{\mathfrak{r}}=\mathfrak{r}/\mathfrak{q}$ over k is 1. Let $\bar{x}$, $\bar{y}$ and $\bar{z}$ be the residue classes of x, y and z respectively modulo $\mathfrak{q}$. Then $\bar{\mathfrak{r}}$ is algebraic over $k[\bar{x}]$. Therefore $\bar{\mathfrak{r}}$ is a spot of rank 1. But, in the completion of $\bar{\mathfrak{r}}$, $\bar{z}$ is identified with $(\bar{y}+\sum a_i\bar{x}^i)^2$. Since $\sum a_i\bar{x}^i$ is transcendental over $k(\bar{x})$, we see that either $\bar{z}$ or $\bar{y}$ is transcendental over $k(\bar{x})$, which contradicts the fact that $\mathfrak{r}$ is algebraic over $k[\bar{x}]$.

Thus we have proved that $\mathfrak{r}$ is Noetherian. Since the maximal ideal is generated by x, y, we see that $\mathfrak{r}$ is a regular local ring.

Furthermore, we shall prove

Lemma. $z\mathfrak{r}$ is a prime ideal in the regular local ring $\mathfrak{r}$.

Proof. Since $\mathfrak{r}[1/x]$ is a ring of quotients of $k[x, y, z]$, z is a prime element of $\mathfrak{r}[1/x]$. Since $z\mathfrak{r}$ and $x\mathfrak{r}$ have no common prime divisor, we see that $z\mathfrak{r}$ is a prime ideal.

(3) The proof of the example.

Since $\mathfrak{r}$ is a regular local ring, $\mathfrak{o}$ is Noetherian. Since z is a prime element of $\mathfrak{r}$, we see that $\mathfrak{o}$ is normal, and it is easy to see that $\mathfrak{o}$ is a local ring. Thus $\mathfrak{o}$ is a normal local ring. The completion of $\mathfrak{o}$ is $k\{x, y\}[W]/(W^2-z)$. But $W^2-z=(W-(y+w))(W+(y+w))$. Thus the zero ideal of the completion of $\mathfrak{o}$ has two prime divisors.

Kyoto University and
Harvard University

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