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p-primary components of homotopy groups

I. Exact sequences in Steenrod algebra

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The structure of the Steenrod algebra $\mathscr{S}^*$ mod p [1] gives important tools for the calculation of the homotopy groups. In this section, the exactness of the several $\mathscr{S}^*$-homomorphisms is studied, and it will be applied to prove the triviality of mod p Hopf invariant in the next section and also to verify the homotopy groups in those sections which follow further.

§ Notations.

Throughout this paper, p denotes an odd prime and $\mathscr{S}^*$ denotes the Steenrod algebra mod p [1] [3]. $\mathscr{S}^*$ is a graded Z_p-algebra $\sum_i \mathscr{S}^i$ which is generated multiplicatively by the Bockstein operator $\Delta \in \mathscr{S}^1$ and Steenrod's reduced powers $\mathscr{P}^t \in \mathscr{S}^{2t(p-1)}$, $t=0, 1, 2, \cdots$.

For the simplicity of the descriptions, we shall use the following notations.

(1.1) $\mathscr{P}(\Delta^{\varepsilon_0}, r_1, \Delta^{\varepsilon_1}, r_2, \cdots, r_n, \Delta^{\varepsilon_n}) = \Delta^{\varepsilon_0}\mathscr{P}^{r_1}\Delta^{\varepsilon_1}\mathscr{P}^{r_2} \cdots \mathscr{P}^{r_n}\Delta^{\varepsilon_n}$,

where ε_i and r_i are non-negative integers. From the relation

$$\Delta^2 = \Delta\Delta = 0,$$

the monomial (1.1) vanishes if one of $\varepsilon_i \geqq 2$. If $\varepsilon_i = 0$, we may omit Δ^{ε_i} in (1.1) since Δ^0 means the identity. If $\varepsilon_i = 1$, we write Δ^{ε_i} by Δ. Also if $r_i = 0$, then we may replace "$\Delta^{\varepsilon_{i-1}}, r_i, \Delta^{\varepsilon_i}$" and "$\Delta^{\varepsilon_{i-1}}\mathscr{P}^{r_i}\Delta^{\varepsilon_i}$" by "$\Delta^{\varepsilon_{i-1}+\varepsilon_i}$" since $\mathscr{P}^0$ is the identity.

A monomial (1.1) is said to be *admissible* if ε_i are 0 or 1, $r_n > 0$ and if $r_i \geqq pr_{i+1} + \varepsilon_i$ for $i = 1, 2, \cdots, n-1$. Then the admissible monomials form an additive Z_p-base of $\mathcal{S}^*$ [1] [2].

Let A^* be a left (*resp.* right) $\mathcal{S}^*$-module and let α be an element of $\mathcal{S}^*$. We define a homomorphism

$$\alpha_* \ (resp.\ \alpha^*) \quad : \quad A^* \to A^*$$

by setting $\alpha_*(a)=\alpha a$ (*resp.* $\alpha^*(a)=a\alpha$), $a\in A^*$. If A^* is a two sided $\mathcal{S}^*$-module, then α_*(*resp.* α^*) is a right (resp. left) $\mathcal{S}^*$-homomorphism. Obviously

$$(\alpha\beta)^* = \alpha_*\beta_*\,, \quad (\alpha\beta)^* = \beta^*\alpha^* \quad and \quad \alpha_*\beta^* = \beta^*\alpha_*$$

for $\alpha, \beta \in \mathcal{S}^*$. In particular, we denote that

$$R(r) = (r+1)\,\Delta\mathcal{P}^1 - r\mathcal{P}^1\Delta = (r+1)\,\mathcal{P}(\Delta, 1) - r\mathcal{P}(1, \Delta)\,,$$

and we shall treat the induced homomorphisms

$$R(r)_* \quad and \quad R(r)^* \quad : \quad \mathcal{S}^* \to \mathcal{S}^*\,.$$

We denote that

$$\alpha A^* = \{\alpha a \mid a\in A^*\} = \alpha_*(A^*)\,,$$
$$A^*\alpha = \{a\alpha \mid a\in A^*\} = \alpha^*(A^*)\,.$$

Since $\Delta\Delta=0$, a left (*resp.* right) $\mathcal{S}^*$-module A^* is a complex with respect to the coboundary operator Δ_* (*resp.* Δ^*). Denote by

$$H_\Delta(A^*) \qquad (resp.\ H^\Delta(A^*))$$

the cohomology group of the complex (A^*, Δ_*) (*resp.* (A^*, Δ^*)).

An admissible monomial (1.1) is Δ_*-cocycle (*resp.* Δ^*-cocycle) if and only if $\varepsilon_0=0$ (*resp.* $\varepsilon_n=0$), and it is Δ_*-cobounded (*resp.* Δ^*-cobounded). It follows

(1.2) $\quad H_\Delta(\mathcal{S}^*) = H^\Delta(\mathcal{S}^*) = 0\,, \quad H^\Delta(\Delta\mathcal{S}^*) = H_\Delta(\mathcal{S}^*\Delta) = \{\Delta\}$ *and* $H^\Delta(\mathcal{S}^*/\Delta\mathcal{S}^*) = H_\Delta(\mathcal{S}^*/\mathcal{S}^*\Delta) = \{1\}\,.$

It is convenient to regard that $\mathcal{S}^*/\Delta\mathcal{S}^*$ (*resp.* $\mathcal{S}^*/\mathcal{S}^*\Delta$) is spanned by the admissible monomials (1.1) of $\varepsilon_0=0$ (*resp.* $\varepsilon_n=0$). Then we define two right $\mathcal{S}^*$-homomorphisms

$$R' : \mathcal{S}^*/\Delta\mathcal{S}^* + \mathcal{S}^*/\Delta\mathcal{S}^* \to \mathcal{S}^*\,,$$
$$R\ : \mathcal{S}^* \to \mathcal{S}^*/\Delta\mathcal{S}^* + \mathcal{S}^*/\Delta\mathcal{S}^*\,,$$

by the formulas $R'(\alpha, \beta)=\mathscr{P}^1\Delta\alpha+\Delta\mathscr{P}^1\Delta\beta$, $\alpha, \beta \in \mathscr{S}^*/\Delta\mathscr{S}^*$ and $R(\alpha)=(\mathscr{P}^1\Delta\alpha, -\mathscr{P}^1\alpha)$, $\alpha \in \mathscr{S}^*$.

§ Exact sequences of right $\mathscr{S}^*$-homomorphisms.

Any monomial (1.1) may be normalized to a sum of admissible monomials (uniquely) by use of the Adem's relations [1] [2]:

$$\mathscr{P}(r, s)=\sum_i (-1)^{r+i}\binom{(s-i)(p-1)-1}{r-pi}\mathscr{P}(r+s-i, i) \text{ if } r<ps,$$

$$(1.3)\quad \mathscr{P}(r, \Delta, s)=\sum_i (-1)^{r+i}\binom{(s-i)(p-1)}{r-pi}\mathscr{P}(\Delta, r+s-i, i)$$

$$+\sum_i (-1)^{r+i+1}\binom{(s-i)(p-1)-1}{r-pi-1}\mathscr{P}(r+s-i, \Delta, i) \text{ if } r \leqq ps.$$

For the case $0\leqq r<p$, we have from (1.3)

$$(1.3)'\quad \begin{aligned} \mathscr{P}(r, s)&=\binom{r+s}{r}\mathscr{P}(r+s), \\ \mathscr{P}(r, \Delta, s)&=\binom{r+s-1}{r}\mathscr{P}(\Delta, r+s)+\binom{r+s-1}{s}\mathscr{P}(r+s, \Delta). \end{aligned}$$

In particular, $\mathscr{P}(1, s)=(s+1)\mathscr{P}(s+1)$ and $\mathscr{P}(1, \Delta, s)$ $=s\mathscr{P}(\Delta, s+1)+\mathscr{P}(s+1, \Delta)$.

Proposition 1.1. *The following circular sequence is exact.*

$$\begin{array}{ccccccc} \mathscr{S}^* & \xrightarrow{R(p-2)_*} & \mathscr{S}^* \longrightarrow \cdots & \xrightarrow{R(2)_*} & \mathscr{S}^* & \xrightarrow{R(1)_*} & \mathscr{S}^* \\ & \nwarrow^{R'} & & & & \swarrow_{R} & \\ & & & \mathscr{S}^*/\Delta\mathscr{S}^*+\mathscr{S}^*/\Delta\mathscr{S}^*. & & & \end{array}$$

The groups H^Δ of the kernel-images are spanned by the classes of the following elements:

$H^\Delta(R(r)\mathscr{S}^*)$: $\mathscr{P}^{pi+p-r}\Delta$, $\Delta\mathscr{P}^{pi+p-r}\Delta$, $(1\leqq r\leqq p-2)$,

H^Δ(*image of* R') : $\mathscr{P}^{pi+1}\Delta$, $\Delta\mathscr{P}^{pi+1}\Delta$,

H^Δ(*image of* R) : $(\mathscr{P}^{pi}\Delta, 0)$, $(0, \mathscr{P}^{pi}\Delta)$,

where $i=0, 1, 2, \cdots$.

Proof. It follows from (1.3)′

$R(r)\mathscr{P}(s, t, \cdots)=(r+s+1)\mathscr{P}(\Delta, s+1, t, \cdots)-r\mathscr{P}(s+1, \Delta, t, \cdots)$,

$R(r)\mathscr{P}(s, \Delta, t, \cdots)=(r+s+1)\mathscr{P}(\Delta, s+1, \Delta, t, \cdots)$,

$R(r)\mathscr{P}(\Delta, s, t, \cdots)=(r+1)\mathscr{P}(\Delta, s+1, \Delta, t, \cdots)$,

$R(r)\mathscr{P}(\Delta, s, \Delta, t, \cdots)=0$.

If a monomial in the left side is admissible, then so is in the right side. For the case $1\leqq r\leqq p-2$, the kernel of $R(r)_*$ is generated by the elements $(r+s+1)\mathscr{P}(\Delta, s, t, \cdots)-(r+1)\mathscr{P}(s, \Delta, t, \cdots)$ and $\mathscr{P}(\Delta, s, \Delta, t, \cdots)$. In particular, $R(r+1)$ is in the kernel of $R(r)_*$. Thus $R(r)_*\circ R(r+1)_*=0$. Since $(r+s+1)\mathscr{P}(\Delta, s, t, \cdots)-(r+1)\mathscr{P}(s, \Delta, t, \cdots)=R(r+1)\mathscr{P}(s-1, t, \cdots)$, and $(r+2)(\Delta, s, \Delta, t, \cdots)=R(r+1)\mathscr{P}(\Delta, s-1, t, \cdots)$, then the kernel of $R(r)_*$ is contained in the image of $R(r+1)_*$ if $1\leqq r<p-2$. Therefore the exactness of the sequence

$$\mathscr{S}^* \xrightarrow{R(r+1)_*} \mathscr{S}^* \xrightarrow{R(r)_*} \mathscr{S}^*$$

is established for $1\leqq r<p-2$. The exactness of the sequence

$$\mathscr{S}^*/\Delta\mathscr{S}^*+\mathscr{S}^*/\Delta\mathscr{S}^* \xrightarrow{R'} \mathscr{S}^* \xrightarrow{R(p-2)_*} \mathscr{S}^*$$

follows from the above results on the kernel of $R(p-2)_*$ and from the first two of the following relations obtained from (1.3)$'$.

$$R'(\mathscr{P}(s, t, \cdots), 0)=s\mathscr{P}(\Delta, s+1, t, \cdots)+\mathscr{P}(s+1, \Delta, t, \cdots),$$
$$R'(0, \mathscr{P}(s, t, \cdots))=\mathscr{P}(\Delta, s+1, \Delta, t, \cdots),$$
$$R'(\mathscr{P}(s, \Delta, t, \cdots), 0)=s\mathscr{P}(\Delta, s+1, \Delta, t, \cdots),$$
$$R'(0, \mathscr{P}(s, \Delta, t, \cdots))=0.$$

From these relations, it follows that the kernel of R' is generated by $(\mathscr{P}(s, \Delta, t, \cdots), -s\mathscr{P}(s, t, \cdots))$ and $(0, \mathscr{P}(s, \Delta, t, \cdots,))$. Then the exactness of the sequence $\xrightarrow{R} \xrightarrow{R'}$ follows from the first two of the following relations.

$$R\mathscr{P}(s, t, \cdots)=(\mathscr{P}(s+1, \Delta, t, \cdots), -(s+1)\mathscr{P}(s+1, t, \cdots)),$$
$$R\mathscr{P}(\Delta, s, t, \cdots)=(0, -\mathscr{P}(s+1, \Delta, t, \cdots)),$$
$$R\mathscr{P}(s, \Delta, t, \cdots)=(0, -(s+1)\mathscr{P}(s+1, \Delta, t, \cdots)),$$
$$R\mathscr{P}(\Delta, s, \Delta, t, \cdots)=0.$$

Then the kernel of R is generated by $(s+1)\mathscr{P}(\Delta, s, t, \cdots)-\mathscr{P}(s, \Delta, t, \cdots)=R(1)\mathscr{P}(s-1, t, \cdots)$ and $\mathscr{P}(\Delta, s, \Delta, t, \cdots)=\frac{1}{2}R(1)\mathscr{P}(\Delta, s-1, t, \cdots)$. Since $R\circ R(1)_*=0$, we have the exactness of the remainder sequence $\xrightarrow{R(1)_*} \xrightarrow{R}$.

A monomial is Δ^*-cocycle if it is of a form $\mathscr{P}(\cdots, \Delta)$. Let $1\leqq r\leqq p-2$ and consider the generators $(r+s+1)\mathscr{P}(\Delta, s+1, t, \cdots)$

$-r\mathscr{P}(s+1, \Delta, t, \cdots)$ and $\mathscr{P}(\Delta, s+1, \Delta, t, \cdots)$ of $R(r)\mathscr{S}^*$. Then the Δ^*-cocycles of $R(r)\mathscr{S}^*$ are generated by the elements of the following forms:

$$(r+s+1)\mathscr{P}(\Delta, s+1, t, \cdots, \Delta) - r\mathscr{P}(s+1, \Delta, t, \cdots, \Delta),$$
$$\mathscr{P}(\Delta, s+1, \Delta, t, \cdots, \Delta),$$
$$\mathscr{P}(\Delta, s+1, \Delta) \quad \textit{and} \quad r\mathscr{P}(pi-r, \Delta).$$

Obviously the Δ^*-cocycles of the first two forms are Δ^*-cobounded in $R(r)\mathscr{S}^*$. $\mathscr{P}(\Delta, s+1, \Delta)$ is Δ^*-cobounded if $r+s+1 \not\equiv 0$ mod p, since $\mathscr{P}(\Delta, s+1, \Delta) = \frac{1}{r+s+1}((r+s+1)\mathscr{P}(\Delta, s+1) - (r+1)\mathscr{P}(s+1, \Delta))\Delta$. The elements $\mathscr{P}(pi-r, \Delta)$ and $\mathscr{P}(\Delta, pi-r, \Delta)$, $i=1, 2, 3, \ldots$, are not Δ^*-cobounded and their classes form a Z_p-base of $H^\Delta(R(r)\mathscr{S}^*)$. The other results on H^Δ are proved similarly, q.e.d.

Proposition 1. 2. *The following two sequences are exact*:

i) $\mathscr{S}^* \xrightarrow{\mathscr{P}^1_*} \mathscr{S}^* \xrightarrow{\mathscr{P}^{p-1}_*} \mathscr{S}^* \xrightarrow{\mathscr{P}^1_*} \mathscr{S}^*$,

ii) $\mathscr{S}^*/R(1)\mathscr{S}^* \xrightarrow{\mathscr{P}^1_*} \mathscr{S}^*/\Delta\mathscr{S}^* \xrightarrow{\mathscr{P}^{p-1}_*} \mathscr{S}^*/R(1)\mathscr{S}^* \xrightarrow{\mathscr{P}^1_*} \mathscr{S}^*/\Delta\mathscr{S}^*$.

$H^\Delta(\mathscr{P}^1\mathscr{S}^*) = H^\Delta(\mathscr{P}^{p-1}\mathscr{S}^*) = 0$, $H^\Delta((\mathscr{P}^1\mathscr{S}^* + \Delta\mathscr{S}^*)/\Delta\mathscr{S}^*) = \{\mathscr{P}^{pi}\Delta,\ i=1, 2, 3, \cdots\}$ *and* $H^\Delta((\mathscr{P}^{p-1}\mathscr{S}^* + R(1)\mathscr{S}^*/R(1)\mathscr{S}^*) = \{\mathscr{P}^{pi-1},\ i=1, 2, 3, \cdots\}$.

Proof. By (1. 3)′,

$$\mathscr{P}(1)\mathscr{P}(s, t, \cdots) = (s+1)\mathscr{P}(s+1, t, \cdots),$$
$$\mathscr{P}(1)\mathscr{P}(s, \Delta, t, \cdots) = (s+1)\mathscr{P}(s+1, \Delta, t, \cdots),$$
$$\mathscr{P}(1)\mathscr{P}(\Delta, s, t, \cdots) = s\mathscr{P}(\Delta, s+1, t, \cdots) + \mathscr{P}(s+1, \Delta, t, \cdots),$$
$$\mathscr{P}(1)\mathscr{P}(\Delta, s, \Delta, t, \cdots) = s\mathscr{P}(\Delta, s+1, \Delta, t, \cdots).$$

Then the kernel of $\mathscr{P}(1)_*$ is generated by $\mathscr{P}(pi+p-1, t, \cdots) = \mathscr{P}(p-1)\mathscr{P}(pi, t, \cdots)$, $\mathscr{P}(pi+p-1, \Delta, t, \cdots) = \mathscr{P}(p-1)\mathscr{P}(pi, \Delta, t, \cdots)$, $\mathscr{P}(\Delta, pi, t, \cdots) - \mathscr{P}(pi, \Delta, t, \cdots) = \mathscr{P}(p-1)\mathscr{P}(\Delta, pi-p+1, t, \cdots)$ and $\mathscr{P}(\Delta, pi, \Delta, t, \cdots) = \mathscr{P}(p-1)\mathscr{P}(\Delta, pi-p+1, \Delta, t, \cdots)$. As a consequence we have the exactness of the sequence

$$\mathscr{S}^* \xrightarrow{\mathscr{P}(p-1)_*} \mathscr{S}^* \xrightarrow{\mathscr{P}(1)_*} \mathscr{S}^*.$$

The cokernel $\mathscr{S}^*/\mathscr{P}(1)\mathscr{S}^*$ of $\mathscr{P}(1)_*$ has a base which consists of the admissible monomials $\mathscr{P}(pi, t, \cdots)$, $\mathscr{P}(pi, \Delta, t, \cdots)$, $\mathscr{P}(\Delta, pi+1, t, \cdots)$ and $\mathscr{P}(\Delta, pi+1, \Delta, t, \cdots)$. From (1.3)$'$, it follows that these elements of the base are mapped by $\mathscr{P}(p-1)_*$ to the elements $\mathscr{P}(pi+p-1, t, \cdots)$, $\mathscr{P}(pi+p-1, \Delta, t, \cdots)$, $\mathscr{P}(\Delta, pi+p, t, \cdots)$ and $\mathscr{P}(\Delta, pi+p, \Delta, t, \cdots)$ respectively. Thus $\mathscr{P}(p-1)_*$ maps $\mathscr{S}^*/\mathscr{P}(1)\mathscr{S}^*$ isomorphically into $\mathscr{S}^*$, and then the exactness of the sequence

$$\mathscr{S}^* \xrightarrow{\mathscr{P}(1)_*} \mathscr{S}^* \xrightarrow{\mathscr{P}(p-1)_*} \mathscr{S}^* .$$

is proved.

Next consider the sequence ii). Concerning the above images of $\mathscr{P}(1)_*$, in the biginning of the proof, mod. by $\Delta\mathscr{S}^*$, we have that the kernel of $\mathscr{P}(1)_*: \mathscr{S}^* \to \mathscr{S}^*/\Delta\mathscr{S}^*$ is generated by the element $\mathscr{P}(pi+p-1, t, \cdots) = \mathscr{P}(p-1)\mathscr{P}(pi, t, \cdots)$, $(s+1)\mathscr{P}(\Delta, s, t, \cdots) - \mathscr{P}(s, \Delta, t, \cdots) = R(1)\mathscr{P}(s-1, t, \cdots)$ and $\mathscr{P}(\Delta, s, \Delta, t, \cdots) = R(1)\mathscr{P}(\Delta, s-1, t, \cdots)$. Then the sequence

$$\mathscr{S}^* \xrightarrow{\mathscr{P}(p-1)_*} \mathscr{S}^*/R(1)\mathscr{S}^* \xrightarrow{\mathscr{P}(1)_*} \mathscr{S}^*/\Delta\mathscr{S}^*$$

is exact. The admissible monomials $\mathscr{P}(pi, t, \cdots)$ from a base of the cokernel $\mathscr{S}^*/(\mathscr{P}(1)\mathscr{S}^* + \Delta\mathscr{S}^*)$. Since $R(\mathscr{P}(p-1)) = (\mathscr{P}(1, \Delta, p-1), -\mathscr{P}(1, p-1)) = (\mathscr{P}(p, \Delta), 0)$ and since $\mathscr{P}(p, \Delta)\mathscr{P}(pi, t, \cdots) = \mathscr{P}(pi+p, \Delta, t, \cdots)$ mod $\Delta\mathscr{S}^*$, it holds $(R\circ\mathscr{P}(p-1)_*)\mathscr{P}(pi, t, \cdots) = (\mathscr{P}(pi+p, \Delta, t, \cdots), 0)$. Then $R\circ\mathscr{P}(p-1)_*$ maps $\mathscr{S}^*/(\mathscr{P}(1)\mathscr{S}^* + \Delta\mathscr{S}^*)$ isomorphically into $\mathscr{S}^*/\Delta\mathscr{S}^* + \mathscr{S}^*/\Delta\mathscr{S}^*$. By Proposition 1.1, R carries $\mathscr{S}^*/R(1)\mathscr{S}^*$ isomorphically into $\mathscr{S}^*/\Delta\mathscr{S}^* + \mathscr{S}^*/\Delta\mathscr{S}^*$. Therefore $\mathscr{P}(p-1)_*$ maps $\mathscr{S}^*/(\mathscr{P}(1)\mathscr{S}^* + \Delta\mathscr{S}^*)$ isomorphically into $\mathscr{S}^*/R(1)\mathscr{S}^*$, and the sequence

$$\mathscr{S}^* \xrightarrow{\mathscr{P}(1)_*} \mathscr{S}^*/\Delta\mathscr{S}^* \xrightarrow{\mathscr{P}(p-1)_*} \mathscr{S}^*/R(1)\mathscr{S}^*$$

is exact.

The factor group $(\mathscr{P}^1\mathscr{S}^* + \Delta\mathscr{S}^*)/\Delta\mathscr{S}^*$ is generated by the classes of $(s+1)\mathscr{P}(s+1, t, \cdots)$ and $\mathscr{P}(s+1, \Delta, t, \cdots)$. As is seen in the previous proof, $H^\Delta((\mathscr{P}^1\mathscr{S}^* + \Delta\mathscr{S}^*)/\Delta\mathscr{S}^*) = \{\mathscr{P}(pi, \Delta), i = 1, 2, \cdots\}$. From the exact sequence ii), we have an exact sequence of Δ^*-complexes:

$$0 \to (\mathscr{P}^1\mathscr{S}^* + \Delta\mathscr{S}^*)/\Delta\mathscr{S}^* \to \mathscr{S}^*/\Delta\mathscr{S}^*$$
$$\to (\mathscr{P}^{p-1}\mathscr{S}^* + R(1)\mathscr{S}^*)/R(1)\mathscr{S}^* \to 0 .$$

From the cohomology exact sequence associated with this sequence and from (1.2), there is an isomorphism

$$H^{d}((\mathcal{P}^{p-1}\mathcal{S}^{*}+R(1)\,\mathcal{S}^{*})/R(1)\,\mathcal{S}^{*})$$
$$\approx H^{d}((\mathcal{P}^{1}\mathcal{S}^{*}+\Delta\mathcal{S}^{*})/\Delta\mathcal{S}^{*})+H^{d}(\mathcal{S}^{*}/\Delta\mathcal{S}^{*})\,.$$

By this isomorphism $\mathcal{P}(pi+p-1)$ corresponds to $\mathcal{P}(pi, \Delta)$ (for $i\geqq 1$) or 1 (for $i=0$). Thus $H^{d}((\mathcal{P}^{p-1}\mathcal{S}^{*}+R(1)\,\mathcal{S}^{*})/R(1)\,\mathcal{S}^{*}) =\{\mathcal{P}(pi+p-1),\ i=0, 1, 2, \cdots\}$. The proof of $H^{d}(\mathcal{P}^{1}\mathcal{S}^{*}) =H^{d}(\mathcal{P}^{p-1}\mathcal{S}^{*})=0$ is similar and easy, q.e.d.

Denote that

$$M_t=\Delta\mathcal{S}^{*}+\mathcal{P}^{1}\mathcal{S}^{*}+\mathcal{P}^{p}\mathcal{S}^{*}+\cdots+\mathcal{P}^{p^{t-1}}\mathcal{S}^{*}\quad (M_0=\Delta\mathcal{S}^{*})\,.$$

Lemma 1.3. i) *M_t is spanned by the admissible monomials which are not of the forms $\mathcal{P}(a_0p^t, a_1p^{t-1}, \cdots, a_{t-1}p, a_t, \cdots)$, where $a_0\geqq a_1\geqq \cdots \geqq a_t\geqq 0$ and we omit $a_rp^{t-r}, \cdots, a_t, \cdots$ if $a_r=0$.*

ii) *$\mathcal{P}(q_1, q_2, \cdots, q_{t-s})\,M_s\subset M_t$ for $0\leqq s<t$.*

Proof. $M_0=\Delta\mathcal{S}^{*}$ is spanned by the admissible monomials $\mathcal{P}(\Delta, r, \cdots)$. From the proof of Proposition 1.2, it follows that $M_1/M_0=(\mathcal{P}^{1}\mathcal{S}^{*}+\Delta\mathcal{S}^{*})/\Delta\mathcal{S}^{*}$ is spanned by the admissible monomials $\mathcal{P}(s, r, \cdots)$ and $\mathcal{P}(r, \Delta, t, \cdots)$ such that $s\not\equiv 0 \bmod p$. Then i) is true for M_0 and M_1. i) implies that $\mathcal{P}(q, \Delta)\in M_1$. Thus $\mathcal{P}(q)\,M_0=\mathcal{P}(q, \Delta)\,\mathcal{P}^{*}\subset M_1\mathcal{P}^{*}=M_1$.

Now suppose that i) and ii) are true for M_s, $s\leqq t$. Then it is sufficient to prove that i) and ii) are true for M_{t+1}. We shall verify the image M_{t+1}/M_t of $\mathcal{P}(p^t)_{*}$. Since $\mathcal{P}(p^t)\,M_{t-1}\subset M_t$, it is sufficient to compute $\mathcal{P}(p^t, a_0p^{t-1}, a_1p^{t-2}, \cdots, a_{t-1}, \cdots) \bmod M_t$. Let $s\leqq t$ and consider the relation

$$\mathcal{P}(p^s, ap^{s-1})=\sum_{i=0}^{p^{s-1}}(-1)^{i+1}\binom{(ap^{s-1}-i)(p-1)-1}{p^s-pi}\mathcal{P}(p^s+ap^{s-1}-i, i)$$

of (1.3). If the term $\mathcal{P}(p^s+ap^{s-1}-i, i)$ is not in M_s, then $p^s+ap^{s-1}-i\equiv 0 \bmod p^s$ and $i\equiv 0 \bmod p^{s-1}$ by the assertion i) for M_s. This is possible only if $a=bp$ or $a=bp+1$ for some integer b, and then the non-trivial relations $\bmod M_s$ are the followings.

$$\begin{aligned}&\mathcal{P}(p^s, bp^s)\equiv(b+1)\,\mathcal{P}((b+1)\,p^s) && \bmod M_s\,,\\ &\mathcal{P}(p^s, bp^s+p^{s-1})\equiv\mathcal{P}((b+1)\,p^s, p^{s-1}) && \bmod M_s\,.\end{aligned}\tag{1.4}$$

From ii), we remark that $\alpha\equiv\beta \bmod M_s$ implies $\mathcal{P}(c_0p^t, \cdots, c_{t-s-1}p^{s+1})\,\alpha\equiv\mathcal{P}(c_0p^t, \cdots, c_{t-s-1}p^{s+1})\,\beta \bmod M_t$. Then repeating the relation (1.4) and concerning the relation $\mathscr{P}(1, \Delta, s) \equiv \mathscr{P}(s+1, \Delta) \bmod M_0$, it follows that $\mathscr{P}(p^t, a_0p^{t-1}, \cdots, a_{t-1}, \cdots)$ is not in M_t only if it has one of the following forms: $(0 \leqq r \leqq t)$

$$\begin{aligned} &\mathscr{P}(p^t, b_0p^t+p^{t-1}, \cdots, b_{r-1}p^{t-r+1}+p^{t-r}, b_rp^{t-r}, \cdots, b_{t-1}p, b_t, \cdots) \\ &\quad \equiv (b_r+1)\, \mathscr{P}((b_0+1)\,p^t, \cdots, (b_r+1)\,p^{t-r}, b_{r-1}p^{t-r-1}, \cdots, b_t, \cdots) \quad \bmod M_t, \\ &\mathscr{P}(p^t, b_0p^t+p^{t-1}, \cdots, b_{t-1}p+1, \Delta, b_t, \cdots) \\ &\quad \equiv \mathscr{P}((b_0+1)\,p^t, \cdots, (b_{t-1}+1)\,p, b_t+1, \Delta, \cdots) \quad \bmod M_t. \end{aligned} \tag{1.5}$$

Then M_{t+1}/M_t is spanned by the admissible monomials $\mathscr{P}(c_0p^t, c_1p^{t-1}, \cdots, c_{t-1}p, c_t, \Delta^\varepsilon, \cdots)$ such that one of c_i is not divisible by p or $\varepsilon=1$. It follows from this and from the assertion i) for M_t that i) is true for M_{t+1}.

By i), $\mathscr{P}(ap^{t+1}, \Delta) \in M_{t+1}$ and $\mathscr{P}(ap^{t+1}, p^i) \in M_{t+1}$ for $0 \leqq i \leqq t-1$, then $\mathscr{P}(ap^{t+1})\, M_t \subset M_{t+1}$. If $q \not\equiv 0 \bmod p^{t+1}$, then $\mathscr{P}(q) \in M_{t+1}$ and $\mathscr{P}(q)\, M_t \subset M_{t+1}$. Thus $\mathscr{P}(q_1, \cdots, q_{t-s+1})\, M_s = \mathscr{P}(q_1)\, \mathscr{P}(q_2, \cdots, q_{t-s+1})\, M_s \subset \mathscr{P}(q_1)\, M_t \subset M_{t+1}$, and then ii) is proved, q.e.d.

Proposition 1.4. *The kernel of the homomorphism*

$$\mathscr{P}^{p^t}_* : \mathscr{S}^* \longrightarrow \mathscr{S}^*/M_t$$

is $M_{t-1} + \mathscr{P}^{2p^{t-1}}\mathscr{S}^* + (2\mathscr{P}^{p^t+p^{t-1}} - \mathscr{P}^{p^t}\mathscr{P}^{p^{t-1}})\, \mathscr{S}^* + \mathscr{P}^{(p-1)p^t}\mathscr{S}^*$ *for* $t \geqq 1$.

Proof. Set $B = M_{t-1} + \cdots + \mathscr{P}^{(p-1)p^t}\mathscr{S}^*$. The following relations are verified from (1.3) and by Lemma 1.3.

$$\mathscr{P}(p^t, 2p^{t-1}) = \sum_{i=0}^{p^{t-1}} * \mathscr{P}(p^t+2p^{t-1}-i, i) \equiv 0 \quad \bmod M_t,$$

$$\begin{aligned} &2\mathscr{P}(p^t, p^t+p^{t-1}) - \mathscr{P}(p^t, p^t, p^{t-1}) \\ &= 2\sum_{i=0}^{p^{t-1}} * \mathscr{P}(2p^t+p^{t-1}-i, i) - \sum_{j=0}^{p^{t-1}} \sum_{i=0}^{[j/p]} * \mathscr{P}(2p^t+p^{t-1}-i-j, j, i) \\ &\equiv 2\binom{p^t(p-1)-1}{0} \mathscr{P}(2p^t, p^{t-1}) + \binom{p^t(p-1)-1}{p^t} \mathscr{P}(2p^t, p^{t-1}) \bmod M_t \\ &= 0, \end{aligned}$$

$$\begin{aligned} &\mathscr{P}(p^t, (p-1)\,p^t) = \sum_{i=0}^{p^{t-1}} * \mathscr{P}(p^{t+1}-i, i) \\ &\equiv -\binom{p^t(p-1)^2-1}{p^t} \mathscr{P}(p^{t+1}) = 0 \quad \bmod M_t. \end{aligned}$$

These and ii) of Lemma 1.3 imply that $\mathscr{P}(p^t)\, B \subset M_t$. Then it is sufficient to prove that $\mathcal{S}^*/B$ is mapped isomorphically into $\mathcal{S}^*/M_t$ by $\mathcal{P}(p^t)_*$.

First we consider the image of $\mathcal{P}(2p^{t-1})_*: \mathcal{S}^* \to \mathcal{S}^*/M_{t-1}$. By Lemma 1.3, $\mathcal{P}(2p^{t-1}, \Delta)$, $\mathcal{P}(2p^{t-1}, p^i) \in M_{t-1}$ for $i=0, 1, 2, \cdots, t-3$. Then $\mathcal{P}(2p^{t-1}) M_{t-2} \subset M_{t-1}$. Thus the image of $\mathcal{P}(2p^{t-1})_*$ in $\mathcal{S}^*/M_t$ is generated by $\mathcal{P}(2p^{t-1}, a_0 p^{t-2}, \cdots, a_{t-2}, \cdots)$ mod M_{t-1} where $a_0 \geqq \cdots \geqq a_{t-2} \geqq 0$. Consider the relation $\mathcal{P}(2p^s, ap^{s-1}) = \sum * \mathcal{P}(2p^s + ap^{s-1} - i, i)$, $0 \leqq i \leqq 2p^{s-1}$, of (1.3). Then, by Lemma 1.3, the non-trivial relations mod M_s are

$$\mathcal{P}(2p^s, bp^s) = \binom{b+2}{2} \mathcal{P}((b+2)p^s) \qquad \text{mod } M_s,$$

$$\mathcal{P}(2p^s, bp^s + p^{s-1}) = (b+1)\mathcal{P}((b+2)p^s, p^{s-1}) \qquad \text{mod } M_s,$$

and $$\mathcal{P}(2p^s, bp^s + 2p^{s-1}) = \mathcal{P}((b+2)p^s, 2p^{s-1}) \qquad \text{mod } M_s.$$

Analogous discussions of the proof of Lemma 1.3 lead us to the following (1.6) from these relations and from (1.4).

(1.6) $M_{t-1} + \mathcal{P}(2p^{t-1})\mathcal{S}^*$ *is spanned by the admissible monomials which are not of the forms* $\mathcal{P}(b_0 p^t + p^{t-1}, \cdots, b_{t-1}p+1, \Delta, \cdots)$ *and* $\mathcal{P}(b_0 p^t + p^{t-1}, \cdots, b_{r-1}p^{t-r+1} + p^{t-r}, b_r p^{t-r}, \cdots, b_{t-1}p, b_t, \cdots)$ *where* $0 \leqq r \leqq t$ *and* $b_0 \geqq b_1 \geqq \cdots \geqq b_t \geqq 0$.

B was given by

$$B = M_{t-1} + \mathcal{P}(2p^{t-1})\mathcal{S}^* + (2\mathcal{P}(p^t + p^{t-1}) - \mathcal{P}(p^t, p^{t-1}))\mathcal{S}^* + \mathcal{P}((p-1)p^t)\mathcal{S}^*$$

and let C be a submodule of $\mathcal{S}^*$ spanned by the admissible monomials

$$\mathcal{P}(b_0 p^t + p^{t-1}, \cdots, b_{t-1}p+1, \Delta, b_t, \cdots)$$

and $$\mathcal{P}(c_0 p^t + p^{t-1}, \cdots, c_{r-1}p^{t-r+1} + p^{t-r}, c_r p^{t-r}, \cdots, c_t, \cdots)$$

such that $c_0 + 1 \equiv 0, \cdots, c_s + 1 \equiv 0$, $c_r + 1 \not\equiv 0$ mod p and $c_{s+1} = \cdots = c_r$ for some $0 \leqq r \leqq t$, $s < r$.

By (1.5), it is verified easily that $\mathcal{P}(p^t)_*$ maps C isomorphically into $\mathcal{S}^*/M_t$ and also onto M_{t+1}/M_t. Therefore, for the proof of the proposition, it is sufficient to prove the equality

$$B + C = \mathcal{S}^*.$$

Or, by (1.6), it is sufficient to prove that an admissible monomial $\mathscr{P}(c_0p^t+p^{t-r-1},\cdots,c_{r-1}p^{t-r+1}+p^{t-r},c_rp^{t-r},\cdots,c_t,\cdots)$ belongs to $B+C$ if it satisfies one of the following three conditions.

a) $c_s+1\not\equiv 0$, $c_r+1\not\equiv 0 \bmod p$ *and* $c_s>c_r$ *for some* $0\leqq s<r$,

b) $c_s+1\not\equiv 0$ *and* $c_r+1\equiv 0 \bmod p$ *for some* $0\leqq s<r$.

c) $c_0+1\equiv 0,\cdots,c_{r-1}+1\equiv 0$ *and* $c_r+1\equiv 0 \bmod p$.

For the simplicity we set $Q_s=2\mathscr{P}(p^s+p^{s-1})-\mathscr{P}(p^s,p^{s-1})$. By (1.3) and by (1.6), we compute the following relations:

$$Q_s\mathscr{P}(bp^s)\equiv(b+2)\mathscr{P}((b+1)p^s+p^{s-1})-\mathscr{P}((b+1)p^s,p^{s-1}) \quad \bmod M_{s-1}+\mathscr{P}(2p^{s-1})\mathscr{S}^*,$$

$$Q_s\mathscr{P}(bp^s+p^{s-1}+p^{s-2})\equiv\mathscr{P}((b+1)p^s+p^{s-1})Q_{s-1} \quad \bmod M_{s-1}+\mathscr{P}(2p^{s-1})\mathscr{S}^*.$$

Applying these relations and (1.4) to $Q_t\mathscr{P}((c_0-1)p^t+p^{t-1}+p^{t-2},\cdots,(c_{s-1}-1)p^{t-s+1}+p^{t-s}+p^{t-s-1},\ (c_s-1)p^{t-s},c_{s+1}p^{t-s-1}+p^{t-s-2},\cdots,c_{r-1}p^{t-r+1}+p^{t-r},c_rp^{t-r},\cdots,c_t,\cdots)$ we have the following relation $(0\leqq s<r\leqq t)$

$$\begin{aligned}(c_s+1)\,&\mathscr{P}(c_0p^t+p^{t-1},\cdots,c_sp^{t-s}+p^{t-s-1},\cdots,c_{r-1}p^{t-r+1}+p^{t-r},c_rp^{t-r},\cdots,c_t,\cdots)\\ &\equiv(c_r+1)\mathscr{P}(c_0p^t+p^{t-1},\cdots,c_{s-1}p^{t-s+1}+p^{t-s},c_sp^{t-s},(c_{s+1}+1)p^{t-s-1},\cdots,\\ &\quad (c_r+1)p^{t-r},c_{r+1}p^{t-r-1},\cdots,c_t,\cdots) \quad \bmod B.\end{aligned}$$

Consider an admissible monomial satisfying the condition a) in which we may suppose that $c_s>c_{s+1}$ and that $c_q=c_s$ if $q<s$ and $c_q+1\not\equiv 0 \bmod p$. Then the last relation shows that the monomial is equivalent mod B to an element of C, and it belongs to $B+C$. It follows directly from the last relation that an admissible monomial satisfying b) belongs to $B\subset B+C$.

By (1.3) and by (1.6) we have a relation mod $M_{s-1}+\mathscr{P}(2p^{s-1})\mathscr{S}^*$

$$\mathscr{P}((p-1)p^s,bp^{s-1}+p^s)\equiv\mathscr{P}(bp^{s+1}+(p-1)p^s+p^{s-1},\ (p-1)p^{s-1}).$$

In the case c), we compute the following relation from the above one.

$$\begin{aligned}&\mathscr{P}(c_0p^t+p^{t-1},\cdots,c_{r-1}p^{t-r+1}+p^{t-r},c_rp^{t-r},c_{r+1}p^{t-r-1},\cdots,c_t,\cdots)\\ &\equiv\mathscr{P}((p-1)p^t)\mathscr{P}((c_0-p+2)p^t,\cdots,(c_{r-1}-p+2)p^{t-r+1},\ (c_r-p+1)\\ &\quad p^{t-r},c_{r+1}p^{t-r-1},\cdots,c_t,\cdots)-\mathscr{P}(c_0p^t+p^{t-1},\cdots,c_{r-1}p^{t-r+1}+p^{t-r},(c_r-1)\\ &\quad p^{t-r}+p^{t-r-1},\ (p-1)p^{t-r-1})\mathscr{P}(c_{r+1}p^{t-r-1},\cdots,c_t,\cdots)\\ &\qquad \bmod M_{t-1}+\mathscr{P}(2p^{t-1})\mathscr{S}^*,\end{aligned}$$

Since $\mathscr{P}(c_0p^t+p, \cdots, c_{r-1}p^{t-r+1}+p^{t-r}, (c_r-1)p^{t-r}+p^{t-r-1}, (p-1)p^{t-r-1})$ satisfies b), it belongs to B. Then the last term of the above relation belongs to $B\mathscr{S}^*=B$. Therefore the relation shows that an admissible monomial satisfying c) belongs to $B\subset B+C$.

Consequently we have proved $B+C=\mathscr{S}^*$ and then the proposition is established, q.e.d.

§ Exact sequences of left $\mathscr{S}^*$-homomorphisms.

Let

$$c : \mathscr{S}^* \longrightarrow \mathscr{S}^*$$

be the anti-automorphism (conjugation) of [3]. c is determined by the following properties.

$$(1.7)\qquad \begin{aligned} &c(\alpha\beta)=(-1)^{rs}c(\beta)\,c(\alpha), \qquad \alpha\in\mathscr{S}^r,\ \beta\in\mathscr{S}^s,\\ &c(\Delta)+\Delta=0 \ \ \textit{and} \ \ \sum_{i+j=t}\mathscr{P}^i c(\mathscr{P}^j)=0, \qquad t>0. \end{aligned}$$

First we remark that (1.7) implies

$$(1.7)' \qquad c^2=1\,(c^{-1}=c) \ \ \textit{and} \ \ \sum_{i+j=t}c(\mathscr{P}^i)\,\mathscr{P}^j=0, \qquad t>0.$$

Proof. Obviously $c^2(\Delta)=\Delta$ and $c^2(\mathscr{P}^1)=\mathscr{P}^1$. By (1.7),

$$\sum_{i+j=t}(c^2(\mathscr{P}^i)-\mathscr{P}^i)\,c(\mathscr{P}^i)=c(\sum_{i+j=t}\mathscr{P}^j c(\mathscr{P}^i))-\sum_{i+j=t}\mathscr{P}^i c(\mathscr{P}^j)=0.$$

Then the equality $c^2(\mathscr{P}^t)-\mathscr{P}^t=0$ is proved inductively. Since c^2 is a ring homomorphism, it follows that $c^2=1$.

Next the second equality is true for $t=1$. Suppose that it is true for $t<r$. Then

$$\begin{aligned}\sum_{i+j=r}c(\mathscr{P}^i)\,\mathscr{P}^j &= \sum_{i+j=r}c(\mathscr{P}^i)\,\mathscr{P}^j+\sum_{l=1}^{r-1}(\sum_{i+k=r-l}c(\mathscr{P}^i)\,\mathscr{P}^k)\,c(\mathscr{P}^l)\\ &=\sum_{i+k+l=r}c(\mathscr{P}^i)\,\mathscr{P}^k c(\mathscr{P}^l)-c(\mathscr{P}^r)\\ &=\sum_{i=0}^{r-1}c(\mathscr{P}^i)(\sum_{k+l=r-i}\mathscr{P}^k c(\mathscr{P}^l))=0.\end{aligned}$$

Thus the equality $\sum_{i+j=r}c(\mathscr{P}^i)\,\mathscr{P}^j=0$ is proved by the induction, q.e.d.

By (1.3)′ and by (1.7), we have easily

$$(1.8)\quad c(\mathscr{P}^r)=(-1)^r\mathscr{P}^r \ \textit{and} \ c(\mathscr{P}^{p+r})=(-1)^{r+1}\mathscr{P}^p\mathscr{P}^r \ \ \textit{for}\ 0\leqq r<p.$$

Also we have that $c(R(r))=(r+1)\,c(\Delta\mathscr{P}^1)-rc(\mathscr{P}^1\Delta)=(r+1)\mathscr{P}^1\Delta-r\Delta\mathscr{P}^1$. Then we denote that

$$R_r=c(R(r))=(r+1)\,\mathscr{P}^1\Delta-r\Delta\mathscr{P}^1\,.$$

Define two left $\mathscr{S}^*$-homomorphisms

$$R^*\;:\;\mathscr{S}^*\longrightarrow\mathscr{S}^*/\mathscr{S}^*\Delta+\mathscr{S}^*/\mathscr{S}^*\Delta\,,$$
$$'R^*\;:\;\mathscr{S}^*/\mathscr{S}^*\Delta+\mathscr{S}^*/\mathscr{S}^*\Delta\longrightarrow\mathscr{S}^*\,,$$

by the formulas $R^*(\alpha)=(\alpha\Delta\mathscr{P}^1,\,\alpha\mathscr{P}^1)$, $\alpha\in\mathscr{S}^*$ and $'R^*(\alpha,\,\beta)=\alpha\Delta\mathscr{P}^1-\beta\Delta\mathscr{P}^1\Delta$, $\alpha,\,\beta\in\mathscr{S}^*/\mathscr{S}^*\Delta$.

Proposition 1.5. *The following circular sequence is exact.*

$$\begin{array}{c}\mathscr{S}^*\xrightarrow{R^*_{p-1}}\mathscr{S}^*\longrightarrow\cdots\xrightarrow{R^*_2}\mathscr{S}^*\xrightarrow{R^*_1}\mathscr{S}^*\\ \nwarrow{\scriptstyle 'R^*}\qquad\qquad{\scriptstyle R^*}\swarrow\\ \mathscr{S}^*/\mathscr{S}^*\Delta+\mathscr{S}^*/\mathscr{S}^*\Delta\,.\end{array}$$

The group H_Δ of the kernel-images are spanned by the classes of the following elements:

$H_\Delta(\mathscr{S}^*R_r)$: $\Delta c(\mathscr{P}^{pi+p-r})$, $\Delta c(\mathscr{P}^{pi+p-r})\,\Delta$, $(1\leqq r\leqq p-2)$,
$H_\Delta(image\ of\ R^*)$: $\Delta c(\mathscr{P}^{pi+1})$, $\Delta c(\mathscr{P}^{pi+1})\,\Delta$,
$H_\Delta(image\ of\ R^*)$: $(\Delta c(\mathscr{P}^{pi}),\,0)$, $(0,\,\Delta c(\mathscr{P}^{pi}))$,
where $i=0,\,1,\,2,\cdots$.

Proof. The formula $\tilde{c}(\alpha,\,\beta)=(c(\alpha),\,c(\beta))$ defines an anti-automorphism of $\mathscr{S}^*/\mathscr{S}^*\Delta+\mathscr{S}^*/\mathscr{S}^*\Delta$. Then c and $\tilde{c}$ define an anti-isomorphism of the sequence of Proposition 1.1 onto that of this proposition. It follows from Proposition 1.1 that the sequence of this proposition is exact. The kernel-images are the image of those of Proposition 1.1 under c and $\tilde{c}$. c and $\tilde{c}$ induce isomorphisms of H^Δ onto H_Δ. Then the proposition is established, q.e.d.

Similarly, the following proposition is obtained from Proposition 1.2.

Proposition 1.6. *The following two sequences are exact.*

$$\mathscr{S}^*\xrightarrow{(\mathscr{P}^1)^*}\mathscr{S}^*\xrightarrow{(\mathscr{P}^{p-1})^*}\mathscr{S}^*\xrightarrow{(\mathscr{P}^1)^*}\mathscr{S}^*\,,$$

$$\mathscr{S}^*/\mathscr{S}^*R_1\xrightarrow{(\mathscr{P}^1)^*}\mathscr{S}^*/\mathscr{S}^*\Delta\xrightarrow{(\mathscr{P}^{p-1})^*}\mathscr{S}^*/\mathscr{S}^*R_1\xrightarrow{(\mathscr{P}^1)^*}\mathscr{S}^*/\mathscr{S}^*\Delta\,.$$

$H_{\Delta}(\mathcal{S}^*\mathcal{P}^1)=H_{\Delta}(\mathcal{S}^*\mathcal{P}^{p-1})=0$, $H_{\Delta}((\mathcal{S}^*\mathcal{P}^1+\mathcal{S}^*\Delta)/\mathcal{S}^*\Delta)$
$=\{\Delta c(\mathcal{P}^{pi}),\ i=1,2,3,\cdots\}$ *and* $H_{\Delta}((\mathcal{S}^*\mathcal{P}^{p-1}+\mathcal{S}^*R_1)/\mathcal{S}^*R_1)$
$=\{c(\mathcal{P}^{pi-1}),\ i=1,2,3,\cdots\}$.

Put $M_t^*=c(M_t)=\mathcal{S}^*c(\Delta)+\mathcal{S}^*c(\mathcal{P}^1)+\cdots+\mathcal{S}^*c(\mathcal{P}^{p^{t-1}})$.

By Lemma 1.3, $\mathcal{S}^i\subset M_t$ and also $\mathcal{S}^i\subset M_t^*$ for $0<i<p^t$. By (1.7)′, $0=\sum c\mathcal{P}(i)\,\mathcal{P}(p^t-i)\equiv\mathcal{P}(p^t)+c\mathcal{P}(p^t) \bmod M_t$ and $\bmod M_t^*$. Thus we have the followings.

(1.9)
i) $\mathcal{P}(p^t)\equiv -c\mathcal{P}(p^t) \quad \bmod M_t$ *and* $\bmod M_t^*$.
ii) $M_t^*=M_{t-1}^*+\mathcal{S}^*\mathcal{P}^{p^{t-1}}=\mathcal{S}^*\Delta+\mathcal{S}^*\mathcal{P}^1+\cdots+\mathcal{S}^*\mathcal{P}^{p^{t-1}}$.
iii) $(c\mathcal{P}^{p^t})^*=-(\mathcal{P}^{p^t})^*:\mathcal{S}^*\longrightarrow\mathcal{S}^*/M_t^*$.
iv) $\mathcal{P}(2p^t)\equiv c(\mathcal{P}(2p^t)) \quad \bmod M_t$ *and* $\bmod M_t^*$.

The last relation iv) can be verified as follows. By (1.7), $\mathcal{P}(2p^t)+\mathcal{P}(p^t)\,c\mathcal{P}(p^t)+c\mathcal{P}(2p^t)\equiv 0 \bmod M_t$. By (1.3), $\mathcal{P}(p^t)\,\mathcal{P}(p^t)\equiv 2\mathcal{P}(2p^t) \bmod M_t^*$. Then $\mathcal{P}(p^t)\,c\mathcal{P}(p^t)\equiv -c\mathcal{P}(p^t)\,c\mathcal{P}(p^t)$ $\equiv -c(\mathcal{P}(p^t)\,\mathcal{P}(p^t))\equiv -2c\mathcal{P}(2p^t) \bmod M_t$ and the relation iv) follows.

Then operating the anti-automorphism c, it follows from Proposition 1.4 the following proposition.

Proposition 1.7. *The kernel of the homomorphism*

$$(\mathcal{P}^{p^t})^*:\mathcal{S}^*\longrightarrow\mathcal{S}^*/M_t^*$$

is $M_{t-1}^*+\mathcal{S}^*\mathcal{P}^{2p^{t-1}}+\mathcal{S}^*c(2\mathcal{P}^{p^t+p^{t-1}}-\mathcal{P}^{p^t}\mathcal{P}^{p^{t-1}})+\mathcal{S}^*c(\mathcal{P}^{(p-1)p^t})$ *for* $t\geqq 1$.

§ A remark on Steenrod algebra A* mod 2.

It was proved in [4]

Proposition 1.8. (*Negishi*) *Let* $M_t=Sq^1A^*+\cdots+Sq^{2^{t-1}}A^*$, *then the kernel of the homomorphism*

$$(Sq^{2^t})_*:A^*\longrightarrow A^*/M_t$$

is $M_{t-1}+Sq^{2^t}A^*$.

Then by use of the anti-automorphism c, it follows

Proposition 1.9. *Let* $M_t^*=A^*Sq^1+\cdots+A^*Sq^{2^{t-1}}$, *then the kernel of the homomorphism*

$$(Sq^{2^t})^*:A^*\longrightarrow A^*/M_t^*$$

is $M_{t-1}^*+A^*Sq^{2^t}$.

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