On specialization of the Albanese and Picard varieties

By

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Received October 22, 1959
(Communicated by Prof. Akizuki)

This is a continuation of the paper “On specializations of abelian varieties” by G. Shimura and the writer, where we have studied some basic results on our subjects.

In [3] Igusa has proved in the more general form that if a curve $C$ and its specialization $\tilde{C}$ are both non-singular curves, the Jacobian variety $J$ of $C$ is essentially without defect and its specialization $\tilde{J}$ is the Jacobian variety of $\tilde{C}$. As a natural generalization of this theorem it arises a problem whether under some conditions 1) the Albanese and Picard varieties of a $p$-simple variety $V$ are essentially without defect and 2) their specializations are the Albanese and Picard varieties of the specialization $\tilde{V}$ of $V$. Though the main purpose in this paper is to solve these two questions, we cannot give the complete answer to 2). The problem 1) is affirmatively solved in § 2 under the condition that both $V$ and $\tilde{V}$ are projective varieties, non-singular in codimension 1. In § 4 we shall find an affirmative answer of the problem 2) when $V$ is an abelian variety without defect. Besides these results we prove in § 3 that a non-singular specialization of an abelian variety has structure of an abelian variety, the law of composition of which is obtained by the specialization of the law of composition of the former abelian variety. I am communicated by Igusa that Prof. Chow is also having a proof of this theorem.

Throughout the paper we shall freely use the terminologies, the notations and the results in [4].
§ 1. Rational mapping into an abelian variety without defect.

In this paper we shall concern ourselves only with projective varieties, but not with abstract varieties, and we understand, by a variety, a complete variety embedded in a projective space. We shall always denote by \( k \) a field with a discrete valuation \( v = \{0, \mathfrak{p}, \hat{k} \} \) of rank 1, where \( 0, \mathfrak{p} \) and \( \hat{k} \) denote respectively the valuation ring, the maximal ideal of \( 0 \) and the residue field \( 0/\mathfrak{p} \). Any projective variety, defined over \( k \), can be considered having the structure of a \( \mathfrak{p} \)-variety in the natural way and hence we shall always deal with the specialization with respect to \( 0 \) under this structure of \( \mathfrak{p} \)-varieties.

Let \( V \) and \( A \) be respectively a variety and an abelian variety, both defined over \( k \); \( f \) a rational mapping of \( V \) to \( A \), also defined over \( k \). Then we can define a rational mapping \( f^{(n)} : \widetilde{V} \times \cdots \times \widetilde{V} \to A \) by \( f^{(n)}(x_1, \cdots, x_n) = f(x_1) + \cdots + f(x_n) \) for a generic point \( (x_1, \cdots, x_n) \) of \( \widetilde{V} \times \cdots \times \widetilde{V} \) over \( k \). We say that \((f, V) \) generates \( A \), if for a sufficiently large \( n \), \( f^{(n)} \) is surjective to \( A \). If \( V \) is \( \mathfrak{p} \)-simple and \( A \) is an abelian variety without defect for \( \mathfrak{p} \), the specialization \( \tilde{f}^{(n)} \) of \( f^{(n)} \) coincides with the rational mapping \( \tilde{f}^{(n)} : \tilde{V} \times \cdots \times \tilde{V} \to \tilde{A} \), which is obtained, from the specialization \( \tilde{f} \) of \( f \), by the same process as above.

Moreover if we assume that both \( V \) and \( \tilde{V} \) are non-singular, \( f \) is defined everywhere on \( \tilde{V} \) [4, Th 1] and hence the graph \( \tilde{\Gamma}_f \) of \( \tilde{f} \) coincides with the specialization \( \Gamma_f \) of the graph \( \Gamma_f \) of \( f \). This shows that \( \tilde{f} \) is surjective whenever \( f \) is so. Applying these considerations to \( f^{(n)} \) instead of \( f \) we see that \((\tilde{f}, \tilde{V}) \) generates \( \tilde{A} \) whenever \((f, V) \) generates \( A \).

**Proposition 1.** Let \( V \) be a \( \mathfrak{p} \)-simple variety, where both \( V \) and its specialization \( \tilde{V} \) are non-singular in codimension 1; \( A \) an abelian variety without defect for \( \mathfrak{p} \) and \( f \) a rational mapping of \( V \) to \( A \). If \((f, V) \) generates \( A \), \((\tilde{f}, \tilde{V}) \) also generates \( \tilde{A} \). Moreover if both \( V \) and \( \tilde{V} \) are non-singular and \( f \) is surjective, then \( \tilde{f} \) is also surjective.

*(Proof)* We denote, by \( r \) or \( N \), the dimension of \( V \) or of the ambient projective space \( P \) for \( V \). \( L \) (resp. \( L \)) is a generic plane
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of dimension $N-r+1$, over $(k$ resp. over $\bar{k})$ in $P$ (resp. $\bar{P}$); namely $L$ (resp. $\bar{L}$) is defined by a set of linear equations

$$u_0^{(v)}X_0 + u_1^{(v)}X_1 + \cdots + u_N^{(v)}X_N = 0 \quad (v=1, \cdots, r-1)$$

(resp. $\bar{u}_0^{(v)} \bar{X}_0 + \bar{u}_1^{(v)} \bar{X}_1 + \cdots + \bar{u}_N^{(v)} \bar{X}_N = 0 \quad (v=1, \cdots, r-1))$

where $u_i^{(v)}$ (resp. $\bar{u}_i^{(v)}$) $v=1, \cdots, r-1, i=0, 1, \cdots, N$, are independent variables over $k$ (resp. $\bar{k}$). Let $C$ (resp. $\bar{C}$) be a curve which is defined by the intersection of $V$ and $L$ (resp. $\bar{V}$ and $\bar{L}$). $C$ (resp. $\bar{C}$) is a nonsingular curve, and any point of $C$ (resp. $\bar{C}$) is simple on $V$ (resp. $\bar{V}$). Thus since $f$ (resp. $\tilde{f}$) is defined along $C$ (resp. $\bar{C}$), we have the restriction mapping $f_C$ (resp. $\tilde{f}_C$) of $f$ (resp. $\tilde{f}$) on $C$ (resp. $\bar{C}$). As we know [7], $(f, V)$ generates $A$ if and only if $(f_C, C)$ generates $A$. On the other hand the specialization ring $\mathfrak{o}_*=[(u)\to (\bar{u})]$ is a discrete valuation ring and $\bar{C}$ is the specialization of $C$ with respect to $\mathfrak{o}_*$. Since $f$ is defined along $\bar{C}$, we can easily see that $\tilde{f}_C$ is identified with the specialization $\tilde{f}_C$ of $f_C$ with respect $\mathfrak{o}_*$. This shows that if $(f_C, C)$ generates $A$, $(\tilde{f}_C, \bar{C})=(\tilde{f}_\bar{C}, \bar{C})$ generates $\tilde{A}$ because both $C$ and $\bar{C}$ are non-singular.

The fact that $(f_C, \bar{C})$ generates $\tilde{A}$ is equivalent to that $(\tilde{f}, \bar{V})$ generates $\tilde{A}$. This proves the former part of the proposition and the latter part has been already considered. Applying Prop. 1 we shall now give some deformation to the theorem 2 in [4]

**PROPOSITION 2.** Let $A$ be an abelian variety defined over $k$; suppose that the following conditions are satisfied.

D1) There are a prolongation $\{k', o', \psi\}$ of $\{k, o, \psi\}$ and an abelian variety $A^*$ without defect for $\psi'$, defined over $k'$, which is isomorphic to $A$.

D2') There are a $\psi$-simple variety $V$, both $V$ and $\tilde{V}$ nonsingular in codimension 1, and a rational mapping $f$ of $V$ to $A$, both $V$ and $f$ defined over $k$, such that $(f, V)$ generates $A$.

Then there exists an abelian variety $A_1$, defined over $k$, without defect for $\psi$, which is $k$-isomorphic to $A$.

*Proof.* It is sufficient to show that the condition D2) in Th 2 [4] is satisfied. If we replace $V \times \cdots \times V$ and $f^{(v)}$ in places of $V$
and \( f \), the proposition 1 shows that they satisfies D2) for a sufficiently large \( n \).

§ 2. Specialization of the Albanese or Picard varieties.

Igusa has proved in his paper [3]

**Theorem C.** Let \( C \) be a non-singular curve defined over \( k \) such that the specialization \( \tilde{C} \) of \( C \) with respect to \( o \) is also a non-singular curve. Then there exists a model \( J_{\circ} \), defined over \( k \), of the Jacobian variety of \( C \) such that \( J_{\circ} \) is an abelian variety without defect for \( \wp \). In these circumstances the specialization \( J_{\circ} \), with respect to \( o \), of the canonical mapping \( \varphi : C \to J_{\circ} \), is the canonical mapping \( \tilde{C} \to \tilde{J}_{\circ} \).

This may be also derived from Cor. of Th. 3 in [4] because the Chow variety \( C(g) \) of positive \( C \)-cycles on \( C \) of degree \( g \) (= the genus of \( C \)) obviously has a structure of a pre-group \( \wp \)-variety without defect for \( \wp \).

Let \( V \) be a variety, non-singular in codimension 1 and defined over \( k \); \( L \) a generic plane over \( k \) such that \( V \cdot L = C \) is a curve. Since \( C \) is a non-singular curve, we have the Jacobian variety \( J \) of \( C \). If \( f \) is a rational mapping of \( V \) to an abelian variety \( A \), we can define a homomorphism \( \lambda(f_c) : J \to A \), which is the linear extension of \( f_c \) where \( f_c \) is the restriction mapping of \( f \) onto \( C \). Then we know that \( \lambda(f_c) \) is surjective to \( A \) if and only if \((f, V)\) generates \( A \).

Now we shall assume that \( V \) is \( \wp \)-simple and that \( \tilde{V} \) is non-singular in codimension 1. Let \( \tilde{C}, \tilde{J} \) be a generic plane section curve of \( \tilde{V} \) over \( \tilde{k} \), the Jacobian variety of \( \tilde{C} \). Then for suitably chosen models (noted by the same letters \( J, \tilde{J} \)) of \( J, \tilde{J} \) we have a specialization

\[
(C, J) \to (\tilde{C}, \tilde{J}) \quad \text{ref} \quad o'
\]

where \( \{k', o', \wp'\} \) is a prolongation of \( \{k, o, \wp\} \). This shows that \( J \) is an abelian variety without defect for \( \wp' \). In these circumstances applying Th. 4 in the previous paper [4] we have,

**Theorem 1.** Let \( V \) be a \( \wp \)-simple variety, defined over \( k \) such
that both \( V \) and \( \tilde{V} \) are non-singular in codimension 1. Let \( f \) be a rational mapping of \( V \) to an abelian variety \( A \), where \( f \) and \( A \) are defined over \( k \). If \((f, V)\) generates \( A \), there is an abelian variety \( A_1 \), defined over \( k \), without defect for \( \wp \) which is \( k \)-isomorphic to \( A \).

(Proof) Under the same notations as above, if the field of definition for \( A_1 \) is out of considerations, our theorem is a consequence of the facts that \( J \) is an abelian variety without defect for \( \wp' \) and that \( A \) is a homomorphic image of \( \lambda(f_\wp) \). [4, Th. 4]. The assertion for the field of definition will be immediately proved by Prop. 2.

Before applying this theorem to the theory of the Albanese and Picard varieties, we shall recall some well-known facts for the Chow varieties of positive 0-cycles on \( V \). Let \( V \) be a variety defined over \( k \). Then the Chow variety \( V(n) \) of positive 0-cycles on \( V \) of degree \( n \) and the symmetrizing mapping \( \sigma : V \times \cdots \times V \to V(n) \) are defined over \( k \). If we denote, by \( \Gamma_\sigma \), the graph of \( \sigma \), then \([\Gamma_\sigma : V(n)]\) is equal to \( n! \). We shall now assume that \( V \) is \( \wp \)-simple. For the specialization \( \tilde{V} \) of \( V \), we can define \( \tilde{V}(n), \sigma, \Gamma_\sigma \) by the corresponding objects to \( V(n), \sigma, \Gamma_\sigma \) for \( V \). From the definitions we know that \( V \times \cdots \times V \) is \( \wp \)-simple, \( \sigma \) is everywhere defined on \( \tilde{V} \times \cdots \times \tilde{V} \) and the specialization of \( \sigma \) is identified with \( \sigma \), namely \( \tilde{\Gamma}_\sigma = \Gamma_\sigma \). From the equality \([\Gamma_\sigma : V(n)] = [\Gamma_\sigma : \tilde{V}(n)] = n! \) it follows that \( \tilde{V}(n) = \tilde{\Gamma}(n) \).

Theorem 2. Let \( V \) be a \( \wp \)-simple variety defined over \( k \) such that both \( V \) and \( \tilde{V} \) are non-singular in codimension 1. Then there exists a model \( A(V)_1 \) (resp. \( P(V)_1 \)) of the Albanese variety (resp. the Picard variety) of \( V \), such that \( A(V)_1 \) (resp. \( P(V)_1 \)) is an abelian variety, defined over \( k \), without defect for \( \wp \).

(Proof) We may assume that the dimension of \( V \) is greater than 1, because the case of curves has been already considered in Th. C. First we consider the case of the Albanese variety of \( V \). If \( V \) contains a \( k \)-rational simple point we have a pair \((A(V), \wp)\) of the Albanese variety \( A(V) \) of \( V \) and the canonical mapping \( \wp : V \to A(V) \), both \( A(V) \) and \( \wp \) are defined over \( k \). Since this shows that the conditions in Th. 1 are satisfied, the proof is completed. In general case we use \( V(n) \) instead of \( V \) such that \( V(n) \) has a \( k \)-rational simple point. The existence of such a Chow variety
$V(n)$ is obvious and $V(n)$ is a $p$-simple variety, whose specialization $\tilde{V}(n)$ is non-singular in codimension 1. The Albanese variety of $V$ is also that of $V(n)$. Thus the general case is reduced to the special case treated above and this completes the proof for the case of the Albanese variety.

For the Picard variety of $V$, since the Picard varieties of $V$ and of the Albanese variety of $V$ are identified, we may assume that $V$ is an abelian variety $A$ without defect for $p$. It is known that there exists a $k$-rational non-degenerate divisor $X$ on $A$. We shall define the homomorphism $\varphi_X: A \to P(A)$ which is defined by

$$\varphi_X(u) = Cl(X_u - X)$$

where $P(A)$, $X_u$ and $Cl$ denote respectively the Picard variety of $A$, the translation of $X$ by the birational correspondence $x \to x + u$ for $x$ on $A$ and the linear equivalence class of a divisor on $A$, which is considered to be a point of $P(A)$. Since $\varphi_X$ is a surjective homomorphism defined over $k$, our assertion for the Picard variety is a consequence of Th. 4 in [4].

**Corollary.** $V$, $\tilde{V}$ being as in Th. 2, we denote, by $q$, $\tilde{q}$, respectively the Albanese variety of $V$, $\tilde{V}$. Then we have $q \ll \tilde{q}$.

**Remark.** $V$ being as in Th. 2 we shall fix a simple point $t$ on $V$. We denote, by $\Omega^*(V)$, the totality of pairs $(B, g)$ where $B$ is an abelian variety and $g$ is a rational mapping of $V$ to $B$ such that $(g, V)$ generates $B$ and $g(t) = 0$. For two members $(B_1, g_1)$ and $(B_2, g_2)$ in $\Omega^*(V)$ we say that $(B_1, g_1)$ is greater than $(B_2, g_2)$ (noted by $(B_1, g_1) \geq (B_2, g_2)$) if there is a homomorphism $\lambda$ of $B_1$ onto $B_2$ such that $\lambda \circ g_1 = g_2$. If both relations $(B_1, g_1) \geq (B_2, g_2)$ and $(B_2, g_2) \geq (B_1, g_1)$ hold, $(B_1, g_1)$ and $(B_2, g_2)$ are called to be equivalent to each other. If we denote, by $\Omega(V)$, the set of equivalence classes in $\Omega^*(V)$, the relation $\geq$ in $\Omega^*(V)$ induces the fixed relation $\geq$ in $\Omega(V)$. A member $(B, g)$ in $\Omega(V)$ is called to be maximal in $\Omega(V)$ if any member $(B_1, g_1)$ greater than $(B, g)$ such that it holds $\dim B = \dim B_1$ coincides with $(B, g)$ itself.

Th. 1 shows that any equivalence class in $\Omega^*(V)$ contains a member $(B, g)$ such that $B$ is an abelian variety without defect for $\nu'$ where $\{k', o', \nu'\}$ is a prolongation of $\{k, o, \nu\}$. If we define $\tilde{\Omega}^*(\tilde{V})$, $\tilde{\Omega}(\tilde{V})$ for $\tilde{V}$ as $\Omega^*(V)$, $\Omega(V)$ for $V$, the specialization $(\tilde{B}, \tilde{g})$ of $(B, g)$ with respect to $\nu'$ is a member of $\tilde{\Omega}^*(\tilde{V})$ and the
equivalence class of \((\bar{B}, \bar{g})\) depends only upon the equivalence class of \((B, g)\) and does not depend upon the choice of models in the class. This means that for any member \((B, g)\) of \(\Omega(V)\) we can define the specialization \((\bar{B}, \bar{g})\) of \((B, g)\) and that \((\bar{B}, \bar{g})\) can be considered to be a member of \(\tilde{\Omega}(\tilde{V})\). Of course the specialization defined as this, preserves the order relation \(\succ\), but the specialization of a maximal member in \(\Omega(V)\) is not always maximal in \(\tilde{\Omega}(\tilde{V})\). A counter example is given in the remark in [4, §6].

§ 3. Non-singular specialization of an abelian variety

The aim in this section is to prove

**Theorem 3.** Let \(A\) be an abelian variety defined over \(k\). If the specialization \(\bar{A}\) of \(A\) with respect to \(\sigma\) is a non-singular variety, \(A\) is an abelian variety without defect for \(\wp\).

*(Proof)* From Th. 1 follows that there exist an abelian variety \(A_1\) defined over \(k\), without defect for \(\wp\) and an isomorphism \(\theta\), of \(A\) onto \(A_1\), defined over \(k\). Th. 1 in [4] shows that \(\theta\) is defined everywhere on \(A\). The proof for the present theorem is reduced to the following proposition.

**Proposition 3.** Let \(V\) and \(V_1\) be two \(\wp\)-simple varieties such that there exists a birational and everywhere biregular mapping \(f\) of \(V\) onto \(V_1\), where \(V, V_1\) and \(f\) are defined over \(k\). If both specializations \(\bar{V}\) and \(\bar{V}_1\) of \(V\) and \(V_1\) with respect to \(\sigma\) are non-singular and \(f\) is defined everywhere on \(\bar{V}\), then \(f^{-1}\) is also defined everywhere on \(\bar{V}_1\).

*(Proof)* Since \(f\) is defined everywhere on \(\bar{V}\), the specialization \(\bar{\Gamma}_f\), with respect to \(\sigma\), of the graph \(\Gamma_f\) of \(f\) is identified with the graph \(\bar{\tilde{\Gamma}}_f\) of the specialization \(\tilde{f}\) of \(f\) with respect to \(\sigma\). From this follows that \(\tilde{f}\) is a birational mapping of \(\bar{V}\) onto \(\bar{V}_1\). Unless \(f^{-1}\) is everywhere defined, there exists a non-empty bunch \(\bar{\mathcal{E}}\) on \(\bar{V}\) composed of all degenerate subvarieties \(\bar{V}^{(i)}\) by \(\tilde{f}\), namely such that \(\text{dim} f(\bar{V}^{(i)}) < \text{dim} \bar{V}^{(i)}\). From van der Waerden's theorem [12] follows that \(\bar{\mathcal{E}}\) is purely \((r-1)\)-dimensional where \(r\) is the dimension of \(V\). For a component \(\bar{V}\) of \(\bar{\mathcal{E}}\), we can take a proper sub-
variety $\bar{Y}$ of $\bar{Y}$ such that $f(\bar{Y})=\bar{f}(\bar{Y}')$ and that $\bar{Y}'$ is defined over an algebraic extension of $\bar{k}$. Let $X$ be such a positive divisor on $V$, rational over $k$, that the support of the specialization $\bar{X}$ of $X$ with respect to $\bar{\sigma}$ does not contain any component of $\bar{\mathcal{Y}}$ but contains $\bar{Y}'$. Then any component of $\bar{X}$ regularly corresponds to an $(r-1)$-dimensional subvariety on $\bar{V}$ by $f$. Let $X_1, \bar{X}_1$ be respectively regularly corresponding divisors to $X, \bar{X}$ by $f, \bar{f}$. Then the equalities $X_i=pr_{V_1}(1_i\times(X\times V_1))$ and $\bar{X}_i=pr_{\bar{V}_1}(\bar{X}\times \bar{V}_1)$ lead us to the specialization $(X, X_1)\rightarrow(\bar{X}, \bar{X}_1)$ ref. $\sigma$.

On the other hand since it holds $X=pr_{V}(V\times X_1)\cdot 1_f$ we have the equality $\bar{X}=pr_{\bar{V}}(\bar{V}\times \bar{X}_1)\cdot \bar{1}_f$. From our construction $pr_{\bar{V}}(\bar{V}\times \bar{X}_1)\cdot \bar{1}_f$ must have the component $\bar{Y}$ and this contradicts our previous assumption.

§ 4. Specialization of the Picard variety of an abelian variety without defect

Though the purpose in this section is to prove that the specialization of the Picard variety of an abelian variety $A$ without defect is essentially identified with the Picard variety of the specialization $\bar{A}$ of $A$, we shall first make some basic considerations for Picard varieties.

Let $V$ be a variety defined over $k$ and non-singular in codimension 1. For convenience we shall fix a generic point $t$ of $V$ over $k$, and put $k'=k(t)$. Let $C$ be a general plane section curve of $V$, containing $t$, which is defined over a purely transcendental extension $k'(u)$ of $k'$. Let $(J, \varphi)$ be a pair of the Jacobian variety of $C$ and the canonical mapping $\varphi : C \rightarrow J$, such that $\varphi(t)=0$, where both $J$ and $\varphi$ are defined over $k'(u)$.

Denote, by $g_a(V)$ (resp. $g_t(V)$), the group of divisors on $V$ which are algebraically (resp. linearly) equivalent to zero. We shall define a group homomorphism $\Phi$ of $g_a(V)$ into $J$ by the formula $\Phi(X)=S[\varphi(X'\cdot C)]$ for $X$ in $g_a(V)$ where $X'$ is a divisor on $V$ linearly equivalent to $X$ such that the intersection $X'\cdot C$ is defined and $S[a]$ for a 0-cycle $a$ on an abelian variety is defined by $S[a]=a_1+\cdots+a_n-b_1-\cdots-b_m$ if $a=(a_1)+\cdots+(a_n)-(b_1)-\cdots-(b_m)$. We know that the kernel of $\Phi$ is $g_t(V)$ and that $\Phi$ induces an isomorphism between the Picard group $g_a(V)/g_t(V)$ and the image
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of $\Phi$ (c.f. [6]). Though the image of $\Phi$ is not the Picard variety of $V$, there exists a purely inseparable homomorphism of the Picard variety of $V$ onto the image of $\Phi$. Let $(u_1), (u_2), \ldots, (u_m)$ be independent generic specialization of $(u)$ over $k'$; $(C_i, J_i, \varphi_i)$ a specialization of $(C, J, \varphi)$ over $(u)\to(u_i)$ (ref. $k'$). Of course $(J_i, \varphi_i)$ is a pair of the Jacobian variety $C_i$ and the canonical mapping $C_i\to J_i$ where $C_i$ is a general plane section curve of $V$ containing $t$, over $k$. If we define the group homomorphism $\Phi_i : g_a(V)\to J_i$ in the same way as in the definition for $\Phi$, it is known that for a sufficiently large $m$ the image of $\Phi_1\times\cdots\times\Phi_m$ is identified with the Picard variety $P(V)$ of $V$ and the homomorphism $\Phi_1\times\cdots\times\Phi_m$ gives the canonical mapping $g_a(V)/g_a(V)\to P(V)$, where $\Phi_1\times\cdots\times\Phi_m$ is defined by $\Phi_1\times\cdots\times\Phi_m(X) = (\Phi_1(X), \ldots, \Phi_m(X))$ in $J_1\times\cdots\times J_m$ for any $X$ in $g_a(V)$. [1]

We shall now assume that $V$ is $p$-simple and that $V$ is also non-singular in codimension 1. If $t$ is a generic point of $\bar{V}$ over $\bar{k}$, the specialization ring $[t\to\bar{l}; \vartheta] = \vartheta'$ is a discrete valuation ring in $k'$, and $\{k, \vartheta', \vartheta'\}$ is a prolongation of $\{k, \vartheta, \vartheta\}$ where $\vartheta'$ is the maximal ideal of $\vartheta'$. Let $(\bar{a})$ be an isolated specialization of $(u)$ with respect to $\vartheta'$. If we denote, by $\vartheta^*, \vartheta'^*$, the specialization ring $[(u)\to(\bar{u}); \vartheta']$ and the maximal ideal of $\vartheta^*$, then $\vartheta^*$ is a discrete valuation ring in $k'(u)$ and $\{k'(u), \vartheta^*, \vartheta'^*\}$ is a prolongation of $\{k', \vartheta', \vartheta\}$. Since the specialization $\bar{C}$ of $C$ with respect to $\vartheta^*$ is a non-singular curve, we can consider that $J$ is an abelian variety without defect for $\vartheta^*$, and that the specialization $(\bar{J}, \bar{\varphi})$ of $(J, \varphi)$ with respect to $\vartheta^*$ is a pair of the Jacobian variety of $\bar{C}$ and the canonical mapping $\bar{C}\to \bar{J}$. Since $\bar{C}$ is a generic plane section curve of $V$ over $\bar{k}$, we can define $\Phi_1 : g_a(\bar{V})\to \bar{J}$ as above. Thus we have

**Proposition 4.** Let $X$ be a divisor on $V$ which is contained in $g_a(V)$; $\bar{X}$ a specialization of $X$ with respect to $\vartheta^*$. Then

i) $\bar{X}$ is contained in $g_a(\bar{V})$

ii) $\Phi(\bar{X})$ is a uniquely determined specialization of $\Phi(X)$ over $X\to \bar{X}$ (ref. $\vartheta^*$).

The assertion i) is a direct consequence of the Principle of Degeneration [13, 2] and the assertion ii) will be proved straightforward if every intersection $X_i\cdot C$, $X_i\cdot \bar{C}$, $\bar{X}_i\cdot C$, $\bar{X}_i\cdot \bar{C}$ is defined
where \( X = X_1 - X_2, \overline{X} = \overline{X}_1 - \overline{X}_2 \) and \((X_1, X_2) \rightarrow (\overline{X}_1, \overline{X}_2)\) (ref. \(\sigma^*\)). The general case is reduced to this case by the following proposition.

**Proposition 5.** \( V, \bar{V}, C, \bar{C} \) being as above, let \( X_1, X_2 \) be positive divisors on \( V \); \( \xi \) a geometric quantity in the algebraic geometry for \( V \). Assume that we have a specialization \((X_1, X_2, \xi) \rightarrow (\overline{X}_1, \overline{X}_2, \overline{\xi})\) (ref. \(\sigma^*\)). Then there exist divisors \( Y_1, Y_2 \) (resp. \( \bar{Y}_1, \bar{Y}_2 \)) on \( V \) (resp. \( \bar{V} \)) such that \( X_1, X_2 \) (resp. \( \overline{X}_1, \overline{X}_2 \)) are respectively linearly equivalent to \( Y_1, Y_2 \) (resp. \( \bar{Y}_1, \bar{Y}_2 \)).

The intersection \( Y_1 \cdot C \) and \( Y_2 \cdot C \) (resp. \( \bar{Y}_1 \cdot \bar{C}, \bar{Y}_2 \cdot \bar{C} \)) are defined and that it holds \((X_1, X_2, \xi, X_1, Y_2) \rightarrow (\overline{X}_1, \overline{X}_2, \overline{\xi}, \bar{Y}_1, \bar{Y}_2)\) (ref. \(\sigma^*\)).

Since this is a translation of Lemma 1 of Matsusaka [8] to our case and is proved quite similarly, we may omit the proof. In order to apply Prop. 5 to the proof for Prop. 4, we must replace \( \Phi(X) \) in place of \( \Phi(X) \).

Let \( ((u_1), (u_2), \ldots, (u_m)) \) be an isolated specialization of \((u_1), (u_2), \ldots, (u_m))\) with respect to \( \sigma' \); \( v_1, v_2, \ldots, v_m \), respectively the specialization ring \([((u_1), (u_2), \ldots, (u_m)) \rightarrow ((u_1), (u_2), \ldots, (u_m)); \sigma']\), the maximal ideal of \( \sigma '. \) Then \( \sigma' \) is a discrete valuation ring in \( k(\sigma') = k((u_1), \ldots, (u_m)) \). If we denote, by \( \tilde{C}_i, \tilde{J}_i, \tilde{\varphi}_i \), the specialization of \((C_i, J_i, \varphi_i)\) with respect to \( \sigma' \), \( \tilde{J}_i, \tilde{\varphi}_i \) are, respectively, the Jacobian variety of \( \tilde{C}_i \), the canonical mapping \( \tilde{C}_i \rightarrow \tilde{J}_i \). We define \( \tilde{\Phi}_i = \tilde{g}_a(\tilde{V}) \rightarrow \tilde{J}_i \) as above and fix a positive integer \( m \) such that the image of \( \tilde{\Phi}_i \times \cdots \times \tilde{\Phi}_m \) is identified with the Picard variety \( \tilde{P}(\bar{V}) \) of \( \bar{V} \). If \( X \) is a divisor in \( g_a(V) \) and we have a specialization \( X \rightarrow \bar{X} \) (ref. \( \sigma' \)), from Prop. 5 it follows that \( \tilde{\Phi}_1 \times \cdots \times \tilde{\Phi}_m(\bar{X}) \) is a uniquely determined specialization of \( \tilde{\Phi}_1 \times \cdots \times \tilde{\Phi}_m(X) \) over \( X \rightarrow \bar{X} \) ref. \( \sigma' \). This shows that the support \( \tilde{B} \) of the specialization \( \tilde{P}(V) \) of \( P(V) \) with respect to \( \sigma' \) is contained in \( \tilde{P}(\tilde{V}) \). Since \( P(V) \) is an abelian subvariety of an abelian variety \( J_1 \times \cdots \times J_m \) without defect for \( \varphi_i \), from Prop. 8 in [4] follows that \( \tilde{P}(\bar{V}) = \bar{p}^*\tilde{B} \) where \( p \) is the characteristic of \( k \) and \( e \) is a non-negative integer. Moreover if we assume that the dimensions of the Picard varieties of \( V \) and \( \bar{V} \) are equal to each other, we have the equality \( \tilde{P}(\bar{V}) = \bar{p}^*\tilde{P}(\bar{V}) \) for the just defined models \( P(V) \) and \( \tilde{P}(\tilde{V}) \) of the Picard varieties of \( V \) and \( \bar{V} \). Our purpose is to prove that \( e = 0 \)
if $V$ is an abelian variety without defect.

**Theorem 4.** Let $A$ be an abelian variety defined over $k$ without defect for $\wp$. Then,

i) there exists a model $P(A)$, of the Picard variety of $A$, which is an abelian variety, defined over $k$, without defect for $\wp$.

ii) the specialization $\widetilde{P(A)}$ of $P(A)$ with respect to $\wp$ is the Picard variety of the specialization $\widetilde{A}$ of $A$ with respect to $A$.

iii) Moreover if $\Phi: g_\omega(A) \to P(A)$ is the canonical mapping, we can choose the canonical mapping $\widetilde{\Phi}: g_\omega(\widetilde{A}) \to \widetilde{P(A)}$ as follows: for any $X$ in $g_\omega(A)$, $\widetilde{\Phi}(X)$ is the uniquely determined specialization of $\Phi(X)$ over $X \to \widetilde{X}$.

(P) The notations being as in the above discussions except using $A, \widetilde{A}$ instead of $V, \widetilde{V}$. Especially, $P(A)$ is an abelian subvariety, of the product variety of several Jacobian varieties, which is not defined over $k$, and $\Phi$ is the homomorphism $g_\omega(A) \to P(A)$ in Prop. 4. If we take a non-degenerate positive divisor $X$, rational over $k$, on $A$, the specialization $\widetilde{X}$ of $X$ with respect to $\wp$ is non-degenerate on $\widetilde{A}$. Let $u, \tilde{u}$ be respectively generic points of $A, \widetilde{A}$ over $k_1, \tilde{k}_1$, where $k_1 = k(t, (u_1), \ldots, (u_m)) = \wp_1$. As usual the homomorphism $\varphi_X$ (resp. $\tilde{\varphi}_X$) of $A$ (resp. $\widetilde{A}$) onto $P(A)$ (resp. $\widetilde{P(A)}$) is defined by $\varphi_X(u) = \Phi(X_u - X)$ (resp. $\tilde{\varphi}_X = \tilde{\Phi}(\tilde{X}_u - \tilde{X})$) where $X_u$ (resp. $\tilde{X}_u$) is the translation of $X$ (resp. $\tilde{X}$) by the birational correspondence: $x \to x + u$ for $x$ in $A$ (resp. $\tilde{x} \to \tilde{x} + \tilde{u}$ for $\tilde{x}$ in $\widetilde{A}$). Then $\varphi_X$ is defined over $k_1$ and the specialization of $\varphi_X$ with respect to $\wp_1$ coincides with $\tilde{\varphi}_X$. On the other hand from the Nishi’s theorem [9] follows that $l(X) = \tilde{l}(\widetilde{X})$ and from the Frobenius’ theorem which was recently proved in the abstract case by Nishi [10] we know that $\nu(\varphi_X) = \nu(\tilde{X}) = \tilde{\nu}(\tilde{\varphi}_X)$. Thus by considering the projections of the graphs of $\varphi_X$ and $\tilde{\varphi}_X$ to the value-varieties we have the specialization $\nu(\varphi_X)P(A) \to \nu(\tilde{\varphi}_X)\widetilde{P(A)}$ (ref. $\omega_1$). This shows that $\widetilde{P(A)}$ is the specialization of $P(A)$ with respect to $\wp$. Our theorem itself follows from this and the fact that the specialization of an abelian variety does not depend upon the choice of models as far as they are without defect.

**Corollary** Let $A$ and $B$ be two abelian varieties defined over $k$ and without defect for $\wp$; $\lambda$ a homomorphism of $A$ to $B$ defined
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over $k$ and $P(A)$, $P(B)$ the Picard varieties of $A$, $B$ which are both defined over $k$ and without defect for $\wp$. If we denote, by $\lambda$ and $\tilde{\lambda}$, the transpose $P(B)\rightarrow P(A)$ of $\lambda$ and the transpose $\tilde{\lambda}: \overline{P(B)} \rightarrow \overline{P(A)}$ of the specialization $\tilde{\lambda}$ of $\lambda$ with respect to $\wp$, then the specialization $\tilde{\lambda}$ of $\lambda$ with respect to $\wp$ is identified with $\tilde{\lambda}$.

Proof. If we take a non-degenerate positive divisor $X$ on $B$, we have two commutative diagrams

$$
\begin{array}{cc}
A & \rightarrow & B \\
\downarrow \phi_{\lambda^{-1}(X)} & & \downarrow \phi_X \\
P(A) & \leftarrow & P(B)
\end{array}
\quad
\begin{array}{cc}
\tilde{A} & \rightarrow & \tilde{B} \\
\downarrow \tilde{\phi}_{\lambda^{-1}(X)} & & \downarrow \tilde{\phi}_X \\
\overline{P(A)} & \leftarrow & \overline{P(B)}
\end{array}
$$

Since $\lambda^{-1}(X) = \tilde{\lambda}^{-1}(\tilde{X})$, this shows $\tilde{\lambda} = \tilde{\lambda}$.

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BIBLIOGRAPHY