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On the theory of regular functions in banach algebras

By

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Introduction. The present paper is concerned with the general problem of extending the classical theory of regular functions of a complex variable. This problem was discussed by many authors from various directions. Our approach differs from most of the others in two main respects, namely, in the type of domain and range of the functions and in the definition of regularity. We deal with functions which have for their domains and ranges subsets of a commutative Banach algebra with unit element and we use a definition of regularity introduced by E. R. Lorch [1]. It is known [4] that a regular function by this definition is differentiable in the Fréchet sense but not every Fréchet-differentiable function on a commutative Banach algebra is regular in the Lorch sense. Accordingly, the Lorch theory is the richer.

For the most part, the development of the Lorch theory goes parallel with that of the classical theory. As one would expect, the Cauchy integral theorem and formula occupy a central position and yield the Taylor expansion. Our purpose is to discuss his theory in detail and to get more precise consequences. Our investigations contain some results which were not studied by Lorch. For example we have discussed the functions $\log z$ and $\sqrt[m]{z}$ from the view point of analytic continuations. The logarithmic function was also introduced by Lorch, but his definition seems to be artificial.

The main results of this paper is the theory of analytic continuations in which Theorem 3.1 employ an essential role. And the proof of this theorem is not simple by the reason that the Jordan curve theorem does not hold in Banach algebras and consequently we can not make use of the Rouché theorem.

Throughout the present paper, the symbol $\mathfrak{B}$ designates a commutative Banach algebra having a nuit element e.

§ 1. Definitions and elementary theorems.

DEFINITION 1.1 (Lorch). A function $f(z)$ whose domain D (open) and range R are in $\mathfrak{B}$ is said to be differentiable at $z=z_0\in D$, if there exists an element $\alpha\in\mathfrak{B}$ which satisfies the following relation

$$||f(z_0+h)-f(z_0)-\alpha h|| = \circ(||h||)$$

where $\circ$ is the Landau notation. If $f(z)$ is differentiable at $z=z_0$, then it is easy to prove that an element α which satisfies the above relation is unique. Hence we denote this α by $f'(z_0)$ and call it the derivative of $f(z)$ at $z=z_0$ or we say that $f(z)$ has derivative $f'(z_0)$ at $z=z_0$. If $f(z)$ is differentiable at every point in D, then it is called to be regular in D. For any regular function $f(z)$ defined on D, $f'(z)$ is also a function defined on D and is called the derivative of $f(z)$. If $f'(z)$ is regular in D, we define $f''(z)=(f'(z))'$. Similarly, we can define the derivative of i-th order of $f(z)$ by $f^{(i)}(z)=(f^{(i-1)}(z))'$ if $f^{(i-1)}(z)$ is regular in D.

THEOREM 1.1. *Let* $p(z)=\sum_{n=0}^{\infty}a_n(z-a)^n$ *(a and* a_i'*s are elements in* $\mathfrak{B}$*) be a power series. Let* $\mu=\limsup_{n\to\infty}||a_n||^{1/n}$, *and put*

$$(1.1)\qquad \rho=\begin{cases}1/\mu & \text{when } \mu\neq 0\\ \infty & \text{when } \mu=0\\ 0 & \text{when } \mu=\infty.\end{cases}$$

Then $p(z)$ *defines a regular function in the domain* $D=\{z;\ ||z-a||<\rho\}$. *And we have*

$$(1.2)\qquad p'(z)=\sum_{n=1}^{\infty}na_n(z-a)^{n-1}\qquad \text{for every } z\in D$$

more generally,

$$(1.3)\qquad p^{(i)}(z)=\sum_{n=i}^{\infty}n(n-1)\cdots(n-i+1)a_n(z-a)^{n-i}\quad \text{for every } z\in D.$$

Hence we have

$$(1.4)\qquad p^{(n)}(a)=n!a_n,\quad n=1,2,\cdots.$$

The proof is similar to that of the usual power series and is omitted.

REMARK. For any $r>\rho$, there exists an element z_0 such that $||z_0-a||=r$ and the series $\sum_{n=1}^{\infty} a_n(z_0-a)^n$ is divergent.

DEFINITION 1.2. $a_0+a_1(z-a)+\cdots+a_n(z-a)^n+\cdots$ (a and a_i's are elements in $\mathfrak{B}$) be power series. And let ρ be as in Theorem 1.1. We call ρ the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$.

DEFINITION 1.3. Let Γ be a rectifiable curve in $\mathfrak{B}$ which is parametrized by an equation $z=z(t)$, $0\leqq t\leqq 1$. (The concepts "curve", "rectifiable" and "length" are defined in the usual way.) Let $f(z)$ be a continuous function defined on Γ and having its values in $\mathfrak{B}$. We then define

$$(1.5)\qquad \int_\Gamma f(z)dz = \lim_{n\to\infty}\sum_{i=1}^{n} f(z(t_i'))[z(t_i)-z(t_{i-1})]$$

where $0=t_0<t_1<\cdots<t_n=1$, $t_{i-1}\leqq t_i'\leqq t_i$, $i=1, 2, \cdots$ and $\max_i (t_i-t_{i-1})\to 0$. (The existence of the integral is proved in the usual manner.) The following inequality is evident

$$(1.6)\qquad ||\int_\Gamma f(z)dz|| \leqq \max_{z\in\Gamma} ||f(z)||\cdot l(\Gamma)$$

where $l(\Gamma)$ is the length of Γ.

THEOREM 1.2. *Let $\varphi(z)$ be a continuous function defined on a rectifiable curve $\Gamma\subset\mathfrak{B}$ and having its values in $\mathfrak{B}$. Let D be the set of elements z such that $\xi-z$ has an inverse in $\mathfrak{B}$ for every $\xi\in\Gamma$. We define*

$$(1.7)\qquad f(z) = \int_\Gamma \frac{\varphi(\xi)}{\xi-z}d\xi \qquad \textit{for } z\in D.$$

Let a be an element of D. If we set $1/r=\mathrm{Min}_{\xi\in\Gamma} ||(\xi-a)^{-1}||$, then the set $\{z\,;\,||z-a||<r\}$ is contained in D and we have

$$(1.8)\qquad f(z) = \sum_{n=0}^{\infty}(z-a)^n\int_\Gamma \frac{\varphi(\xi)}{(\xi-a)^{n+1}}d\xi \qquad \textit{for } ||z-a||<r.$$

Hence D is an open subset of $\mathfrak{B}$ and $f(z)$ is regular there. Moreover we have

$$(1.9)\qquad f^{(n)}(a) = n!\int_\Gamma \frac{\varphi(\xi)}{(\xi-a)^{n+1}}d\xi\,, \qquad n=0, 1, 2, \cdots.$$

It is easy to see that $\xi-\lambda e$ ($\xi\in\Gamma$) has an inverse if $|\lambda|>\underset{\xi\in\Gamma}{\mathrm{Max}}\,||\xi||$. Hence D is not empty. The theorem is proved similarly as in classical function theory, and the details are omitted.

THEOREM 1.3 (*Cauchy*). *Let D be a convex open subset of $\mathfrak{B}$. If $f(z)$ is regular in D and Γ is an arbitrary rectifiable closed curve in D, then $\int_\Gamma f(z)dz=0$.*

This is proved in [1].

COROLLARY. *Let Γ_0 and Γ_1 be two rectifiable closed curves in an open subset $D\subset\mathfrak{B}$. If $f(z)$ is regular in D and Γ_0 is homotopic* to Γ_1 in D, then*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz\,.$$

For a proof of this result, see my note [9].

§ 2. The integral formula and expansion theorem.

In the sequel we shall use the following new notations.

NOTATION 2.1. Let D be an open subset of $\mathfrak{B}$ nad a an element of D. The symbol $\rho(a\,;D)$ designates the radius of the maximal open sphere which is contained in D and having its center at a.

NOTATION 2.2. Let $p(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ be a power series. The symbol $\rho(p(z))$ designates the radius of convergence of the power series $p(z)$.

LEMMA 2.1. *Let $f(z)$ be a regular function defined on an open subset $D\subset\mathfrak{B}$ and a an element of D. Let Γ be a closed curve defined by an equation $\xi(t)=a+r\exp(2\pi it)e$, $0\leq t\leq 1$, where $0<r<\rho(a\,;D)$ and e is the unit element of $\mathfrak{B}$. Then we have*

$$f(a)=\frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi-a}d\xi\,. \tag{2.1}$$

PROOF. It is easily seen that

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* Let Γ_0 and Γ_1 be parametrized by $z=z_0(t)$ and $z=z_1(t)$, $0\leq t\leq 1$, respectively. Γ_0 is called to be homotopic to Γ_1 in D, if there exists a continuous function $\varphi(s,t)$ $(0\leq s\leq 1,\ 0\leq t\leq 1)$ of two variables s and t such that $\varphi(0,t)\equiv z_0(t)$, $\varphi(1,t)\equiv z_1(t)$, $\varphi(s,0)=\varphi(s,1)$ for every $0\leq s\leq 1$ and $\varphi(s,t)$ is in D for every (s,t) in the unit square.

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\xi)}{\xi-a}d\xi = \frac{1}{2\pi i}\int_{0}^{1}\frac{f(a+r\exp(2\pi it)e)}{r\exp(2\pi it)e}\cdot 2\pi i\cdot r\exp(2\pi it)e\ dt$$
$$= \int_{0}^{1} f(a+r\exp(2\pi it)e)dt\,.$$

Let D_0 be the set of elements z such that $z\in D$ and $z-a$ has an inverse in $\mathfrak{B}$. Then D_0 is evidently an open subset of D. Let $\Gamma_\varepsilon(0<\varepsilon<r)$ be a curve which is defined by an equation $\xi(t)=a+\varepsilon\exp(2\pi it)e$, $0\leqq t\leqq 1$. We can easily see that Γ is homotopic to Γ_ε in D_0 and the above function $f(\xi)/(\xi-a)$ is regular in D_0. Hence by the corollary to Theorem 1.3 we have

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\xi)}{\xi-a}d\xi = \frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{f(\xi)}{\xi-a}d\xi = \int_{0}^{1} f(a+\varepsilon\exp(2\pi it)e)dt\,.$$

Letting $\varepsilon\to 0$, we obtain (2.1).

LEMMA 2.2. *Let $f(z)$ be a regular function defined on an open subset $D\subset\mathfrak{B}$ and a an element of D. Let Γ be as in the above lemma. Then we have*

$$(2.2)\qquad f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\xi)}{\xi-z}d\xi \qquad \textit{for } \|z-a\|<r/2\,.$$

PROOF. Let Γ_0 and Γ_1 be two curves defined by the equations $\xi(t)=z+\frac{r}{2}\exp(2\pi it)e(\|z-a\|<r/2)$ and $\xi(t)=a+\frac{r}{2}\exp(2\pi it)e$, $0\leqq t\leqq 1$, respectively. According to Lemma 2.1, $f(z)=\frac{1}{2\pi i}\int_{\Gamma_0}\frac{f(\xi)}{\xi-z}d\xi$. Let D^* be the set of elements ξ such that $\xi\in D$ and $\xi-z$ has an inverse. It is evident D^* is an open subset of D. Setting $\varphi(s,t)=a+s(z-a)+\frac{r}{2}\exp(2\pi it)e$, we can easily see that Γ_0 is homotopic to Γ_1 in D^*. Hence by the Corollary to Theorem 1.3 we have

$$(2.3)\qquad f(z) = \frac{1}{2\pi i}\int_{\Gamma_0}\frac{f(\xi)}{\xi-z}d\xi = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\xi)}{\xi-z}d\xi\,.$$

Since Γ_1 is clearly homotopic to Γ in D^*, we have also

$$(2.4)\qquad \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\xi)}{\xi-z}d\xi = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\xi)}{\xi-z}d\xi\,.$$

Form (2.3) and (2.4) we obtain (2.2).

LEMMA 2.3 (*Expansion theorem in the weak sense*). *Let $f(z)$ be* *a regular function defined on an open subset $D\subset\mathfrak{B}$. Then $f(z)$ has derivatives of all orders at every point in D. Let a be an arbitrary element in D. Then we have*

$$(2.5)\qquad f(z)=f(a)+f'(a)(z-a)+\cdots+\frac{f^{(n)}(a)}{n!}(z-a)^n+\cdots$$
$$\textit{for } \|z-a\|<\rho(a;D)/2.$$

PROOF. Let z be an arbitrary element in the set $\{z;\|z-a\|<\rho(a;D)/2\}$. Then there exists a number r such that $r<\rho(a;D)$ and $\|z-a\|<r/2$. Let Γ be a closed curve defined by an equation $\xi(t)=a+r\exp(2\pi it)e$, $0\leqq t\leqq 1$. Then by Lemma 2.2 we have

$$f(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(\xi)}{\xi-z}d\xi\qquad\text{for } \|z-a\|<r/2.$$

Hence by Theorem 1.2 we obtain (2.5).

LEMMA 2.4 (*Theorem of identity*). *Let $f(z)$ and $g(z)$ be two regular functions defined on a connected open subset $D\subset\mathfrak{B}$. If $f(z)\equiv g(z)$ on some sphere $S\subset D$, then $f(z)$ coincides with $g(z)$ in D.*

PROOF. By using Lemma 2.3 this can be proved quite similary as in classical fuction theorey.

THEOREM 2.1 (*Integral formula*). *Let $f(z)$ be a regular function defined on an open subset $D\subset\mathfrak{B}$ and a an element in D. Let r be an arbitrary number such that $0<r<\rho(a;D)$ and Γ be a curve defined by an equation $\xi(t)=a+r\exp(2\pi it)e$, $0\leqq t\leqq 1$. Then we have*

$$(2.6)\qquad f(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(\xi)}{\xi-z}d\xi\qquad\textit{for } \|z-a\|<r.$$

PROOF. Let D^* be the set of elements z such that $z\in D$ and $\xi-z$ has an inverse for every $\xi\in\Gamma$. It is evident that D^* is an open subset of D and contains the sphere $\{z;\|z-a\|<r\}$. If we define

$$(2.7)\qquad g(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(\xi)}{\xi-z}d\xi\qquad\text{for } z\in\{z;\|z-a\|<r\}\subset D^*,$$

then $g(z)$ is regular in $\{z;\|z-a\|<r\}$. On the other hand by Lemma 2.2 we have

$$(2.8)\qquad f(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(\xi)}{\xi-z}d\xi\qquad\text{for } \|z-a\|<r/2.$$

From (2.7) and (2.8) we see that $f(z)\equiv g(z)$ on the sphere $\{z;\|z-a\|<r/2\}$. Since the sphere $\{z;\|z-a\|<r\}$ is connected, we obtain (2.6) from Lemma 2.4. Our theorem is thereby proved.

THEOREM 2.2 (*Taylor's expansion*). *Let $f(z)$ be a regular function defined on an open subset $D\subset\mathfrak{B}$ and a an element of D. Then we have*

$$f(z)=f(a)+f'(a)+\cdots+\frac{f^{(n)}(a)}{n!}(z-a)^n+\cdots \quad \text{for } \|z-a\|<\rho(a\,;D)\,. \tag{2.9}$$

PROOF. Let z be an element in $\{z\,;\|z-a\|<\rho(a\,;D)\}$. Then there exists a number r such that $\|z-a\|<r<\rho(a\,;D)$. Let Γ be a curve defined by an equation $\xi(t)=a+r\exp(2\pi it)e$, $0\leqq t\leqq 1$. By the above theorem we have

$$f(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(\xi)}{\xi-z}d\xi\,.$$

From this and Theorem 1.2 we obtain (2.9).

THEOREM 2.3 (*Cauchy's inequalities*). *Let $f(z)$ be a regular function defined on an open svbset $D\subset\mathfrak{B}$ and a an element of D. Suppose that $\|f(z)\|\leqq M$ for every $z\in D$. Then we have*

$$\|f^{(n)}(a)\|\leqq n!\,M/\rho(a\,;D)^n\,,\qquad n=1,2,\cdots. \tag{2.10}$$

PROOF. This is proved by Theorem 2.1 as in the usual manner.

COROLLARY (*Liouville*). *If $f(z)$ is a regular function defined on the whole space $\mathfrak{B}$ and such that $\|f(z)\|\leqq M$ for every $z\in\mathfrak{B}$, then $f(z)$ is a constant.*

§ 3. Inverse functions.

LEMMA 3.1. *Let $f(z)$ be a regular function defined on an open subset $D\subset\mathfrak{B}$ and a an element of D. If $f'(a)$ has an inverse, then $f(D)$ contains a neighborhood of $f(a)$.*

PROOF. By Taylor's expansion theorem we have

$$f(z)=a_0+a_1(z-a)+\cdots+a_n(z-a)^n+\cdots\qquad\text{for}\quad\|z-a\|<\rho(a\,;D)\,.$$

Since $f'(z)$ is continuous on D and $f'(z)$ has an inverse at $z=a$, $f'(z)$ has an inverse on a suitable neighborhood of a. Hence without loss of generality we may assume that

$$\|f(z)\|\leqq M,\ \ \|f'(z)^{-1}\|\leqq M\qquad\text{for}\quad\|z-a\|\leqq R \tag{3.1}$$

where R is a sufficient small number such that $0<R<\rho(a\,;D)$. We choose a number r such that $r<R/2$ and let ε be a positive number satisfying the following conditions

(3. 2) $$\varepsilon M/r<\frac{1}{2}, \quad 2M(M/r)^2\varepsilon<\frac{1}{2}.$$

We shall show that for every b with $\|b-f(a)\|<\varepsilon$ there exists an element z_0 such that

(3. 3) $$f(z_0)=b, \quad \|z_0-a\|<r.$$

First we set $z_1=a+(b-f(a))f'(a)^{-1}$. We shall estimate the value of $f(z)$ at $z=z_1$.

$$f(z_1)=a_0+a_1(b-f(a))f'(a)^{-1}+\cdots+a_n[(b-f(a))f'(a)^{-1}]^n+\cdots.$$

From $a_0=f(a)$ and $a_1=f'(a)$ we have

(3. 4) $$f(z_1)-b=a_2[(b-f(a))f'(a)^{-1}]^2+\cdots + a_n[(b-f(a))f'(a)^{-1}]^n+\cdots.$$

and hence

$$\|f(z_1)-b\|\leqq\|a_2\|\cdot\|b-f(a)\|^2\cdot\|f'(a)^{-1}\|^2+\cdots + \|a_n\|\cdot\|b-f(a)\|^n\cdot\|f'(a)^{-1}\|^n+\cdots$$

(using the relations $\|a_n\|\leqq M/r^n$, $n=1,2,\cdots$ (see (2. 10))

$$\leqq\frac{M}{r^2}\varepsilon^2M^2+\cdots+\frac{M}{r^n}\varepsilon^nM^n+\cdots=\frac{M}{r^2}\varepsilon^2M^2\sum_{n=0}^{\infty}\left(\frac{\varepsilon M}{r}\right)^n$$

(using (3. 2))

$$<2M\varepsilon^2\frac{M^2}{r^2}=2M\left(\frac{M}{r}\right)^2\varepsilon^2<\varepsilon/2 \quad \text{(see (3. 2))}.$$

On the other hand

(3. 5) $$\|a-z_1\|=\|(b-f(a))f'(a)^{-1}\|\leqq\varepsilon M<r/2 \quad \text{(see (3. 2))}.$$

The function $f(z)$ is also expanded in the following form

(3. 6) $$f(z)=b_0+b_1(z-z_1)+\cdots+b_n(z-z_1)^n+\cdots \quad \text{for } \|z-z_1\|<r.$$

The sphere $\{z\,;\|z-z_1\|<r\}$ is clearly contained in the sphere $\{z\,;\|z-a\|<R\}$. Hence we have

(3. 7) $$\|f'(z_1)^{-1}\|\leqq M, \quad \|b_n\|\leqq M/r^n \quad \text{(see (3. 1), Theorem 2. 3))}.$$

We set $z_0=z_1+(b-f(z_1))f'(z_1)^{-1}$ and estimate the value of $f(z)$ at $z=z_2$.

$$f(z_2)=b_0+b_1(b-f(z_1)f'(z_1)^{-1}+\cdots+b_n[(b-f(z_1))f'(z_1)^{-1}]^n+\cdots.$$

By using the relations $b_0=f(z_1)$, $b_1=f'(z_1)$ and $\|b-f(z_1)\|<\varepsilon/2$ (see (3. 4)), we have similarly

$$\|f(z_2)-b\|<\varepsilon/4\,, \quad \|z_1-z_2\|<r/4\,. \tag{3.8}$$

The function $f(z)$ is also written in the following form

$$f(z) = c_0+c_1(z-z_2)+ \cdots +c_n(z-z_2)^n+ \cdots .$$

If we set $z_3=z_2+(b-f(z_2))f'(z_2)^{-1}$, then it is also proved that

$$\|f(z_3)-b\|<\varepsilon/8\,, \quad \|z_2-z_3\|<r/8\,. \tag{3.9}$$

Repeating this process indefinitely we have a sequence $z_1, z_2, \cdots, z_n, \cdots$ of elements of $\mathfrak{B}$ such that

$$\|f(z_n)-b\|<\varepsilon/2^n\,, \quad \|z_{n-1}-z_n\|<r/2^n\,, \qquad n = 1, 2, \cdots . \tag{3.10}$$

The sequence $z_1, z_2, \cdots, z_n, \cdots$ is clearly a Cauchy sequence and hence it converges to an element z_0. It is evident that $f(z_0)=b$ and

$$\|z_0-a\| \leqq \|a-z_1\|+\|z_1-z_2\|+ \cdots +\|z_{n-1}-z_n\|+ \cdots \quad <r\,. \tag{3.11}$$

Our lemma is thereby completely proved.

LEMMA 3.2. *Under the same assumptions in the preceding lemma, there exists a neighborhood U of a such that $f(z)$ is one-to-one on U.*

PROOF. By Taylor's expansion theorem $f(z)$ is expanded in the following form

$$f(z) = a_0+a_1(z-a)+ \cdots +a_n(z-a)^n+ \cdots .$$

for any element z in $\{z\,;\|z-a\|<\rho(a\,;D)\}$. Let R be a number such that $0<R<\rho(a\,;D)$. Then there exists a number M such that

$$\|f(z)\| \leqq M \qquad \text{for} \quad \|z-a\| \leqq R\,.$$

By Cauchy's inequalities we have

$$\|a_n\| \leqq M/R^n\,, \qquad n = 0, 1, 2, \cdots . \tag{3.12}$$

We choose a number r such that $r<R/2$. Suppose that $f(a+h)=f(a+g)$ for two different points h and g with $\|h\|<r$ and $\|g\|<r$. Since $f(a+h)=\sum_{n=0}^{\infty} a_nh^n=\sum_{n=0}^{\infty} a_ng^n=f(a+g)$, we have

$$\begin{aligned} a_1(h-g) &= -\{a_2(h^2-g^2)+ \cdots +a_n(h^n-g^n)+ \cdots\} \\ &= -(h-g)\{a_2(h+g)+ \cdots +a_n(h^{n-1}+h^{n-2}g+ \cdots +g^{n-1})+ \cdots\} \end{aligned}$$

Hence

$$||a_1(h-g)|| \leqq ||h-g|| \sum_{n=2}^{\infty} ||a_n|| nr^{n-1} \leqq ||h-g|| \sum_{n=1}^{\infty} n\frac{M}{R^n} r^{n-1}$$
$$= \left(||h-g|| \frac{M}{R}\right) \cdot \sum_{n=2}^{\infty} n\left(\frac{r}{R}\right)^{n-1}$$

(using the relation $1/(1-x)^2 = 1+2x+ \cdots +nx^{n-1}+ \cdots$)

$$=(||h-g|| M/R) \cdot \left\{1/\left(1-\frac{r}{R}\right)^2 - 1\right\} = (||h-g|| M/R) \cdot (r(2R-r)/(R-r)^2)$$

(using $r < R/2$)

$$< (||h-g|| M/R) \cdot (r2R/(R/2)^2) = ||h-g|| 8Mr/R^2 .$$

Consequently we have

$$(3.12) \quad ||a_1(h-g)|| < ||h-g|| 8Mr/R^2 = ||a_1(h-g)a_1^{-1}|| 8Mr/R^2$$
$$\leqq ||a_1(h-g)|| \cdot ||a_1^{-1}|| 8Mr/R^2 .$$

From the above relation we obtain

$$(3.13) \qquad r \geqq R^2/(8M \cdot ||f'(a)^{-1}||) \qquad (a_1 = f'(a))$$

Hence if we set $r_0 = \text{Min}\ \{R/2,\ R^2/(8M \cdot ||f'(a)^{-1}||)$, then it is easily seen that

$$(3.14) \quad ||z_1 - a|| < r_0,\ ||z_2 - a|| < r_0 \text{ and } z_1 \neq z_2 \text{ imply } f(z_1) \neq f(z_2) .$$

Our lemma is thereby proved.

THEOREM 3.1. *Let $f(z)$ be a regular function defined on an open subset $D \subset \mathfrak{B}$. If $f'(z)$ has an inverse at $z = a \in D$, then there exists an open neighborhood U of a such that $f(U)$ is open and $f(z)$ is one-to-one on U. Hence, to every point $w \in f(U)$ an element $z \in U$ satisfying $f(z) = w$ is uniquely determined, which defines a function $z = \varphi(w)$ whose domain and range are $f(U)$ and U respectively. Then $\varphi(w)$ is regular at $w = f(a)$ and we have*

$$(3.15) \qquad \varphi'(b) = f'(a)^{-1} \qquad \textit{where} \quad b = f(a) .$$

PROOF. The first half of the theorem is evident from the above two lemmas. We shall prove the last half. If we set

$$f(a+h) - f(a) - hf'(a) = \delta(a, h) ,$$

we have clearly $\lim_{||h|| \to 0} ||\delta(a, h)|| / ||h|| = 0$. Let $f(a) = b$ and $f(a_1) = b_1$. Then we have

$$(3.16) \qquad f(a_1) - f(a) - (a_1 - a) f'(a) = \delta(a, a_1 - a) .$$

This implies that

(3.17) $$b_1-b-(a_1-a)f'(a) = \delta(a, a_1-a), \quad \text{that is,}$$
$$\varphi(b_1)-\varphi(b)-(b_1-b)f'(a)^{-1} = -\delta(a, a_1-a)f'(a)^{-1}.$$

In order to prove (3.15) it is sufficient to show that the following equality holds

(3.18) $$\lim_{b_1\to b} ||\delta(a, a_1-a)f'(a)^{-1}|| / ||b_1-b|| = 0.$$

On the other hand from (3.17) we have

(3.19) $$||b_1-b||\cdot||f'(a)^{-1}|| \geqq ||a_1-a|| - ||\delta(a, a_1-a)||\cdot||f'(a)^{-1}||.$$

Since $\lim_{||h||\to 0} ||\delta(a, h)||/||h||=0$, there exists a number $\rho>0$ such that

(3.20) $$||a_1-a|| < \rho \quad \text{implies} \quad ||\delta(a, a_1-a)||\cdot||f'(a)^{-1}|| < ||a_1-a||/2.$$

Hence we have from (3.19) and (3.20)

(3.21) $$2||f'(a)^{-1}|| \geqq ||a_1-a|| / ||b_1-b|| \qquad \text{for} \quad ||a_1-a|| < \rho.$$

Without loss of generality we may assume that $f'(z)$ has an inverse at every point $z \in U$. By lemma 3.1 and 3.2 $f(z)$ is a one-to-one open mapping from U onto $f(U)$. Hence $\varphi(w)$ is continuous on $f(U)$. This shows that $\lim b_1=b$ implies $\lim a_1=a$. Thus we have

$$\lim_{b_1\to b} \frac{||\delta(a, a_1-a)f'(a)^{-1}||}{||b_1-b||} \leqq \lim_{b_1\to b} \frac{||\delta(a, a_1-a)||\cdot||f'(a)^{-1}||}{||a_1-a||} \cdot \frac{||a_1-a||}{||b_1-b||}$$
$$\leqq \lim_{a_1\to a} (||\delta(a, a_1-a)||\cdot||f'(a)^{-1}||/||a_1-a||)\cdot 2||f'(a)^{-1}|| = 0 \quad (\text{see } (3.21)).$$

Our theorem is thereby completely proved.

COROLLARY (*Theorem of Nagumo*). *Let $\mathfrak{G}$ be the set of all regular elements of $\mathfrak{B}$ and $\mathfrak{G}_1$ the component of $\mathfrak{G}$ containing e. In order that an element a belongs to $\mathfrak{G}_1$ it is necessary and sufficient that a is expressible in the following from*

$$a = e+\frac{z}{1!}+\frac{z^2}{2!}+ \cdots +\frac{z^n}{n!}+ \cdots.$$

PROOF. If we define $f(z)=e+\frac{z}{1!}+\frac{z^2}{2!}+ \cdots +\frac{z^n}{n!}+ \cdots$ for every $z \in \mathfrak{B}$, then by Theorem 1.1 $f(z)$ is regular in $\mathfrak{B}$ and $f'(z)=f(z)$. As is easily seen $f'(z)$ $(=f(z))$ has an inverse $f(-z)$, $f(z)$ is an open mapping from $\mathfrak{B}$ into itself (see Lemma 3.1). Let $\mathfrak{G}_1'=f(\mathfrak{B})$. $\mathfrak{G}_1'$ is clearly an open subgroup of $\mathfrak{G}$ and consequently $\mathfrak{G}_1'$ is also a closed subgroup of $\mathfrak{G}$. On the other hand $\mathfrak{G}_1'$ is connected. So we can easily see that $\mathfrak{G}_1'=\mathfrak{B}_1$. The corollary in thereby proved.

DEFINITION 3.1. The function $f(z)=e+\frac{z}{1!}+\frac{z^2}{2!}+\cdots+\frac{z^n}{n!}+\cdots$, $z\in\mathfrak{B}$, is called the exponential function and is denoted by $\exp(z)$.

NOTATION 3.1. In the future the symbol $\mathfrak{G}$ always denotes the set of all regular elements of $\mathfrak{B}$ and $\mathfrak{G}_1$ denotes the component of $\mathfrak{G}$ containing e.

§ 4. Analytic continuations.

DEFINITION 4.1. A power series $p(z)=a_0+a_1z+\cdots+a_n(z-a)^n+\cdots$ is called a function element if its radius of convergence is positive, and a is called the center of the function element $p(z)$.

Another concepts "direct analytic continuation", "analytic continuation" are defined similarly as in classical function theory.

DEFINITION 4.2. Let $p(z)$ be a function element. The totality of function elements which are the analytic continuations of $p(z)$ is called the analytic function defined by $p(z)$.

NOTATION 4.1. The symbol $p_a(z)$ designates a function element having its center at a.

LEMMA 4.1 (*Theorem of invariance of analytic relations*). *Let $F(z, w)$ be a function defined for $z\in\Delta_1$, $w\in\Delta_2$ (where Δ_1 and Δ_2 are connected open domein in $\mathfrak{B}$) and satisfying the following conditions*:

1) *$F(z, w)$ is a continuous function with respect to two variables $z\in\Delta_1$ and $w\in\Delta_2$.*

2) *$F(z, w)$ is a regular function of $z\in\Delta_1$ for any fixed $w\in\Delta_2$ and similarly a regular function of $w\in\Delta_2$ for any fixed $z\in\Delta_1$.*

3) *The functions $F_z(z, w)$ (derivatives with respect to z) and $F_w(z, w)$ (derivatives with respect to w) are continuous functions in two variables $z\in\Delta_1$ and $w\in\Delta_2$ respectively.*

Let $p(z)$ and $q(z)$ be two function elements havihg their centers at the same point a. Let Γ be a curve which starts from the point a and is parametrized by an equation $z=\xi(t)$, $0\leqq t\leqq 1$. Suppose that the function elements $p(z)$ and $q(z)$ can be continued analytically along the same curve Γ by corresponding to every point $\xi(t)\in\Gamma$ function elements $p_{\xi(t)}(z)$, $q_{\xi(t)}(z)$ which have their centers at $\xi(t)\in\Gamma$. Let us further assume that for every point $\xi(t)\in\Gamma$ there exists a positive number $R(t)$ such that if $\|z-\xi(t)\|<R(t)$, $p_{\xi(t)}(z)$ and $q_{\xi(t)}(z)$

are well defined and $p_{\xi(t)}(z)\in\Delta_1$, $q_{\xi(t)}(z)\in\Delta_2$. *If the relation* $F(p(z), q(z))\equiv 0$ *holds on some neighborhood of* a, *then we have* $F(p_{\xi(t)}(z), q_{\xi(t)}(z))\equiv 0$ *on the maximal domain in which both* $p_{\xi(t)}(z)$ *and* $q_{\xi(t)}(z)$ *are well defined simultaneously and* $p_{\xi(t)}(z)\in\Delta_1$, $q_{\xi(t)}(z)\in\Delta_2$.

The classical proof applies, mutatis mutandis, to this case.

LEMMA 4.2 (*Theorem of monodromy*). *Let* D *be an open convex subset of* $\mathfrak{B}$ *and* a *an element of* D. *If a function element* $p_a(z)$ (*see Notation* 4.1) *can be continued analytically along every curve in* D *starting from the point* a, *then there exists a regular function* $f(z)$ *defined on* D *such that* $f(z)\equiv p_a(z)$ *on some neighborhood of* a.

This is proved quite similarly as in classical function theory, and the details are omitted.

Let $f(w)$ be a regular function defined on the whole space $\mathfrak{B}$. Let D be the set of those points w for which $f'(w)$ has an inverse. Then it is easy to see that D is an open subset of $\mathfrak{B}$. Therefore, D is the sum of disjoint maximal open connected sets, $D=\bigcup_{\alpha} D_\alpha$, the components of D. We choose a component D_α and denote it by D^*. To avoid complications, we confine ourselves to D^*.

DEFINITION 4.3. Let $\xi\in D^*$ and $f(\xi)=a$. Since $f'(\xi)$ has an inverse, there exists by Theorem 3.1 a regular function $\varphi(z)$ defined on a neighborhood of a such that $f(\varphi(z))=z$ and $\varphi(a)=\xi$. We have then

$$\varphi(z) = \xi+a_1(z-a)+ \cdots +a_n(z-a)^n+ \cdots . \tag{4.1}$$

on some neighborhood of a. We denote this power series by $p_a(z)$ (see Notation 4.1) and call it the inverse function element of $f(w)$ at $w=\xi$.

It is evident that $z-f(p_a(z))=0$ on some neighborhood of a. Hence by the theorem of identity we have

$$z-f(p_a(z)) = 0 \qquad \text{for} \quad \|z-a\|<\rho(p_a(z)) \tag{4.2}$$

(see Notation 2.2).

The analytic function $\Psi(z)$ defined by the inverse function element of $f(w)$ at an arbitrary point $w=\xi\in D^*$ is called the inverse analytic function of $f(w)$ with respect to D^*. For the validity of this definition we must show that the analytic function $\Psi(z)$ is defined independently upon the selection of inverse function element of $f(w)$. But this is easily proved in the usual manner.

Let $p_b(z)$ (see Notation 4.1) be a function element of the analytic function $\Psi(z)$ having its center at b. Then there exists a curve Γ such that $p_b(z)$ is the analytic continuation of $p_a(z)$ along Γ. If we define $F(z, w)=z-f(w)$, then the function $F(z, w)$ satisfies the conditions 1), 2) and 3) in Lemma 4.1. Since $F(z, p_a(z))=0$ (see (4.2)), by Lemma 4.1, we have

$$F(z, p_b(z)) = 0\,, \qquad \text{that is,} \quad z-f(p_b(z)) = 0\,. \tag{4.3}$$

Letting $p_b(b)=\eta$, we can easily prove that $\eta \in D^*$ and $p_b(z)$ is an inverse function element of $f(w)$ at $w=\eta \in D^*$.

From these discussions we can say, in short, the totality of inverse function elements of $f(w)$ at all the points of D^* defines an analytic function which is called the inverse analytic function of $f(w)$ with respect to D^*.

Let $H=f(D^*)$. H is a connected open subset of $\mathfrak{B}$ (see Lemma 3.1). From above it is evident that the center of any function element of $\Psi(z)$ is contained in H and conversely for any element a in H there exists a function element of $\Psi(z)$ having its center at a. And it is easy to see that any function element of $\Psi(z)$ cannot be continued analytically beyond the domain H.

Let $p_a(z)$ be a function element of the analytic function $\Psi(z)$ having its center at a. For any element b such that $\|b-a\|< \rho(p_a(z))$ there exists a function element $p_b(z)$ of $\Psi(z)$ which is the direct analytic continuation of $p_a(z)$ and having its center at b. This implies that

$$\rho(p_a(z)) \leqq \rho(a\,; H) \qquad \text{(see Notations 2.1 and 2.2).} \tag{4.4}$$

If the equality $\rho(p_a(z))=\rho(a\,; H)$ holds for every $a \in H$ and every function element $p_a(z)$ of $\Psi(z)$, then any function element $p(z)$ of $\Psi(z)$ can be continued analytically along every curve in H starting from the center of $p(z)$.

From the above arguments we have the following

THEOREM 4.1 *Let $f(w)$ be a regular function defined on the whole space $\mathfrak{B}$. Let D be the set of those elements w for which $f'(w)$ has an inverse. Being an open set, D is the sum of disjoint maximal open connected sets, $D=\bigcup_{\alpha} D_\alpha$, the components of D. We choose a component D_α of D and denote it by D^*. We have then*

(i) *The totality of inverse function elements of $f(w)$ at the points of D^* (see Definition 4.3) defines an analytic function $\Psi(z)$ which is called the inverse analytic function of $f(w)$ with respect to D^*.*

(ii) *For any function element* $p(z)$ *of the analytic function* $\Psi(z)$ *it holds that*

$$f(p(z)) = z\,. \tag{4.5}$$

(iii) *If we set* $H=f(D^{*})$, *then* H *is a connected open subset of* $\mathfrak{B}$. *And the center of any function element of the analytic function* $\Psi(z)$ *is contained in* H *and for any element* a *in* H *there exists a function element of* $\Psi(z)$ *having its center at* a. (*Any function element cannot be continued analytically beyond the domain* H).

(iv) *For any function element* $p_a(z)$ *of the analytic function* $\Psi(z)$ *having its center at* a *we have*

$$\rho(p_a(z)) \leqq \rho(a\,;H) \qquad (\textit{see Notations } 2.1 \textit{ and } 2.2). \tag{4.6}$$

If the equality $\rho(p_a(z)=\rho(a\,;H)$ *holds for every* $a\in H$ *and every function element* $p_a(z)$ *of* $\Psi(z)$ *having its center at* a, *then any function element* $p(z)$ *of* $\Psi(z)$ *can be continued analytically along every curve in* H *starting from the center of* $p(z)$.

In Theorem 4.1 we set $f(w)=\exp(w)$. Then $f'(w)$ $(=f(w))$ has always an inverse. Hence in this case we have $D=\mathfrak{B}$ and $H=f(\mathfrak{B})=\mathfrak{G}_1$ (see the Corollary to Theorem 3.1).

DEFINITION 4.4. The inverse analytic function of the exponential function $\exp(w)$ is called the logarithmic function and is denoted by $\log z$.

COROLLARY 1 *to* THEOREM 4.1. *If* $p(z)$ *is a function element of the analytic function* $\log z$, *then*

$$\exp(p(z)) = z\,. \tag{4.7}$$

Let $p_a(z)$ *be any function element of* $\log z$ *having its center at* a. *Then we have always*

$$\rho(p_a(z)) = \rho(a\,;\mathfrak{G}_1)\,. \tag{4.8}$$

Therefore any function element $p(z)$ *of* $\log z$ *can be continued analytically along every curve in* $\mathfrak{G}_1$ *starting from the center of* $p(z)$, *and* $p(z)$ *cannot be continued analytically beyond the domain* $\mathfrak{G}_1$. *Let* $q(z)$ *be an analytic continuation of* $p(z)\in\log z$ *along a curve* $\Gamma\subset\mathfrak{G}_1$ *having its beginning point at* a *and end point at* b. *We have then*

$$q(b) = p(a)+\int_{\Gamma}\frac{1}{z}dz\,. \tag{4.9}$$

PROOF. Let $p_a(z)$ be a function element of the analytic function

$\log z$ and S the open sphere such that $S=\{z\,;\|z-a\|<\rho(a\,;\mathfrak{G}_1)\}$. Since S is a connected open subset of $\mathfrak{B}$ and the function $1/z$ is regular in S, we have $\int_\Gamma \frac{1}{z}dz=0$ for every rectifiable closed curve $\Gamma\subset S$. Hence we can define a function $f(z)$ such that

$$(4.10)\qquad f(z)=\int_a^z \frac{1}{z}dz$$

where the right hand of the above equation denotes the integral of the function $1/z$ along any curve contained in S and having its beginning point at a and end point at z. Then it is easily proved that $f(z)$ is regular in S and $f'(z)=1/z$. On the other hand from (4.7) we can easily see that $p_a'(z)=1/z$. As in classical function theory we have

$$(4.11)\qquad p_a(z)=p_a(a)+f(z)\qquad \text{on a neighborhood of } a.$$

This implies that $\{z\,;\|z-a\|<\rho(p_a(z))\}\supset S=\{z\,;\|z-a\|<\rho(a\,;\mathfrak{G}_1)$, that is, $\rho(p_a(z))\geqq\rho(a\,;\mathfrak{G}_1)$. From this and (4.6) we have $\rho(p_a(z))=\rho(a\,;\mathfrak{G}_1)$. The last assertion of the Corollary is proved quite similarly as in classical function theory.

In Theorem 4.1 we shall consider a regular function $f(w)=w^m$ (where $m\neq 1$ is a positive integer). In order that $f'(w)=mw^{m-1}$ has an inverse, it is necessary and sufficient that w belongs to $\mathfrak{G}$ (see Notation 3.1). Hence in this case the set D coincides with $\mathfrak{G}$ and any component of $\mathfrak{G}$ is a coset of $\mathfrak{G}$ modulo $\mathfrak{G}_1$.

DEFINITION 4.5. The inverse analytic function of the function $f(w)=w^m$ ($m\neq 1$ is a positive integer) with respect to $\mathfrak{G}_\alpha$ (a component of $\mathfrak{G}$) is called the radical function of order m and is denoted by $\sqrt[m]{z}(\mathfrak{G}_\alpha)$. In particular, the analytic function $\sqrt[m]{z}(\mathfrak{G}_1)$ is called the principal radical function of order m and is denoted simply by $\sqrt[m]{z}$.

COROLLARY 2 *to* THEOREM 4.1. *If $L(z)$ is a function element belonging to the analytic function* $\log z$, *then the function* $\exp(L(z)/m)$ *is a function element belonging to the principal radical function* $\sqrt[m]{z}$. *Conversely, any function element $p(z)$ belonging to the analytic function $\sqrt[m]{z}$ can be expressed in such a form, that is, $p(z)=$* $\exp(L(z)/m)$ (*where $L(z)$ is a function element belonging to* $\log z$). *Hence for any function element $p_a(z)$ belonging to $\sqrt[m]{z}$ we have*

$$(4.12)\qquad \rho(p_a(z))=\rho(a\,;\mathfrak{G}_1)\qquad (\textit{notice that } f(\mathfrak{G}_1)=\mathfrak{G}_1).$$

Thus any function element $p(z)$ belonging to the analytic function $\sqrt[m]{z}$ can be continued analytically along every curve in $\mathfrak{G}_1$ starting from the center of $p(z)$. For the radical function $\sqrt[m]{z}(\mathfrak{G}_\alpha)$ $(\mathfrak{G}_\alpha \neq \mathfrak{G}_1)$, any function element $p(z)$ belonging to $\sqrt[m]{z}(\mathfrak{G}_\alpha)$ can be also continued analytically along every curve in $\mathfrak{G}_\beta$ starting from the center of $p(z)$, where $\mathfrak{G}_\beta$ is the coset of $\mathfrak{G}$ modulo $\mathfrak{G}_1$ containing the center of $p(z)$.

PROOF. The first assertion of the corollary is proved similarly as in classical function theory. We shall prove the last half of the corollary. Let $p_a(z)$ (see Notation 4.1) be an arbitrary function element belonging to the radical function $\sqrt[m]{z}(\mathfrak{G}_\alpha)$ $(\mathfrak{G}_\alpha \neq \mathfrak{G}_1)$. Let $\mathfrak{G}_\alpha = \xi\mathfrak{G}_1$. Then it is easy to see that $a \in \{\xi^m x^m\,;\, x \in \mathfrak{G}_1\} = \xi^m\mathfrak{G}_1 = \mathfrak{G}_\beta$ (a is the center of the function element $p_a(z)$). Let

$$a = \xi^m b\,, \qquad b \in \mathfrak{G}_1\,. \tag{4.13}$$

On the other hand a is also expressed in the following form

$$a = \eta^m\,, \qquad \eta \in \mathfrak{G}_\alpha = \xi\mathfrak{G}_1\,. \tag{4.14}$$

Suppose that

$$\eta = \xi c\,, \qquad c \in \mathfrak{G}_1\,. \tag{4.15}$$

From (4.12), (4.13) and (4.14) we have

$$c^m = b\,. \tag{4.16}$$

Let S be the maximal open sphere which is contained in $\mathfrak{G}_\beta$ and having its center at a. Then $\xi^{-m}S$ is an open convex subset of $\mathfrak{G}_1$ which contains an element $\xi^{-m}a = b$. From (4.16) there exists a function element $p(z) \in \sqrt[m]{z}$ such that $p(b) = c$. Furthermore, from (4.12) it is easy to see that $p(z)$ can be continued analytically along every curve in $\xi^{-m}S$ starting from the point $b \in \xi^{-m}S$. From this and the theorem of monodromy there exists a regular function $f(z)$ which is defined on $\xi^{-m}S$ and such that $f(z) \equiv p(z)$ on some neighborhood of b. Hence $f(z)$ satisfies the condition that

$$f(z)^m = z\,. \tag{4.17}$$

Define

$$\varphi(z) = \xi f(\xi^{-m}z) \qquad \text{for} \quad z \in S\,. \tag{4.18}$$

The function $\varphi(z)$ is regular in S and $\varphi(z)^m = \xi^m f(\xi^{-m}z^m) = \xi^m \cdot \xi^{-m} z = z$. We have further $\varphi(a) = \xi f(\xi^{-m}a) = \xi f(b) = \xi c = \eta$. Hence it is easy to see that $\varphi(z)$ coincides with $p_a(z)$ on some neighborhood of a. This implies that $\{z; \|z-a\|<\rho(p_a(z))\}\supset S$ (damain of $\varphi(z)$). Our corollary is thereby completely proved (see last part of Theorem 4. 1).

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