MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXXIII, Mathematics No. 2, 1960.

On the homotopy-commutativity of groups and loop spaces

By

Masahiro SUGAWARA

(Received August 20, 1960)

Introduction. It is well known that the loop space $\Omega(B)$ in B is homotopy-commutative if B is an H-space. Furthermore, it follows that the group G, which is also a CW-complex, is homotopy-commutative if its classifying space B_G is an H-space, because there is an H-homomorphism $f: G \to \Omega(B_G)$ which is also a weak homotopy equivalence (cf. [9], Theorem 1 and also [12], Theorem 2). It is our purpose of this paper to study about the inverses of these facts.

In the first part, the notion of the strong homotopy-commutativity is considered, and it is proved that the strong homotopy-commutativity of $\Omega(B)$ or G and being B or B_G an H-space are equivalent (Theorems 4.2 and 4.3).

In the second part, an exact sequence of the sets of homotopy classes for a fibre space with certain conditions are considered (Theorem 6.5), and the image of the map $\pi(X, Y) \to \pi(\Omega X, \Omega Y)$ is studied (Lemma 7.4). Finally, it is proved that only the homotopy-commutativity of $\Omega(B)$ or G is equivalent to being B or B_G an H-space for certain kinds of spaces, (Theorems 8.1 and 8.2).

Part I. Strong homotopy-commutativities

1. Commutative groups. Let G be a countable CW-group, and $p: E \to B$ an universal bundle with group G where a classifying space B is a countable CW-complex.[1] If G is *commutative*, then the map

1) A group G is called a countable CW-group if G is a countable CW-complex such that the map $g \to g^{-1}$ of $G \to G$ and the multiplication $G \times G \to G$ are both cellular maps. Milnor, [3], Theorem 5.1.(1), proved that such a group has a countable CW-complex as a classifying space.

(1.1) $$\mu: G\times G\rightarrow G, \quad \mu(x, y) = xy,$$

of the product group $G\times G$ into G is a *homomorphism*. Therefore, it is easy to see that there are maps

$$\bar{\mu}: (E_\infty(G\times G), G\times G)\rightarrow(E_\infty(G), G), \quad \tilde{\mu}: X_\infty(G\times G)\rightarrow X_\infty(G)$$

such that $\bar{\mu}|G\times G=\mu$, $p(G)\circ\bar{\mu}=\tilde{\mu}\circ p(G\times G)$, where $p(G): E_\infty(G)\rightarrow X_\infty(G)$ is the universal bundle with group G, constructed by Milnor, [3], § 3. Because $p\times p: E\times E\rightarrow B\times B$ is also an universal bundle with group $G\times G$, we obtain maps

$$\bar{M}: (E\times E, G\times G)\rightarrow(E, G), \quad M: B\times B\rightarrow B$$

such that $\bar{M}|G\times G=\mu$, $p\circ\bar{M}=M\circ(p\times p)$.

Define $\bar{L}_i: E\times E\rightarrow E\times E$, $L_i: B\times B\rightarrow B\times B$, $i=1, 2$, by

$$\bar{L}_i(u_1, u_2) = (\bar{M}(u_1, u_2), u_i), \quad L_i(v_1, v_2) = (M(v_1, v_2), v_i).$$

Then, $\bar{L}_i|G\times G=l_i: G\times G\rightarrow G\times G$ is given by $l_i(x_1, x_2)=(x_1x_2, x_i)$, which is a homotopy equivalence. Therefore $L_i: (B\times B, (*, *))\rightarrow(B\times B, (*, *))$ $(*=p(G))$ is also a homotopy equivalence, for $i=1, 2$.

On the other hand, we have

Lemma 1.2. *If there is a map $\mu: (F\times F, (e, e))\rightarrow(F, e)$ for a CW-complex F containing a point e such that the map*:

$$l_i: (F\times F, (e, e))\rightarrow(F\times F, (e, e)), \quad l_i(x_1, x_2) = (\mu(x_1, x_2), x_i),$$

is a homotopy equivalence, for $i=1, 2$, then F is an H-space.

Proof. In the proofs of Theorem 4 of [11], the existence of an unit in the conditions of H-space is not used, and so we have the same conclusions for F of this lemma, i.e., there is a weak homotopy equivalence $p: (F*F, F)\rightarrow(\bar{S}F, *)$, where $F*F$ is the join of two copies of F, $\bar{S}F$ the suspension of F and $*$ its point. Therefore, F is an H-space by [11], Theorem 1, noticing that the product of CW-complexes has the same homotopy type of a CW-complex, ([4], Proposition 3). q.e.d.

By the above considerations, we have the following well known theorem:

Theorem 1.3. *A classifying space, which is also a countable CW-complex, of a commutative countable CW-group is an H-space.*

2. Strongly homotopy-multiplicative maps. Let H, H' be topological monoids, i.e., associative H-spaces. If $f: H\rightarrow H'$ is a *homotopy-multiplicative map* (or an H-homomorphism), then there is a map

$M_1\colon H^2\times I\to H'$,[2] $M_1(x_0, x_1, 0) = f(x_0x_1)$, $M_1(x_0, x_1, 1) = f(x_0)f(x_1)$.

In this paper, we shall say that f is *strongly homotopy-multiplicative* if there exist maps $M_n\colon H^{n+1}\times I^n\to H'$, $n=0, 1, \cdots$, such that

$$
\begin{aligned}
&M_0(x) = f(x)\,,\\
&M_n(x_0, \cdots, x_n, t_1, \cdots, t_n)\\
(2.1)\quad &= M_{n-1}(x_0, \cdots, x_{i-1}x_i, \cdots, x_n, t_1, \cdots, \hat{t}_i, \cdots, t_n),^{3)} \qquad \text{for } t_i = 0\,,\\
&= M_{i-1}(x_0, \cdots, x_{i-1}, t_1, \cdots, t_{i-1})M_{n-i}(x_i, \cdots, x_n, t_{i+1}, \cdots, t_n)\,, \qquad \text{for } t_i = 1\,.
\end{aligned}
$$

Dold and Lashof, [2], §§2–3, constructed the universal principal quasifibration $\mathfrak{E}_H=(E_H, p_H, B_H, H)$ for any topological monoid H, as follows:

$E_0=H$, $B_0=b_0$ a point, $p_0\colon E_0\to B_0$ is the trivial map.

A point of E_n is described by $y|t|x$ where $y\in E_{n-1}$, $x\in H$, $t\in I$ and $y|0|x=0|x=(n, x)\in n\times H$, $y|1|x=yx|1=yx\in E_{n-1}$. The product $E_n\times H\to E_n$ is $(y|t|x)x_1=y|t|xx_1$.

A point of B_n is described by $y\perp t$ where $y\in E_{n-1}$, $t\in I$ and $y\perp 0=b_n$ (a point), $y\perp 1=p_{n-1}(y)\in B_{n-1}$.

The projection $p_n\colon E_n\to B_n$, $p_n(y|t|x)=y\perp t$.

$E_H=\bigcup E_n$, $B_H=\bigcup B_n$, $p_H|E_n=p_n$. (The topologies of these spaces are slightly stronger than the usual product, identification and limit topologies, but, for a countable CW-complexes, it may be taken the usual topologies.)

Lemma 2.2. *Let H, H' be monoids and $f\colon H\to H'$ be a strongly homotopy-multiplicative map. Then, there exist maps:*

$$\bar{f}\colon (E_H, H)\to (E_{H'}, H')\,,\quad \tilde{f}\colon B_H\to B_{H'}\,,$$

such that $\bar{f}|H=f$, $p_{H'}\circ\bar{f}=\tilde{f}\circ p_H$.

Proof. Denote by E'_n, B'_n, p'_n the spaces and the maps, constructed from H' instead of H in the above.

Set $\bar{f}_0=f$, $\tilde{f}_0(B_0)=B'_0$.

Assume inductively that $\bar{f}_{n-1}\colon E_{n-1}\to E'_{n-1}$, $\tilde{f}_{n-1}\colon B_{n-1}\to B'_{n-1}$ are

---

2) $H^n=H\times\cdots\times H$, n-times, and $I=[0, 1]$.

3) $\hat{t}_i$ means that t_i is removed.

defined and also there are maps $M_m^{n-1}: E_{n-1}\times H^m\times I^m\to E'_{n-1}$, $m=0, 1, \cdots$, such that

$$
\begin{aligned}
&p'_{n-1}\circ\bar{f}_{n-1} = \tilde{f}_{n-1}\circ p_{n-1},\\
&M_0^{n-1}(y) = \bar{f}_{n-1}(y),\\
&M_m^{n-1}(y, x_1, \cdots, x_m, t_1, \cdots, t_m)\\
&= M_{m-1}^{n-1}(yx_1, \cdots, x_m, t_2, \cdots, t_m), \qquad \text{for } t_1=0,\\
(2.3)\quad &= M_{m-1}^{n-1}(y, \cdots, x_{i-1}x_i, \cdots, x_m, t_1, \cdots, \hat{t}_i, \cdots, t_m), \qquad \text{for } t_i=0,\ i>1,\\
&= M_{i-1}^{n-1}(y, \cdots, x_{i-1}, t_1, \cdots, t_{i-1})M_{m-i}(x_i, \cdots, x_m, t_{i+1}, \cdots, t_m), \qquad \text{for } t_i=1,\\
&p'_{n-1}\circ M_m^{n-1}(y, x_1, \cdots, x_m, t_1, \cdots, t_m) = p'_{n-1}\circ\bar{f}_{n-1}(y),
\end{aligned}
$$

where M_k is the map of (2.1).

Define $\bar{f}_n: E_n\to E'_n$, $\tilde{f}_n: B_n\to B'_n$, $M_m^n: E_n\times H^m\times I^m\to E'_n$, $m=0, 1, \cdots$, by

$$
\begin{aligned}
\bar{f}_n(y|t|x) &= \bar{f}_{n-1}(y)|2t|f(x), &&\text{for } 0\leqslant t\leqslant 1/2,\\
&= M_1^{n-1}(y, x, 2-2t), &&\text{for } 1/2\leqslant t\leqslant 1,\\
\tilde{f}_n(y\perp t) &= \bar{f}_{n-1}(y)\perp 2t, &&\text{for } 0\leqslant t\leqslant 1/2,\\
&= \tilde{f}_{n-1}\circ p_{n-1}(y), &&\text{for } 1/2\leqslant t\leqslant 1,
\end{aligned}
$$

$$
\begin{aligned}
&M_m^n(y|t|x, x_1, \cdots, x_m, t_1, \cdots, t_m)\\
&= \bar{f}_{n-1}(y)|2t|M_m(x, x_1, \cdots, x_m, t_1, \cdots, t_m), \qquad \text{for } 0\leqslant t\leqslant 1/2,\\
&= M_{m+1}^{n-1}(y, x, x_1, \cdots, x_m, 2-2t, t_1, \cdots, t_m), \qquad \text{for } 1/2\leqslant t\leqslant 1.
\end{aligned}
$$

Then, simple calculations show that these are extensions of $\bar{f}_{n-1}$, $\tilde{f}_{n-1}$, M_m^{n-1}, and satisfy the inductive assumptions (2.3). q.e.d.

3. Strong homotopy-commutativities. If G is a countable CW-group, then $\mathfrak{E}_G=(E_G, p_G, B_G)$ of §2 and $(E_\infty(G), p(G), X_\infty(G))$ of §1 are the same by [2], §4. Therefore, the same proofs of §1 are valid by Lemma 2.2, if $\mu: G\times G\to G$ of (1.1) is strongly homotopy-multiplicative.

We shall say that a monoid H with unit e is *strongly homotopy-commutative* if there exist maps $C_n: (H^{2n}\times I^n, e^{2n}\times I^n)\to(H, e)$, $n=1, 2, \cdots$, such that

$$
\begin{aligned}
&C_1(x, y, 0) = xy, \quad C_1(x, y, 1) = yx,\\
&C_n(x_1, \cdots, x_n, y_1, \cdots, y_n, t_1, \cdots, t_n)
\end{aligned}
$$

$$
(3.1)\quad
\begin{aligned}
&= x_1 C_{n-1}(x_2, \cdots, x_n, y_1 y_2, \cdots, y_n, t_2, \cdots, t_n), && \text{for } t_1 = 0, \\
&= C_{n-1}(x_1, \cdots, x_{i-1}x_i, \cdots, x_n, y_1, \cdots, y_i y_{i+1}, \cdots, y_n, t_1, \\
&\qquad \cdots, \hat{t}_i, \cdots, t_n), && \text{for } t_i = 0,\ 1 < i < n, \\
&= C_{n-1}(x_1, \cdots, x_{n-1}x_n, y_1, \cdots, y_{n-1}, t_1, \cdots, t_{n-1})\, y_n, && \text{for } t_n = 0, \\
&= C_{i-1}(x_1, \cdots, x_{i-1}, y_1, \cdots, y_{i-1}, t_1, \cdots, t_{i-1}) y_i x_i \\
&\quad C_{n-i}(x_{i+1}, \cdots, x_n, y_{i+1}, \cdots, y_n, t_{i+1}, \cdots, t_n), && \text{for } t_i = 1.\text{[4]}
\end{aligned}
$$

If G is strongly homotopy-commutative, then the maps M_n: $(G^2)^{n+1} \times I^n \to G$, $n=0, 1, \cdots$, defined by

$$
\begin{aligned}
&M_0(x, y) = xy = \mu(x, y), \\
&M_n((x_0, y_0), \cdots, (x_n, y_n), t_1, \cdots, t_n) \\
&\quad = x_0 C_n(x_1, \cdots, x_n, y_0, \cdots, y_{n-1}, t_1, \cdots, t_n) y_n,
\end{aligned}
$$

satisfy (2.1) and so μ is strongly homotopy-multiplicative. Therefore, by the above considerations, we have

Lemma 3.2. *A classifying space, which is also a countable CW-complex, of a strongly homotopy-commutative countable CW-group G, is an H-space.*

Now, we shall prove

Lemma 3.3. *Let H, H' be monoids, H be also a countable CW-complex, and $f: H \to H'$ be a strongly-multiplictive weak homotopy equivalence. If H' is strongly homotopy-commutative, then so is H.*

Proof. Denote by C'_n the maps of (3.1) for H'.

Define $C_1: H^2 \times \dot{I} \to H$ by $C_1(x, y, 0) = xy$, $C_1(x, y, 1) = yx$, and $\tilde{C}_1: H^2 \times (\dot{I} \times I \cup I \times 1) \to H'$ by

$$
\begin{aligned}
\tilde{C}_1(x, y, t_1, t) &= M_1(x, y, t), && \text{for } t_1 = 0, \\
&= M_1(y, x, t), && \text{for } t_1 = 1, \\
&= C'_1(f(x), f(y), t_1), && \text{for } t = 1,
\end{aligned}
$$

where M_1 is the map of (2.1). Extend C_1, $\tilde{C}_1$ to $C_1: H^2 \times I \to H$, $\tilde{C}_1: H^2 \times I^2 \to H'$ so that $f \circ C_1 = \tilde{C}_1 | H^2 \times I \times 0$. Such extensions are possible, because f is a weak homotopy equivalence and $H^2 \times I^2$ is a CW-complex.

Define inductively $C_n: H^{2n} \times \dot{I}^n \to H$ by (3.1), and $\tilde{C}_n: H^{2n} \times (\dot{I}^n \times I \cup I^n \times 1) \to H'$ as follows:

$$
\tilde{C}_n | H^{2n} \times \dot{I}^n \times 0 = f \circ C_n, \quad \tilde{C}_n | H^{2n} \times I^n \times 1 = C'_n \circ f^*,
$$

---

4) The existence of C_1 shows the usual homotopy-commutativity of H.

and $C_n|H^{2n}\times \dot{I}^n\times I$ by C_i, $i\leqslant n-1$, and M_i, $i\leqslant 2n-1$, of (2.1), where $f^*(x_1,\cdots,y_n,t_1,\cdots,t_n)=(f(x_1),\cdots,f(y_n),t_1,\cdots,t_n)$; and extend these to C_n: $H^{2n}\times I^n\to H$ and $\tilde{C}_n$: $H^{2n}\times I^{n+1}\to H'$ so that $f\circ C_n=\tilde{C}_n|H^{2n}\times I^n\times 0$. q.e.d.

4. Loop spaces of classifying spaces. Let $\Omega'(B)$ be the loop space in B in the sense of Moore, [5], §2:

$$\Omega'(B)=\{(l,r)\,|\,r\geqslant 0,\ l\colon [0,r]\to B,\ l(0)=l(r)=\text{base point}\},$$

which is a monoid and has the same homotopy type as the usual loop space $\Omega(B)$.

Lemma 4.1. *Let B be a classifying space of a group G, then there is a strongly homotopy-multiplicative map $f\colon G\to\Omega'(B)$, which is also a weak homotopy equivalence.*

Proof. Let $p\colon E\to B$ be an universal bundle with group G, and let $k_s\colon E\to E$ be a contraction of E into e, the unit of G. It is proved that the map $f\colon G\to\Omega'(B)$, defined by

$$f(x)(s)=p\circ k_s(x),\qquad 0\leqslant s\leqslant 1,$$

is a weak homotopy equivalence and homotopy-multiplicative, in [9], Theorem I.

Set $J^{n+1}=\{(t_1,\cdots,t_n,s)\,|\,(t_1,\cdots,t_n)\in I^n,\ 0\leqslant s\leqslant 1+t_1+\cdots+t_n\}\subset I^n\times[0,n+1]$, and define inductively $\bar{M}_n\colon G^{n+1}\times\dot{J}^{n+1}\to E$, $n=0,1,\cdots$, by

$$\bar{M}_0(x,s)=k_s(x),$$

$$\begin{aligned}
&\bar{M}_n(x_0,\cdots,x_n,t_1,\cdots,t_n,s)\\
&=\bar{M}_{n-1}(x_0,\cdots,x_{i-1}x_i,\cdots,x_n,t_1,\cdots,\hat{t}_i,\cdots,t_n,s), && \text{for } t_i=0,\\
&=\bar{M}_{i-1}(x_0,\cdots,x_{i-1},t_1,\cdots,t_{i-1},s)x_i\cdots x_n, && \text{for } t_i=1,\ 0\leqslant s\leqslant 1+t_1+\cdots+t_{i-1},\\
&=\bar{M}_{n-i}(x_i,\cdots,x_n,t_{i+1},\cdots,t_n,s-1-t_1-\cdots-t_{i-1}), && \text{for } t_i=1,\ 1+t_1+\cdots+t_{i-1}\leqslant s\leqslant 1+t_1+\cdots+t_n,\\
&=x_1\cdots x_n, && \text{for } s=0,\\
&=e, && \text{for } s=1+t_1+\cdots+t_n,
\end{aligned}$$

and extend it to $\bar{M}_n\colon G^{n+1}\times J^{n+1}\to E$ by the contraction k_s of E.

Define $M_n\colon G^{n+1}\times I^n\to\Omega'(B)$ by

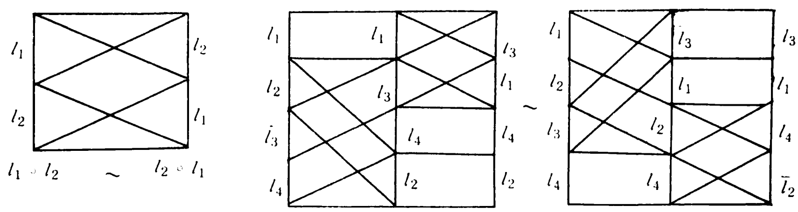
$$M_n(x_0,\cdots,x_n,t_1,\cdots,t_n)(s)=p\circ\bar{M}_n(x_0,\cdots,x_n,t_1,\cdots,t_n,s),\quad \text{for } s\in[0,1+t_1+\cdots+t_n].$$

It is easy to see that these satisfy (2. 1). q.e.d.

Theorem 4. 2. *The loop space $\Omega(B)$ in a countable CW-complex B is strongly homotopy-commutative if, and only if, B is an H-space.*

Proof. By [3], Theorem 5. 2.(3), B is a classifying space of some countable CW-group G. If $\Omega(B)$ is strongly homotopy-commutative, then $\Omega'(B)$ and hence G are also so by Lemmas 3. 3 and 4. 1. Therefore, B is an H-space by Lemma 3. 2.

Conversely, suppose B is an H-space with an unit e, which is the base point. A map C_1: $(\Omega(B))^2\times I\to\Omega(B)$ of (3. 1) is defined by the homotopy $\{l_1\circ l_2=(l_1\circ e)\cdot(e\circ l_2)\sim(e\circ l_1)\cdot(l_2\circ e)=l_2\circ l_1\}$, where e means the constant loop, $\circ$ is the loop- and $\cdot$ the induced multiplication of that of B, (showed in the first figure below). The existence of C_2 of (3. 1) is easily seen from the following rough figures, and so on. q.e.d.



By the sufficiency of this theorem and Lemmas 3. 3 and 4. 1 and also Lemma 3. 2, it follows immediately

Theorem 4. 3. *A classifying space, which is also a countable CW-complex, of a countable CW-group G is an H-space if, and only if, G is strongly homotopy-commutative.*

Part II. Sequences of the sets of homotopy classes

5. The sequence of Puppe. For any spaces X and Y with base point, denote by $\pi(X, Y)$ the set of homotopy classes of maps with base point of X into Y, and by $0\in\pi(X, Y)$ the class represented by the constant map.

For any map f: $X\to Y$, let

$$X\xrightarrow{f}Y\xrightarrow{Pf}C_f\xrightarrow{Qf}SX\xrightarrow{Sf}SY\longrightarrow\cdots$$

be the sequence of maps defined as follows: C_f is the mapping cone of f, i.e., the identification space of $X\times I\cup Y$ by $X\times 1\ni(x, 1)=f(x)\in Y$ and $X\times 0\cup *\times I=f(*)$. SX is the reduced suspension of X, i.e., the identification space $X\times I/X\times \dot{I}\cup *\times I$. Pf is the injection, Qf the projection, and Sf the suspension of f.

Puppe, [8], Satz 6, proved that the above sequence induces the *exact* sequence of the sets of homotopy classes:

$$(5.1)\qquad \begin{array}{l} \pi(X, V) \xleftarrow{f^*} \pi(Y, V) \xleftarrow{(Pf)^*} \pi(C_f, V) \xleftarrow{(Qf)^*} \\ \pi(SX, V) \xleftarrow{(Sf)^*} \pi(SY, V) \leftarrow \cdots \end{array}$$

for any space V.

6. Sequences for fibre spaces. Let $p: E\to B$ be a fibre space in the sense of Serre such that B is m-connected, $m\geqslant 1$, the fibre $F=p^{-1}(*)$ is n-connected, and the spaces considered are of the same homotopy types of CW-complexes.

Then, by [10], p. 469 and [13], Lemma 5.2, the induced homomorphism p^* of the cohomology groups with coefficient in an abelian group G is an isomorphism

$$(6.1)\qquad p^*: H^k(B; G) \xrightarrow{\approx} H^k(E, F; G), \qquad \text{for } k\leqslant m+n+1,$$

and a monomorphism for $k=m+n+2$.

Let C be the mapping cone of the inclusion map $i: F\to E$, then we have isomorphisms:

$$H^k(E, F; G) \xleftarrow{\approx} H^k(E/F; G) \xrightarrow{\approx} H^k(C; G)$$

for any k, because E is a CW-complex and F is its subcomplex, up to a homotopy type. Therefore, combining with (6.1), the map:

$$(6.2)\qquad \bar{p}: C\to B, \quad \bar{p}|E=p, \quad \bar{p}(F\times I)=*,$$

induces an isomorphism

$$\bar{p}^*: H^k(B; G) \xrightarrow{\approx} H^k(C; G), \qquad \text{for } k\leqslant m+n+1,$$

and a monomorphism for $k=m+n+2$. Denote by $K(G, k)$ the Eilenberg-MacLane space, and identify $H^k(X; G)$ with $\pi(X, K(G, k))$, and we have *an isomorphism*

$$(6.3)\qquad \bar{p}^*: \pi(B, K(G, k)) \xrightarrow{\approx} \pi(C, K(G, k)), \qquad \textit{for } k\leqslant m+n+1,$$

and a monomorphism for $k=m+n+2$.

Lemma 6.4. *If* V *is a* CW*-complex such that* $\pi_i(V)=0$ *for*

$i > m+n+1$, then the induced map $\bar{p}^*: \pi(B, V) \to \pi(C, V)$ is onto; and, if $\pi_i(V)=0$ for $i > m+n+2$, then $\operatorname{Ker} \bar{p}^* = 0$.

Proof. We shall prove by the induction on j where $\pi_i(V)=0$ for $i > j \leqslant m+n+1$.

For $j=1$: Since B is simply connected, we can assume that $B^1 = *$, the base point, and so any map $B \to V$ is homotopic to the constant map because $\pi_i(V)=0$ for $i \geqslant 2$; and samely for C.

Induction, $j-1 \to j$ for $j \leqslant m+n+1$: Using the Postnikov system, we can represent V as a principal fibre space, [7], with fibre $K = K(\pi_j(V), j)$ and base W where

$$\pi_i(W) = \pi_i(V) \text{ for } i \leqslant j-1, \quad \pi_i(W)=0 \text{ for } i \geqslant j.$$

Therefore, we have the following commutative diagram

$$\begin{array}{ccccccccc}
\pi(B, \Omega W) & \xrightarrow{\bar{\chi}_*} & \pi(B, K) & \xrightarrow{j_*} & \pi(B, V) & \xrightarrow{q_*} & \pi(B, W) & \xrightarrow{\chi_*} & \pi(B, \chi(K)) \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle e} \\
\pi(C, \Omega W) & \xrightarrow{\bar{\chi}_*} & \pi(C, K) & \xrightarrow{j_*} & \pi(C, V) & \xrightarrow{q_*} & \pi(C, W) & \xrightarrow{\chi_*} & \pi(C, \chi(K))
\end{array}$$

where $\chi(K) = K(\pi_j(V), j+1)$ and each low is the exact sequence of [6], Lemma 2.1, and $a, \cdots, e$ are $\bar{p}^*$.

c is onto: Let $\alpha \in \pi(C, V)$. By the inductive assumptions, d is onto, and we have $\beta \in \pi(B, W)$ such that $d(\beta) = q_*(\alpha)$. Since e is 1-1 by (6.3), $\chi_*(\beta)=0$, and so there is $\gamma \in \pi(B, V)$ such that $q_*(\gamma) = \beta$. Since $q_*(c(\gamma)) = q_*(\alpha)$, there is $\delta \in \pi(C, K)$ such that $\mu_*(\delta, c(\gamma)) = \alpha$ where μ_* is

$$\pi(X, K) \times \pi(X, V) = \pi(X, K \times V) \xrightarrow{\mu_*} \pi(X, V),$$

by [7], Lemma 4.1. By (6.3), there is $\varepsilon \in \pi(B, K)$ such that $b(\varepsilon) = \delta$.

$$c(\mu_*(\varepsilon, \gamma)) = \mu_*(b(\varepsilon), c(\gamma)) = \mu_*(\delta, c(\gamma)) = \alpha$$

shows that c is onto.

$\operatorname{Ker} c = 0$: We can prove this by the same proofs of the Five Lemma, and this is valid for $j = m+n+2$. q.e.d.

In the commutative diagram

$$\begin{array}{ccccc}
\pi(F, V) & \xleftarrow{i^*} & \pi(E, V) & \xleftarrow{p^*} & \pi(B, V) \\
 & & \nwarrow{\scriptstyle (Pi)^*} & & \downarrow{\scriptstyle \bar{p}^*} \\
 & & & & \pi(C, V),
\end{array}$$

the sequence $\overset{i^*}{\longleftarrow} \overset{(Pi)^*}{\longleftarrow}$ of (5.1) for $i: F \to E$ is exact, and so the horizontal sequence is exact if $\bar{p}^*$ is onto. Therefore, by Lemma 6.4, we have

Theorem 6.5. *Let $p: E \to B$ be a fibre space in the sence of Serre such that B is m-connected, $m \geqq 1$, the fibre $F = p^{-1}(*)$ is n-connected, and the spaces considered are of the same homotopy types of CW-complexes. If V is a CW-complex such that $\pi_i(V) = 0$ for $i > m+n+1$, then the sequence of the sets of homotopy classes*

$$\pi(F, V) \overset{i^*}{\longleftarrow} \pi(E, V) \overset{p^*}{\longleftarrow} \pi(B, V)$$

is exact, where $i: F \to E$ is the injection.

If, in addition, $\pi(B, V)$ and $\pi(C, V)$ have group structures with unit 0 *and $\bar{p}^*: \pi(B, V) \to \pi(C, V)$ is a homomorphism where C is the mapping cone of i and $\bar{p}$ is the map of* (6.2), *then there is an exact sequence*

$$\pi(F, V) \overset{i^*}{\longleftarrow} \pi(E, V) \overset{p^*}{\longleftarrow} \pi(B, V) \overset{\bar{p}^{*-1} \circ (Qi)^*}{\longleftarrow} \pi(SF, V) \overset{(Si)^*}{\longleftarrow} \pi(SE, V).$$

7. The map $\Omega: \pi(B, V) \to \pi(\Omega(B), \Omega(V))$. Let B be a CW-complex and

$$\Omega(B) * \Omega(B) \overset{p}{\longrightarrow} \bar{S}\Omega(B) \overset{k}{\longrightarrow} B \tag{7.1}$$

be the sequence such that $\Omega(B)*\Omega(B)$ is the join of two copies of $\Omega(B)$, $\bar{S}\Omega(B)$ the suspension of $\Omega(B)$ and p, k are given by

$$p(l, l'; t) = (l \circ l'; t), \quad k(l; t) = l(t), \quad \text{for } l, l' \in \Omega(B), \quad t \in I.$$

By [1], I, § 3, there are a fibre space $E' \to B'$ with fibre F' and a following diagram

$$\begin{array}{ccccc} \Omega(B)*\Omega(B) & \overset{p}{\longrightarrow} & \bar{S}\Omega(B) & \overset{k}{\longrightarrow} & B \\ \downarrow & & \downarrow & & \downarrow \\ F' & \overset{\subset}{\longrightarrow} & E' & \longrightarrow & B' \end{array} \tag{7.2}$$

such that the squares are commutative, up to a homotopy, and the vertical maps are homotopy equivalences.

If B is n-connected, then $\Omega(B)$ is $(n-1)$-connected, and so $\Omega(B)*\Omega(B)$ is $2n$-connected, by [3], Lemma 2.3. Therefore, by Theorem 6.5, we have *the exact sequence*:

$$\pi(\Omega(B)*\Omega(B), V) \overset{p^*}{\longleftarrow} \pi(\bar{S}\Omega(B), V) \overset{k^*}{\longleftarrow} \pi(B, V) \tag{7.3}$$

if $\pi_i(V)=0$ *for* $i>3n+1$, since these spaces have the same homotopy types of CW-complexes according to [4].

Lemma 7.4. *Let* $\Omega:\ \pi(B, V)\rightarrow\pi(\Omega(B), \Omega(V))$ *be the map defined by*

$$(\Omega\varphi)(l)(t) = \varphi(l(t)), \qquad \textit{for } \varphi:\ B\rightarrow V,\quad l\in\Omega(B),\ t\in I.$$

If B *is a* n*-connected* CW*-complex and* V *is a* CW*-complex such that* $\pi_i(V)=0$ *for* $i>3n+1$, *then*

$$\operatorname{Im}\Omega = \pi'(\Omega(B), \Omega(V)),$$

where $\pi'(\Omega(B), \Omega(V))$ *is the set of all the homotopy classes of homotopy-multiplicative maps.*[5)]

Proof. $\operatorname{Im}\Omega\subset\pi'(\Omega(B), \Omega(V))$ is clear.

In the commutative diagram

$$\begin{array}{ccccc}
\pi(\Omega(B)*\Omega(B), V) & \overset{p^*}{\longleftarrow} & \pi(\bar{S}\Omega(B), V) & \overset{k^*}{\longleftarrow} & \pi(B, V)\\
 & & \uparrow\lambda & \swarrow\Omega & \\
 & & \pi(\Omega(B), \Omega(V)) & &
\end{array}$$

where $(\lambda\varphi)(l\,;t)=\varphi(l)(t)$, the horizontal sequence is exact by (7.3), and λ is 1-1 and onto as well known. Therefore, it is only necessary to prove $p^*\circ\lambda(\pi'(\Omega(B), \Omega(V)))=0$.

Let $\varphi:\ (\Omega(B), *)\rightarrow(\Omega(V), *)$ be a homotopy-multiplicative map, i.e., the two maps, of $\Omega(B)\times\Omega(B)$ into $\Omega(V)$:

$$(l, l')\rightarrow\varphi(l\circ l'),\quad (l, l')\rightarrow\varphi(l)\circ\varphi(l'),$$

are homotopic. Then, for the maps $\bar{\varphi}:\ \Omega(B)*\Omega(B)\rightarrow\Omega(V)*\Omega(V)$, $\tilde{\varphi}:\ \bar{S}\Omega(B)\rightarrow\bar{S}\Omega(V)$ defined by

$$\bar{\varphi}(l, l'\,;t) = (\varphi(l), \varphi(l')\,;t),\quad \tilde{\varphi}(l\,;t) = (\varphi(l)\,;t),$$

the two maps $p'\circ\bar{\varphi}$ and $\tilde{\varphi}\circ p$ are homotopic, where $p':\ \Omega(V)*\Omega(V)\rightarrow\bar{S}\Omega(V)$ is the map of (7.1) for V instead of B. Hence, we have the commutative diagram:

$$\begin{array}{ccccc}
 & & & \pi(V, V) & \\
 & & \swarrow k'^* & & \searrow\Omega\\
\pi(\Omega(V)*\Omega(V), V) & \overset{p'^*}{\longleftarrow} & \pi(\bar{S}\Omega(V), V) & \overset{\lambda}{\longleftarrow} & \pi(\Omega(V), \Omega(V))\\
\downarrow\bar{\varphi}^* & & \downarrow\tilde{\varphi}^* & & \downarrow\varphi^*\\
\pi(\Omega(B)*\Omega(B), V) & \overset{p^*}{\longleftarrow} & \pi(\bar{S}\Omega(B), V) & \overset{\lambda}{\longleftarrow} & \pi(\Omega(B), \Omega(V))
\end{array}$$

---

5) Cf. § 2.

where k' is the map of (7.1) for V. Since the diagram (7.2) for V shows that $k'\circ p'$ is homotopic to the constant map, $p^*\circ\lambda\{\varphi\} = \bar{\varphi}^*\circ p'^*\circ k'^*\{1\} = 0$, where 1 is the identity map. q.e.d.

8. Applications for homotopy-commutativities.

Theorem 8.1. *Let B be a CW-complex such that*

$$\pi_i(B) = 0, \quad \textit{for } i \leqslant n,\ i > 3n+1.$$

Then, $\Omega(B)$ is homotopy-commutative if, and only if, B is an H-space.

Proof. Let μ: $\Omega(B\times B)=\Omega(B)\times\Omega(B)\to\Omega(B)$ be the map defined by

$$\mu(l) = \Omega p_1(l)\circ\Omega p_2(l),$$

where p_i: $B\times B\to B$ is the projection onto the i-th factor.

Suppose that $\Omega(B)$ is homotopy-commutative, then μ is homotopy-multiplicative. Therefore, we have a map M: $(B\times B, (*, *))\to(B, *)$ such that ΩM is homotopic to μ, by Lemma 7.4. It is easily seen that M satisfies the assumptions of Lemma 1.2, since μ does, and so B is an H-space. q.e.d.

If B is a classifying space of a countable CW-group, the map f: $G\to\Omega(B)$ of Lemma 4.1 is a weak homotopy equivalence. Since $\Omega(B)$ is the same homotopy type of a CW-complex, it is also a homotopy equivalence. Therefore, the homotopy-commutativities of $\Omega(B)$ and G are equivalent, and we have

Theorem 8.2. *Let G be a countable CW-group such that*

$$\pi_i(G) = 0, \quad \textit{for } i \leqslant n,\ i > 3n+3.$$

Then, G is homotopy-commutative if, and only if, its classifying space, being a countable CW-complex, is an H-space.

Corollary 8.3. *Under the conditions of Theorems* 8.1 *and* 8.2, *the homotopy-commutativity of G or $\Omega(B)$ implies the strong one.*

REFERENCES

[1] W. D. Barcus and J.-P. Meyer: *The suspension of a loop space,* Amer. J. Math., 80 (1958), 895-920.

[2] A. Dold and R. Lashof: *Principal quasifibrations and fibre homotopy equivalences of bundles,* Illinois J. Math., 3 (1959), 285-305.

[3] J. Milnor: *Constructions of universal bundles.* II, Ann. of Math., 63 (1956), 430-436.

[4] ———: *On spaces having the homotopy type of a CW-complex,* Trans Amer. Math. Soc., 90 (1959), 272-280.

[5] J. C. Moore: *The double suspensions and p-primary components of the homotopy groups of spheres,* Bol. Soc. Mat. Mexicana, 1 (1956), 28-37.

[6] F. P. Peterson: *Functional cohomology operations*, Trans. Amer. Math. Soc., 86 (1957), 197–211.
[7] ———— and E. Thomas: *A note on non-stable cohomological operations*, Bol. Soc. Mat. Mexicana, 3 (1958), 13–18.
[8] D. Puppe: *Homotopiemengen und ihre induzierten Abbildungen*. I, Math. Zeit., 69 (1958), 299–344.
[9] H. Samelson: *Groups and spaces of loops*, Comment. Math. Helv., 28 (1954), 278–287.
[10] J.-P. Serre: *Homologie singulière des espaces fibrés. Applications*, Ann. of Math., 54 (1951), 425–505.
[11] M. Sugawara: *On a condition that a space is an H-space*, Math. J. Okayama Univ., 6 (1956–57), 109–128.
[12] ————: *Note on H-spaces*, Proc. Japan Acad., to appear.
[13] G. W. Whitehead: *On the homology suspension*, Ann. of Math., 62 (1955), 254–268.