MEMOIRS OF THE COLLEGE OF SCIENCE, UNIVERSITY OF KYOTO, SERIES A
Vol. XXXIII, Mathematics No. 2, 1960.

On P. J. Myrberg's approximation theorem on Fuchsian groups

By

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(Received August 15, 1960)

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1. Introduction.

P. J. Myrberg [3] proved an approximation theorem for Fuchsian groups having fundamental domain of finite non-euclidean area. The purpose of the present paper is to extend it for Fuchsian groups of divergence type and, further, show that the validity of his approximation theorem implies conversely the divergence type of Fuchsian groups. As is known ([10]), a Fuchsian group is of divergence type if and only if the corresponding Riemann surface is of class O_G. Finally in 5 and 6 we shall state some related results and problems.

The author wishes to dedicate this paper to the late Professor M. Tsuji who gave him kind suggestions, and express his hearty thanks to Professors A. Kobori and Y. Komatu for their constant encouragement during this research.

2. Main Theorem.

We begin with stating our main result. Let G be a Fuchsian group of linear transformations:

$$S_n: \quad z' = e^{i\alpha_n} \cdot \frac{z-a_n}{1-\bar{a}_n z} \qquad (|a_n|<1) \ (n=0, 1, 2, \cdots),$$

which leave $|z|<1$ invariant and let D_0 be its normal fundamental domain which contains $z=0$. The quantity

$$\sigma(D_0) = 4\iint_{D_0} \frac{r dr d\theta}{(1-r^2)^2}, \qquad z = re^{i\theta},$$

is called its *non-euclidean area.* The boundary of D_0 consists of arcs on circles which are orthogonal to $|z|=1$ and a closed set Λ_0 on $|z|=1$, which may be empty. If we identify the equivalent points on the sides of D_0, then D_0 can be considered as a Riemann surface F_G. If $\sigma(D_0)<\infty$, then either D_0 lies entirely in $|z|<1$ with its boundary, or D_0 has a finite number of sides in $|z|<1$ and a finite number of vertices on $|z|=1$ where two sides of D_0 touch each other. Hence the corresponding Riemann surface F_G is a closed Riemann surface or a Riemann surface which is obtained from a closed Riemann surface by taking off a finite number of points ([5], [7], [10]).

Let a_n $(n=0, 1, 2, \cdots)$ be equivalent points of $z=0$ under G, then either i) $\Sigma(1-|a_n|)=\infty$, or ii) $\Sigma(1-|a_n|)<\infty$. We call G a Fuchsian group *of divergence type* or *of convergence type* according to the case i) or ii), respectively. It is well known that there exists *Green's function* on F_G when and only when G is of convergence type. If $\sigma(D_0)<\infty$, then G is of divergence type.

Main Theorem. *Let G be a Fuchsian group, let $\gamma(e^{i\theta})$ be the radius of $|z|=1$ terminating at $e^{i\theta}$, and let $\{\gamma_n(e^{i\theta})\}$ be the set of arcs which are equivalent to $\gamma(e^{i\theta})$ under G.*

(I) *If G is of divergence type, then there exists a measurable set E of measure 2π on $|z|=1$ satisfying the following property: For any $e^{i\theta}\in E$ and an arbitrarily given circular arc C in $|z|<1$ which intersects $|z|=1$ orthogonally at its two end points on $|z|=1$, it is possible to find $\{n_\nu\}$ such that*

$$\gamma_{n_\nu}(e^{i\theta}) \to C \quad (\nu\to\infty), \quad \textit{uniformly in euclidean metric.}^{1,2)}$$

(II) *If G is of convergence type, then there is no measurable set E of measure 2π on $|z|=1$ satisfying the above mentioned property.*

3. Preliminary lemmas.

For the proof of Main Theorem, we need several lemmas.

If G is of divergence type, the corresponding Riemann surface F_G belongs to O_G, so that as known well, $F_G\in O_{HB}$, hence follows

Lemma 1. *If G is of divergence type, then there exists no measurable set E on $|z|=1$ which is invariant under G and $0<mE<2\pi$. Therefore, if $mE>0$, then $mE=2\pi$.*

1) Exactly speaking, the convergence is in the sense of Fréchet.

2) In other words, 'almost all geodesic lines on a Riemann surface $F\in O_G$ of constant negative curvature are quasi-ergodic.'

Lemma 2 ([12]). *If G is of divergence type, then for any $e^{i\theta}$ on $|z|=1$, the set $\{S_n(e^{i\theta})\}$ $(n=0,1,2,\cdots)$ is everywhere dense on $|z|=1$.*

Lemma 3 ([6], [13]). *Let z be any point of $|z|<1$. We denote its equivalent in D_0 by $\zeta_0(z)$. Let $l(e^{i\theta},\omega)$ $(-\pi/2<\omega<\pi/2)$ be a segment through $e^{i\theta}$, contained in $|z|<1$, making an angle ω with the radius of $|z|=1$ through $e^{i\theta}$. The equivalent of $l(e^{i\theta},\omega)$ in D_0 consists of at most a countable number of arcs which we denote by $l_0(e^{i\theta},\omega)$.*

Then there exists a measurable set E of measure 2π on $|z|=1$ which satisfies either of the following conditions:

i) *If G is of divergence type and $e^{i\theta}\in E$, then $l_0(e^{i\theta},\omega)$ is everywhere dense in D_0 for any ω $(-\pi/2<\omega<\pi/2)$;*

ii) *If G is of convergence type, then $\lim|\zeta_0(z)|=1$, when $z\to e^{i\theta}\in E$ from the inside of any Stolz domain with vertex at $e^{i\theta}$.*

From Lemma 3 we have easily the following

Lemma 4. *If G is of divergence type, then there exists a set E of measure 2π on $|z|=1$ which satisfies the following condition.*

Let $\Delta_0: |z|\leqq\rho$ be a small disc contained in D_0 and Δ_n be its equivalents under G. Let $l(e^{i\theta},\omega)$ $(-\pi/2<\omega<\pi/2)$ be a segment through $e^{i\theta}$ making an angle ω with the radius of $|z|=1$ through $e^{i\theta}$. If $e^{i\theta}\in E$, then $l(e^{i\theta},\omega)$ intersects infinitely many Δ_n for any small $\rho>0$.

Let $\Delta_0=S_n(\Delta_n)$, where $S_n: z'=e^{i\alpha_n}\cdot\dfrac{z-a_n}{1-\bar{a}_n z}\in G$, then $\Delta_n: \left|\dfrac{z-a_n}{1-\bar{a}_n z}\right|\leqq\rho$, and we can easily see that the radius r_n of Δ_n is equal to $\dfrac{\rho(1-|a_n|^2)}{1-|a_n|^2\rho^2}$. If Δ_n and $l(e^{i\theta},\omega)$ intersect each other, we have $|e^{i\theta}-a_n|=O(1-|a_n|)$, so that $r_n=|e^{i\theta}-a_n|\cdot O(\rho)$.

Hence, if we choose $\rho_1>\rho_2>\cdots$, $\rho_n\to 0$, for ρ, then by Lemma 4, for any $e^{i\theta}\in E$, there exist infinitely many $a_n\to e^{i\theta}$, such that $\dfrac{1-a_n e^{-i\theta}}{1-\bar{a}_n e^{i\theta}}\to e^{i\omega'}(-\pi<\omega'<\pi)$. Hence, writing ω instead of ω', we have

Lemma 5. *If G is of divergence type, then there exists a set E of measure 2π on $|z|=1$ which satisfies the following condition: If $e^{i\theta}\in E$, then for any ω $(-\pi<\omega<\pi)$, there exist infinitely many $a_n\to e^{i\theta}$, such that*

$$\frac{1-a_n e^{-i\theta}}{1-\bar{a}_n e^{i\theta}}\to e^{i\omega}, \qquad (-\pi<\omega<\pi).$$

Lemma 6. *Let G be of divergence type and E be a set of measure 2π on $|z|=1$ which is defined in Lemma 5. Let $e^{i\theta}\in E$ and let $S_k: z'=e^{i\gamma_k}\cdot\frac{z-c_k}{1-\bar{c}_k z}\in G$ $(k=1, 2, 3, \cdots)$, such that $c_k\to e^{i\theta}$ in a Stolz domain $\Omega(e^{i\theta}, \omega_0): \left|\arg\left(\frac{1-ze^{-i\theta}}{1-\bar{z}e^{i\theta}}\right)\right|<\omega_0(<\pi)$ and $\gamma_k\to\gamma$. We denote the set of such γ by $M(\omega_0, \theta)$. Then for any $\varepsilon>0$, we can choose $\omega_0=\omega_0(\varepsilon, \theta)<\pi$, such that for any φ in $[0, 2\pi]$, the inequality*

$$|\varphi-\gamma|<\varepsilon \qquad \text{holds for a suitable} \quad \gamma\in M(\omega_0, \theta).$$

Proof. We may assume that $\theta=0$. Let

$$S_m: \quad z' = e^{i\alpha_m}\cdot\frac{z-a_m}{1-\bar{a}_m z} \qquad (m=1, 2, \cdots), \tag{1}$$

$$S_n': \quad z' = e^{i\beta_n}\cdot\frac{z-b_n}{1-\bar{b}_n z} \qquad (n=1, 2, \cdots) \tag{2}$$

belong to G, then $S_n' S_m$ belongs to G, where

$$S_n' S_m: \quad z' = e^{i\gamma_k}\cdot\frac{z-c_k}{1-\bar{c}_k z} \qquad (k=1, 2, \cdots), \tag{3}$$

$$e^{i\gamma_k} = e^{i(\alpha_m+\beta_n)}\cdot\frac{1+\bar{a}_m b_n e^{-i\alpha_m}}{1+a_m\bar{b}_n e^{i\alpha_m}}, \quad c_k = \frac{a_m+b_n e^{-i\alpha_m}}{1+\bar{a}_m b_n e^{-i\alpha_m}}. \tag{3'}$$

By Lemma 5, we may assume that $a_m\to 1$, $\frac{1-a_m}{1-\bar{a}_m}\to 1$ $(m\to\infty)$ and $\alpha_m\to\alpha$. If we fix b_n and let $m\to\infty$, then by (3′), $c_k\to 1$ and $\gamma_k\to\gamma$. Suppose that

$$\frac{1-c_k}{1-\bar{c}_k}\to e^{i\omega} \qquad (|\omega|<\pi), \tag{4}$$

then since

$$\frac{1-c_k}{1-\bar{c}_k} = \frac{(1-a_m)-(1-\bar{a}_m)b_n e^{-i\alpha_m}}{(1-\bar{a}_m)-(1-a_m)\bar{b}_n e^{i\alpha_m}}\cdot\frac{1+a_m\bar{b}_n e^{i\alpha_m}}{1+\bar{a}_m b_n e^{-i\alpha_m}} \to \frac{1-b_n e^{-i\alpha}}{1-\bar{b}_n e^{i\alpha}}\cdot\frac{1+\bar{b}_n e^{i\alpha}}{1+b_n e^{-i\alpha}},$$

we have easily

$$b_n e^{-i\alpha}-\bar{b}_n e^{i\alpha} = (1-|b_n|^2)\cdot\frac{1-e^{i\omega}}{1+e^{i\omega}}. \tag{5}$$

If $b_n=\rho_n e^{i\theta_n}$, then (5) becomes

$$2\rho_n\sin(\theta_n-\alpha) = -(1-\rho_n^2)\tan(\omega/2). \tag{6}$$

Now,

$$2\rho\sin(\theta-\alpha) = -(1-\rho^2)\tan(\omega/2), \quad z=\rho e^{i\theta}, \tag{7}$$

represents a circular arc through $e^{i\alpha}$ and $e^{i(\alpha+\pi)}$, making an angle

$\omega/2$ with the diameter through $e^{i\alpha}$ and $e^{i(\alpha+\pi)}$. This can be proved as follows: We may assume that $\alpha=0$, then if we put $z=x+iy$, (7) becomes

$$x^2+y^2-2y\cot(\omega/2) = 1, \qquad \text{i.e.} \quad x^2+(y-\cot(\omega/2))^2 = \operatorname{cosec}^2(\omega/2),$$

which is a circular arc that passes through $z=1$ and $z=-1$ and makes an angle $\omega/2$ with the real axis.

Hence, it is necessary for (4) that b_n lies on the circular arc (7) and conversely, if b_n lies on (7), then we have $\dfrac{1-c_k}{1-\bar{c}_k} \to e^{i\omega}$.

From (3′), we have

$$e^{i\gamma} = e^{i(\alpha+\beta_n)} \cdot \frac{1+b_n e^{-i\alpha}}{1+\bar{b}_n e^{i\alpha}} = e^{i\beta_n} \cdot \frac{e^{i\alpha}+b_n}{1+\bar{b}_n e^{i\alpha}} = -e^{i\beta_n} \cdot \frac{e^{i(\alpha+\pi)}-b_n}{1-\bar{b}_n e^{i(\alpha+\pi)}}$$
$$= -S_n'(e^{i(\alpha+\pi)}),$$

so that

$$e^{i\gamma} = -S_n'(e^{i(\alpha+\pi)}). \tag{8}$$

Let $c_k \to 1$ in a Stolz domain $\Omega(\omega_0)$: $\left|\arg\left(\dfrac{1-z}{1-\bar{z}}\right)\right| < \omega_0 < \pi$ with vertex at $z=1$ and let $\gamma_k \to \gamma$. Let $M(\omega_0)$ be the set $\{\gamma\}$. Then by (8), $M(\omega_0)$ contains the set $\{-S_n'(e^{i(\alpha+\pi)})\}$ where b_n lies in a domain $D(\omega_0)$ which is bounded by two circular arcs, passing through $e^{i\alpha}$ and $e^{i(\alpha+\pi)}$, and making an angle $\omega_0/2$ and $-\omega_0/2$ with the diameter connecting $e^{i\alpha}$ and $e^{i(\alpha+\pi)}$, respectively. Now, by Lemma 2, if b_n run through all equivalent points of $z=0$, then the set $\{-S_n'(e^{i(\alpha+\pi)})\}$ is everywhere dense on $|z|=1$. While, if $\omega_0 \to \pi$, $D(\omega_0)$ tends to the interior of $|z|=1$. Therefore, for any $\varepsilon>0$, if we choose $\omega_0=\omega_0(\varepsilon)<\pi$ sufficiently near to π, then for any φ in $[0, 2\pi]$, we can find $\gamma \in M(\omega_0)$ such that the inequality $|\varphi-\gamma|<\varepsilon$ holds, q. e. d.

Remark. Since $S_k(0) = -c_k e^{i\gamma_k} = c_k e^{i(\gamma_k+\pi)}$, we see that, for any $\varepsilon>0$, we can choose a suitable $\omega_0<\pi$, for which there exist $c_k \in \Omega(e^{i\theta}, \omega_0)$, $c_k \to e^{i\theta}$, $\gamma_k \to \gamma$, such that $|\varphi-\lim \arg S_k(0)|<\varepsilon$.

Lemma 7. *Let* $\Omega(\omega)$: $\left|\arg\left(\dfrac{1-z}{1-\bar{z}}\right)\right| < \omega < \pi$ *be a Stolz domain whose vertex lies at* $z=1$, *and let* Δ: $\left|\dfrac{z-a}{1-\bar{a}z}\right| \leqq \rho\left(\leqq \dfrac{1}{2}\right)$ *be a disc such that* $a \in \Omega(\omega)$, $1/2 \leqq |a| < 1$, $|a-1| \leqq \cos\omega/2$. *We project* Δ *from* $z=0$ *on* $|z|=1$ *and let* J *be the projection.*

Then J *is contained in an arc* I *on* $|z|=1$ *whose middle point is* $z=1$, *such that*

$$|J|/|I| \geqq \frac{\rho}{6} \cdot \cos(\omega/2) > 0,$$

where $|J|$ denotes the arc length of J. Therefore the parameter of regularity of J with respect to $z=1$ is $\geqq \frac{\rho}{3} \cdot \cos(\omega/2)$ (>0).

Proof. Let $\Delta_0: \left|\frac{z-r}{1-rz}\right| \leqq \rho$, $\left(\frac{1}{2} \leqq r < 1\right)$, be a disc and c be its center and δ be its radius, then

$$c = \frac{r(1-\rho^2)}{1-r^2\rho^2}, \quad \delta = \frac{\rho(1-r^2)}{1-r^2\rho^2}. \tag{1}$$

Hence, if J_0 be the projection of Δ_0 from $z=0$ on $|z|=1$, then

$$|J_0| = 2\sin^{-1}\frac{\delta}{c} = 2\sin^{-1}\frac{\rho(1-r^2)}{r(1-\rho^2)} > \frac{2\rho(1-r^2)}{r(1-\rho^2)} \geqq 4\rho(1-r),$$

so that, if $a = re^{i\varphi}$, then

$$|J| > 4\rho(1-r). \tag{2}$$

If $\varphi > 0$, then we can prove that J is contained in an arc

$$I_0 = \{e^{i\theta};\ |\theta| \leqq \varphi + 4(1-r)\}. \tag{3}$$

For,

$$\varphi + \sin^{-1}\frac{\delta}{c} = \varphi + \sin^{-1}\frac{\rho(1-r^2)}{r(1-\rho^2)} \leqq \varphi + \frac{\pi}{2} \cdot \frac{\rho(1-r^2)}{r(1-\rho^2)}$$

$$\leqq \varphi + \frac{\pi}{4}\frac{(1+1/r)(1-r)}{1-\rho^2} < \varphi + \frac{(1+2)(1-r)}{1-(1/2)^2} = \varphi + 4(1-r). \tag{3'}$$

If $a \in \Omega(\omega)$, $|a-1| \leqq \cos(\omega/2)$, $1/2 \leqq |a| < 1$, then we have easily the following inequality:

$$|1-a| \leqq \frac{2(1-|a|)}{\cos(\omega/2)}.$$

On the other hand,

$$|1-a|^2 = 1+r^2-2r\cos\varphi = (1-r)^2 + 4r\sin^2(\varphi/2) \geqq 4r\sin^2(\varphi/2) \geqq \sin^2(\varphi/2),$$

$$|1-a| \geqq \sin(\varphi/2) \geqq \frac{2}{\pi} \cdot \frac{\varphi}{2} = \frac{\varphi}{\pi}.$$

Hence, we have

$$\varphi \leqq \pi|1-a| \leqq \frac{2\pi(1-|a|)}{\cos(\omega/2)} = \frac{2\pi(1-r)}{\cos(\omega/2)}.$$

Therefore, by (3), J is contained in an arc

$$I(\supseteqq I_0) = \left\{e^{i\theta}\,;\,|\theta| \leqq \left(\frac{2\pi}{\cos(\omega/2)}+4\right)(1-r)\right\}. \tag{4}$$

By (2) and (4), we have

$$|J|/|I| \geqq \rho\frac{\cos(\omega/2)}{\pi+2\cos(\omega/2)} > \frac{\rho}{6}\cdot\cos(\omega/2) > 0\,,$$ q. e. d.

Using Lemma 6 and 7, we can prove the following

Lemma 8. *Let G be a Fuchsian group of divergence type and $\Theta(\theta_0, \varepsilon)$ be a sector domain contained in $|z|<1$, such that*

$$\Theta(\theta_0, \varepsilon) = \{z\,;\,|\arg z-\theta_0|<\varepsilon,\ |z|<1\}$$

and $\{a_n^{\Theta}\}$ be the set of equivalent points of $z=0$ contained in $\Theta(\theta_0, \varepsilon)$, and let $S_n^{\Theta}: z'=e^{i\alpha_n^{\Theta}}\cdot\frac{z-a_n^{\Theta}}{1-\bar{a}_n^{\Theta}z}\in G$.

Let $\Delta: |z|\leqq\rho$ be a small disc contained in D_0 and put $\Delta_n^{\Theta}=S_n^{\Theta}(\Delta)$. We project Δ_n^{Θ} from $z=0$ on $|z|=1$ and let J_n^{Θ} be the projection on $|z|=1$.

Then $\bigcup J_n^{\Theta}$ is a measurable set of measure 2π on $|z|=1$.

Proof. By Lemma 6, there exists a set E of measure 2π on $|z|=1$ satisfying the following condition: For any point $e^{i\theta}\in E$ and for a suitable Stolz domain $\Omega(e^{i\theta}, \omega_0)$ $(\omega_0<\pi)$ whose vertex lies at $e^{i\theta}$, there exist infinitely many points $S_{n_\nu}^{\Theta}(0)$ $(\rightarrow e^{i\theta})$ in $\Omega(e^{i\theta}, \omega_0)$.

Then by Lemma 7, the corresponding $\{J_{n_\nu}^{\Theta}\}$ is *a regular sequence tending to $e^{i\theta}$ in Vitali's sense.*[3] Therefore, *E is covered by $\{J_n^{\Theta}\}$ in Vitali's sense*, and the lemma follows immediately from *Vitali's covering theorem* ([4]), q. e. d.

We shall denote the intersection of E and $\bigcup J_n^{\Theta}$ by $T(\Theta(\theta_0, \varepsilon), \rho)$:

$$T(\Theta(\theta_0, \varepsilon), \rho) \equiv E\frown(\bigcup J_n^{\Theta})\,. \tag{$*$}$$

4. Proof of Main Theorem.

(I) Let G be *of divergence type.* In $(*)$ we put $e^{i\theta_0}=e^{k\pi i/2^{n-1}}$ $(k=0, 1, 2, \cdots, 2^n-1)$, $\varepsilon=\frac{\pi}{2^n}$, and denote briefly $T(n, k, \rho)$ instead of $T\left(\Theta\left(\frac{k\pi}{2^{n-1}}, \rho\right), \frac{\pi}{2^n}\right)$. Put

3) Strictly speaking, $\{(J_{n_\nu}^{\Theta}+e^{i\theta})\}$ is a regular sequence in Vitali's sense and E is covered by $\{(J_n^{\Theta}+e^{i\theta})\}(e^{i\theta}\in E)$ in Vitali's sense. Therefore we may conclude from Vitali's covering theorem that we can choose at most a countable number of closed sets $\{(J_{n_\nu}^{\Theta}+e^{i\theta_\nu})\}$ $(\nu=1, 2, \cdots)$ which cover E except a set of measure zero. Then $\{J_{n_\nu}^{\Theta}\}$ also cover E except a set of measure zero.

$$T(n, \rho) = \bigcap_{k=0}^{2^n-1} T(n, k, \rho), \tag{1}$$

then $T(n, \rho)$ is a set of measure 2π on $|z|=1$. If $n \to \infty$, then $T(n, \rho)$ tends decreasingly to a set $T(\rho)$ of measure 2π;

$$T(1, \rho) \supseteq T(2, \rho) \supseteq \cdots \supseteq T(n, \rho) \supseteq \cdots, \quad T(n, \rho) \to T(\rho), \ (n \to \infty). \tag{2}$$

We take $\rho_1 > \rho_2 > \cdots$, $\rho_\nu \to 0$, for ρ, then $T(\rho_\nu)$ tends decreasingly to a set T^* of measure 2π;

$$T(\rho_1) \supseteq T(\rho_2) \supseteq \cdots \supseteq T(\rho_\nu) \supseteq \cdots, \quad T(\rho_\nu) \to T^* \quad (\nu \to \infty). \tag{3}$$

We shall prove that one may take T^* for E in Main Theorem.

First, we assume that C is a given diameter of $|z|=1$.

Then, by the definition of T^*, for any $e^{i\theta} \in T^*$ we can find $\{n_\nu\}$ such that $\gamma_{n_\nu}(e^{i\theta}) \to C$ $(\nu \to \infty)$, *uniformly.*

This can be proved as follows. If $e^{i\theta} \in T^*$, $e^{i\theta}$ belongs to $T(n, \rho)$ for arbitrary n and ρ. So that, by the definition of $T(n, \rho)$, we can find a set of a_m which are equivalent to $z=0$ under G satisfying the following conditions:

i) Let $S_m: z' = e^{i\alpha_m} \cdot \dfrac{z-a_m}{1-\bar{a}_m z} \in G$ and $\Delta_0: |z| \leqq \rho$, then $S_m^{-1}(\Delta_0) \cap \gamma(e^{i\theta}) \neq \phi$.

ii) If we assume that $e^{i\tau_0}$ and $e^{i(\tau_0+\pi)}$ are two end points of C, then the following inequality holds:

$$|\lim_{m\to\infty} \arg S_m(0) - \tau_0| \leqq \pi/2^{n-1}.$$

Let $S_m(\gamma)$ be the image curve of $\gamma(e^{i\theta})$ by S_m, then $S_m(\gamma)$ is a circular arc orthogonal to $|z|=1$ which starts from $S_m(0)$, intersects Δ_0, and terminates on $|z|=1$. Since n and ρ are arbitrary, we can get the desired conclusion.

Next, we consider the case where C is an arbitrarily given circular arc in $|z|<1$ which intersects $|z|=1$ orthogonally. We can find such a sequence of circular arcs C_m in $|z|<1$ that satisfies the following conditions:

i) $C_m \to C$, *uniformly,* when $m \to \infty$;

ii) Each C_m intersects $|z|=1$ orthogonally at its two end points on $|z|=1$;

iii) Each C_m contains at least one point a_m which is equivalent to $z=0$ under an element S_m of G: $S_m(0)=a_m$, $a_m \in C_m$, $S_m \in G$.

We can approximate C_m by a sequence from $\{\gamma_n(e^{i\theta})\}$. For, since $S_m^{-1}(C_m)$ is a diameter of $|z|=1$, there exist $\gamma_{n_\nu}^{(m)}(e^{i\theta})$ $(\nu=1, 2, \cdots)$ such that

$$\gamma_{n_\nu}^{(m)}(e^{i\theta}) \to S_m^{-1}(C_m) \quad (\nu \to \infty), \quad \textit{uniformly}, \tag{4}$$

consequently

$$\tilde{\gamma}_{n_\nu}^{(m)} \equiv S_m(\gamma_{n_\nu}^{(m)}(e^{i\theta})) \to C_m \quad (\nu \to \infty), \quad \textit{uniformly}. \tag{5}$$

By the above considerations, we can prove by means of a well known diagonal process that C can be approximated by a sequence from $\{\gamma_n(e^{i\theta})\}$.

(II) If G is *of convergence type*, then it is an immediate consequence of Lemma 3 that there exists no measurable set E of measure 2π on $|z|=1$ with the property mentioned in Main Theorem.

Thus we have proved the theorem completely.

5. An application.

As a corollary of Main Theorem, we have the following theorem which can be considered as a precision of the theorem in [12].

Theorem 2. *If G is of divergence type, then there exists a set E of measure 2π on $|z|=1$ which satisfies the following condition. Let*

$$S_n:\ z' = e^{i\alpha_n}\cdot\frac{z-a_n}{1-\bar{a}_n z} \in G \qquad (n=0, 1, 2, \cdots).$$

If we consider the totality of the set $\{a_n\}$ where a_n lies in a Stolz domain $\Omega(e^{i\theta}, \delta)$: $\left|\arg\left(\frac{1-ze^{-i\theta}}{1-\bar{z}e^{i\theta}}\right)\right| < \delta$ whose vertex lies at $e^{i\theta}\in E$, then the corresponding set of α_n is everywhere dense in $[0, 2\pi]$, for any small $\delta > 0$.

Proof. Let L_0 be a diameter of $|z|=1$ through $e^{i\theta}\in E$ and L be any diameter of $|z|=1$. Then, by Main Theorem, we can find n_ν $(\nu=1, 2, 3, \cdots)$, such that $S_{n_\nu}(L_0)\to L$ $(\nu\to\infty)$, so that

$$|S_{n_\nu}(e^{i\theta})-S_{n_\nu}(-e^{i\theta})| \to 2 \qquad (\nu\to\infty). \tag{1}$$

While if $a_{n_\nu}=r_{n_\nu}e^{i\theta_{n_\nu}}$, we have

$$|S_{n_\nu}(e^{i\theta})-S_{n_\nu}(-e^{i\theta})| = \frac{2(1-r_{n_\nu}^2)}{\sqrt{(1-r_{n_\nu}^2)^2+4r_{n_\nu}^2\sin^2(\theta_{n_\nu}-\theta)}}.$$

Hence by (1), one sees

$$|\theta_{n_\nu}-\theta| = o(1-r_{n_\nu}).$$

This means that

$$\lim_{\nu\to\infty} \arg\left(\frac{1-a_{n_\nu}e^{-i\theta}}{1-\bar{a}_{n_\nu}e^{i\theta}}\right) = 0\,. \tag{2}$$

If we assume that L is a diameter through $e^{i\varphi}$, then since

$$S_{n_\nu}(e^{i\theta}) = e^{i(\alpha_{n_\nu}+\theta)}\cdot\frac{1-a_{n_\nu}e^{-i\theta}}{1-\bar{a}_{n_\nu}e^{i\theta}} \to e^{i\varphi}\,,$$

we have

$$\alpha_{n_\nu} \to \varphi-\theta\,. \tag{3}$$

Since φ is an arbitrary point in $[0, 2\pi]$, the set $\{\alpha_{n_\nu}\}$ is everywhere dense in $[0, 2\pi]$, which proves the theorem.

6. Related results and problems.

Let $\eta_1=e^{i\theta}$, $\eta_2=e^{i\varphi}$ be two points on $|z|=1$, then the pair (η_1, η_2) can be considered as a point on a torus Θ;

$$\Theta:\ 0\leqq\theta<2\pi\,,\quad 0\leqq\varphi<2\pi\,.$$

For a measurable set E on Θ, we define its measure $\mu(E)$ by

$$\mu(E) = \iint_E d\theta\, d\varphi\,,\quad \text{so that } \mu(\Theta)=4\pi^2\,.$$

Let S_ν be any substitution of a Fuchsian group G and

$$T_\nu:\ \eta_1' = S_\nu(\eta_1)\,,\quad \eta_2' = S_\nu(\eta_2)\,,$$

then the totality of $\{T_\nu\}$ constitutes a group $\mathcal{G}=G\times G$.

E. Hopf ([1], [2]) proved the following ergodic theorem:

Hopf's ergodic theorem ([1], [2], [9], [10]). *If $\sigma(D_0)<\infty$, then there exists no measurable set E on Θ which is invariant under $\mathcal{G}$ and $0<\mu(E)<4\pi^2$. Hence, if $\mu(E)>0$, then $\mu(E)=4\pi^2$.*

By the same method in [8], we can prove the following proposition.

Proposition. *If Hopf's ergodic theorem holds for a given Fuchsian group G, then Myrberg's approximation theorem also holds for this G.*

By the above proposition and Main Theorem, we have

Theorem 3. *If G is of convergence type, then there exists always a measurable set E on Θ, which is invariant under $\mathcal{G}$ and $0<\mu(E)<4\pi^2$.*

Proof. When G is of convergence type, we see by Main Theorem that Myrberg's approximation theorem does not hold for this G. So that, this theorem follows from the above proposition.

Remark. The late Prof. M. Tsuji gave a direct proof of this theorem, but his proof is yet unpublished.

In conclusion, we propose an unsolved problem:

Problem (*M. Tsuji's conjecture*). *Does Hopf's ergodic theorem hold for G of divergence type?*

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