J. Math. Kyoto Univ. (JMKYAZ)
47-4 (2007), 849–857

Existence and uniqueness of positive periodic solutions for a class of differential delay equations

By

Min-Jei Huang and Duo-Yuan Chen

Abstract

The existence of a unique positive periodic solution for the differential delay equation $x'(t) = -a(t)x(t) + b(t)f(x(t - c(t)))$ is established. Our method is based on Hilbert's projective metric and the contraction mapping principle.

1. Introduction

Much work has been carried out on the existence of positive periodic solutions for various types of functional differential equations (see, e.g., [3], [6], [8] and references therein). One of the most common approaches to a problem of this nature is to use Krasnoselskii's fixed-point theorem ([5]). In this paper we consider the differential delay equation

$$x'(t) = -a(t)x(t) + b(t)f(x(t - c(t))), \tag{1.1}$$

where a, b and c are continuous ω-periodic real-valued functions with $\int_0^\omega a(t)dt > 0$, $\int_0^\omega b(t)dt > 0$ and $b(t) \geq 0$ for $t \in \mathbb{R}$. Under certain conditions on f, we shall show that the equation (1.1) has a unique positive ω-periodic solution. The technique of proof is based on the following observation. Solving (1.1) is equivalent to finding a continuous ω-periodic function $x(t)$ such that

$$x(t) = \int_t^{t+\omega} G(t,s)b(s)f(x(s - c(s)))\,ds, \tag{1.2}$$

where

$$G(t,s) = \frac{e^{\int_t^s a(u)du}}{e^{\int_0^\omega a(u)du} - 1}, \qquad 0 \leq t \leq \omega, \quad t \leq s \leq t + \omega.$$

2000 *Mathematics Subject Classification*(*s*). 34K13; 47H07

Received March 12, 2007

Equation (1.2) is a fixed-point equation for the map

$$(Ax)(t) = \int_t^{t+\omega} G(t,s)b(s)f\left(x(s-c(s))\right)ds.$$

The idea is to introduce Hilbert's projective metric on an appropriate cone in which the contraction mapping principle can be applied.

Much of the work in this paper is motivated by results which appear in Bushell [1] and Potter [7]. In Section 2, we introduce the projective metric and review the basic properties of positive operators. Section 3 contains an estimate of the contraction ratio for the linear integral operator

$$(Lx)(t) = \int_t^{t+\omega} G(t,s)b(s)x(s)ds.$$

The main results are given in Theorems 4.1 and 4.2.

2. Projective metric

Let X be a real Banach space, and let K be a closed solid cone in X, that is, a closed subset K with the properties: (i) $\mathring{K}$, the interior of K, is not empty, (ii) $aK + K \subset K$ for all $a \geq 0$ and (iii) $K \cap (-K) = \{0\}$. Then K defines a reflexive and transitive partial ordering on X by $x \leq y$ if and only if $y - x \in K$.

For $x,\ y \in \mathring{K}$, we let

$$M(x,y) = \inf\{\lambda :\ x \leq \lambda y\} \quad \text{and} \quad m(x,y) = \sup\{\mu :\ \mu y \leq x\}.$$

The Hilbert projective metric is defined in $\mathring{K}$ by

$$d(x,y) = \log \frac{M(x,y)}{m(x,y)}.$$

Lemma 2.1 (Bushell [1]). *For $x,\ y \in \mathring{K}$,*

(a) $0 < m(x,y) \leq M(x,y) < \infty$.

(b) $m(x,y)y \leq x \leq M(x,y)y$.

(c) $d(\lambda x, \mu y) = d(x,y)$ for all $\lambda,\ \mu > 0$.

We define for $r > 0$, $E_r = \left\{x \in \mathring{K} :\ \|x\| = r\right\}$. Then $\{E_r, d\}$ is a metric space for all $r > 0$ ([1]). Moreover,

Lemma 2.2 (Bushell [2], Huang-Huang-Tsai [4]). *Suppose that the norm in X is monotonic, that is, $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$. Then $\{E_r, d\}$ is a complete metric space for all $r > 0$.*

Definition 2.1. Let $T : \mathring{K} \to \mathring{K}$, and let $p \in \mathbb{R}$. We say that

(*a*) T is increasing (respectively, decreasing) if $x, y \in \mathring{K}$ and $x \leq y$ imply $Tx \leq Ty$ (respectively, $Tx \geq Ty$).

(*b*) T is homogeneous of degree p if $T(\lambda x) = \lambda^p Tx$ for all $x \in \mathring{K}$ and $\lambda > 0$.

(*c*) T is p-concave (respectively, p-convex) if $T(\lambda x) \geq \lambda^p Tx$ (respectively, $T(\lambda x) \leq \lambda^p Tx$) for all $x \in \mathring{K}$ and $0 < \lambda < 1$.

If T is homogeneous of degree p, it is clear that T is p-concave and p-convex. Note that T is p-concave (respectively, p-convex) if and only if $T(\mu x) \leq \mu^p Tx$ (respectively, $T(\mu x) \geq \mu^p Tx$) for all $x \in \mathring{K}$ and $\mu > 1$. An important fact is that increasing p-concave and decreasing $(-p)$-convex operators are p-contractions in Hilbert's projective metric.

Theorem 2.1 (Potter [7]). *Let the norm in X be monotonic, and let $p \geq 0$. Suppose that $T : \mathring{K} \to \mathring{K}$ is either increasing p-concave or decreasing $(-p)$-convex. Then*

$$d(Tx, Ty) \leq pd(x, y) \quad \text{for all } x,\ y \in E_r.$$

Definition 2.2. If $T : \mathring{K} \to \mathring{K}$, the projective diameter $\triangle(T)$ of T is defined by

$$\triangle(T) = \sup\left\{d(Tx, Ty) :\ x,\ y \in \mathring{K}\right\},$$

and the contraction ratio $k(T)$ of T is defined by

$$k(T) = \inf\left\{\lambda : d(Tx, Ty) \leq \lambda d(x, y) \text{ for all } x,\ y \in \mathring{K}\right\}.$$

If T is linear, there is a relation between its contraction ratio and its projective diameter.

Theorem 2.2 (Bushell [1]). *If $T : \mathring{K} \to \mathring{K}$ is a linear mapping, then*

$$k(T) = \tanh \frac{\triangle(T)}{4}.$$

3. Preliminaries

Throughout this paper, we shall assume that X is the Banach space of continuous ω-periodic real-valued functions on $\mathbb{R}$ with the norm

$$\|x\| = \sup\{|x(t)| :\ 0 \leq t \leq \omega\},$$

and that K is the closed cone of nonnegative functions in X so that $\mathring{K}$ is the set of positive functions in X. Note that the norm in X is monotonic.

We consider the differential delay equation of the form

$$x'(t) = -a(t)x(t) + b(t)f(x(t - c(t))) \tag{3.1}$$

and suppose that

(H1) $a(t)$, $b(t)$ and $c(t)$ are continuous ω-periodic real-valued functions.

(H2) $\alpha \equiv \int_0^\omega a(t)dt > 0$.

(H3) $\beta \equiv \int_0^\omega b(t)dt > 0$ and $b(t) \geq 0$ for $t \in \mathbb{R}$.

(H4) the function f is continuous and positive.

To study the existence of positive ω-periodic solutions for (3.1), we define the nonlinear integral operator $A : \mathring{K} \to \mathring{K}$ by

$$(Ax)(t) = \int_t^{t+\omega} G(t,s)b(s)f\left(x(s-c(s))\right)ds,$$

where

$$G(t,s) = \frac{e^{\int_t^s a(u)du}}{e^\alpha - 1}, \qquad 0 \le t \le \omega, \quad t \le s \le t+\omega.$$

An elementary computation then shows that the existence of a unique positive ω-periodic solution of (3.1) is equivalent to the existence of a unique fixed point of A in $\mathring{K}$.

Now define $F : \mathring{K} \to \mathring{K}$ by

$$(Fx)(t) = f\left(x(t-c(t))\right) \tag{3.2}$$

and define the linear integral operator $L : \mathring{K} \to \mathring{K}$ by

$$(Lx)(t) = \int_t^{t+\omega} G(t,s)b(s)x(s)ds \tag{3.3}$$

so that $A = L \circ F$.

We shall require some estimates concerning the projective diameter $\triangle(L)$ of L. For this, let

$$a^* = \max_{0\le t\le\omega} a(t) \qquad \text{and} \qquad a_* = \min_{0\le t\le\omega} a(t).$$

Since $\alpha > 0$, we have $a^* > 0$. Let us consider first the case $a_* < 0$. In this case,

$$\frac{e^{a_*\omega}}{e^\alpha - 1} \le G(t,s) \le \frac{e^{a^*\omega}}{e^\alpha - 1} \tag{3.4}$$

for $0 \le t \le \omega$ and $t \le s \le t+\omega$. So, if $x \in \mathring{K}$, we have

$$\frac{e^{a_*\omega}}{e^\alpha - 1}\int_0^\omega b(s)x(s)ds \le (Lx)(t) \le \frac{e^{a^*\omega}}{e^\alpha - 1}\int_0^\omega b(s)x(s)ds$$

for $0 \le t \le \omega$. This implies that

$$M(Lx, x_0) \le \frac{e^{a^*\omega}}{e^\alpha - 1}\int_0^\omega b(s)x(s)ds$$

and

$$m(Lx, x_0) \ge \frac{e^{a_*\omega}}{e^\alpha - 1}\int_0^\omega b(s)x(s)ds,$$

where $x_0(t) \equiv 1$ for all $t \in \mathbb{R}$. It follows from the definition of d that

$$d(Lx, x_0) \le (a^* - a_*)\,\omega \quad \text{for all } x \in \mathring{K},$$

and hence, by the triangle inequality, that

$$d\,(Lx, Ly) \le 2\,(a^* - a_*)\,\omega \quad \text{for all } x,\ y \in \mathring{K}.$$

This shows that

$$\triangle(L) \le 2\,(a^* - a_*)\,\omega.$$

Considering next the case $a_* \ge 0$, we have

$$\frac{1}{e^\alpha - 1} \le G(t,s) \le \frac{e^\alpha}{e^\alpha - 1} \tag{3.5}$$

for $0 \le t \le \omega$ and $t \le s \le t+\omega$. From this, we obtain in the same way as above the result that

$$\triangle(L) \le 2\alpha.$$

Since L is linear, we can now apply Theorem 2.2 to obtain the following lemma.

Lemma 3.1. *Let L be given by* (3.3). *Then the contraction ratio $k(L)$ of L satisfies*

$$k(L) \le \begin{cases} 1/\delta & if\ a_* < 0 \\ 1/\eta & if\ a_* \ge 0, \end{cases}$$

where

$$\delta = \frac{e^{(a^*-a_*)\omega} + 1}{e^{(a^*-a_*)\omega} - 1} \quad \text{and} \quad \eta = \frac{e^\alpha + 1}{e^\alpha - 1}.$$

4. Main results

We consider first the case $f(x) = x^p$ so that (3.1) is now

$$x'(t) = -a(t)x(t) + b(t)\,[x(t - c(t))]^p\,. \tag{4.1}$$

Theorem 4.1. *Suppose that* (H1) *to* (H3) *hold.*

(*a*) *If $a_* < 0$, then the equation* (4.1) *has a unique positive ω-periodic solution provided $-\delta < p < \delta$ and $p \ne 1$.*

(*b*) *If $a_* \ge 0$, then the equation* (4.1) *has a unique positive ω-periodic solution provided $-\eta < p < \eta$ and $p \ne 1$.*

Proof. By our remarks at the beginning of Section 3, the theorem will be proved when it is shown that the operator $A = L \circ F$ has a unique fixed point in $\mathring{K}$. Here, $F : \mathring{K} \to \mathring{K}$ is defined by

$$(Fx)\,(t) = [x(t - c(t))]^p\,,$$

which is homogeneous of degree p.

If $p \ge 0$, then F is increasing. If, on the other hand, $p < 0$, then F is decreasing. In either case, we have

$$d\,(Fx, Fy) \le |p|\,d\,(x,y) \quad \text{for all } x,\ y \in E_1 \tag{4.2}$$

by Theorem 2.1. Now consider the operator $\tilde{A} : E_1 \to E_1$ given by

$$\tilde{A}x = Ax / \|Ax\| .$$

It follows from Lemma 2.1(c), Lemma 3.1 and (4.2) that for all $x,\ y \in E_1$,

$$\begin{aligned} d\left(\tilde{A}x, \tilde{A}y\right) &= d\left(L(Fx), L(Fy)\right) \leq k(L) d\left(Fx, Fy\right) \\ &\leq \left\{ \begin{array}{ll} \frac{|p|}{\delta} d\left(x, y\right) & if\ a_* < 0 \\ \frac{|p|}{\eta} d\left(x, y\right) & if\ a_* \geq 0 \end{array} \right. , \end{aligned}$$

where $|p|/\delta < 1$ and $|p|/\eta < 1$ by our hypotheses on p. Thus, $\tilde{A}$ is a contraction in Hilbert's projective metric. Since $\{E_1, d\}$ is complete, there exists a unique z in E_1 such that $\tilde{A}z = z$. Thus, $Az = \|Az\| z$. If we set $x = \|Az\|^{\frac{1}{1-p}} z$, it follows that x is the unique fixed point of A in K. This proves the theorem. □

Remark 1. Theorem 4.1 may not be true in the linear case $p = 1$. For example, if $b(t) \equiv b > 0$ and $c(t) \equiv 0$ for all t, where $b \neq \alpha/\omega$, then the equation $x'(t) = -a(t)x(t) + bx(t)$ has no positive ω-periodic solution.

We consider next the case where

(H5) $f : [0, \infty) \to (0, \infty)$ is a continuous decreasing function and satisfies

$$f(\lambda x) \leq \lambda^p f(x) \quad \text{for all } x \geq 0 \text{ and } 0 < \lambda < 1.$$

Obviously in (H5) it is necessary that $p \leq 0$. Simple examples of such f are given by $f(x) = (1 + x)^p + (1 + x)^q$, where $p < q \leq 0$ or $f(x) = 1/\log(2 + x)$ for $p = -1$.

Theorem 4.2. *Suppose that* (H1)*,* (H2)*,* (H3) *and* (H5) *hold.*

(a) *If $a_* < 0$, then the equation* (3.1) *has a unique positive ω-periodic solution provided $-\delta < p \leq 0$.*

(b) *If $a_* \geq 0$, then the equation* (3.1) *has a unique positive ω-periodic solution provided $-\eta < p \leq 0$.*

Proof. Since f satisfies condition (H5), the operator F defined by (3.2) is decreasing and p-convex. Thus, by Theorem 2.1,

$$d\left(Fx, Fy\right) \leq |p|\, d\left(x, y\right) \quad \text{for all } x,\ y \in E_r.$$

Now for each $r > 0$, define $A_r : E_r \to E_r$ by

$$A_r x = rAx / \|Ax\| .$$

As in the proof of Theorem 4.1, we find that

$$d\left(A_r x, A_r y\right) \leq \left\{ \begin{array}{ll} \frac{|p|}{\delta} d\left(x, y\right) & if\ a_* < 0 \\ \frac{|p|}{\eta} d\left(x, y\right) & if\ a_* \geq 0 \end{array} \right.$$

for all $x,\ y \in E_r$. So, A_r is a contraction by our hypotheses on p. Since $\{E_r, d\}$ is complete, A_r has a unique fixed point in E_r. Thus, for each $r > 0$, there exists a unique pair $(x_r, \lambda_r) \in E_r \times (0, \infty)$ such that $Ax_r = \lambda_r x_r$, where $\lambda_r = \|Ax_r\| / r$. We now want to prove that there is exactly one r such that $\lambda_r = 1$. This will imply that the operator A has a unique fixed point in $\overset{\circ}{K}$, and so, by our previous remarks, the theorem follows.

Since f is decreasing and $x_r \in E_r$, we have

$$f(r) \leq f\left(x_r(s - c(s))\right) \leq f(0) \quad \text{ for all } s.$$

Thus, by (3.4) and (3.5), we find that

$$\begin{cases} \frac{e^{a_*\omega}}{e^{\alpha}-1} f(r)\beta \leq \|Ax_r\| \leq \frac{e^{a^*\omega}}{e^{\alpha}-1} f(0)\beta & if\ a_* < 0 \\ \frac{1}{e^{\alpha}-1} f(r)\beta \leq \|Ax_r\| \leq \frac{e^{\alpha}}{e^{\alpha}-1} f(0)\beta & if\ a_* \geq 0 \end{cases},$$

where $\beta = \int_0^{\omega} b(t)dt > 0$. In either case, we conclude that

$$\lim_{r \to 0} \lambda_r = \infty \quad \text{and} \quad \lim_{r \to \infty} \lambda_r = 0. \tag{4.3}$$

The continuity of f implies that $A : \overset{\circ}{K} \to \overset{\circ}{K}$ is continuous in the norm topology. By an argument similar to that in the proof of Theorem 3.4 ([4]), we can prove that the mapping $r \mapsto \lambda_r$ is continuous. Moreover, it is strictly decreasing. To prove this, let $r < s$ and for simplicity of notation, let $M = M\left(x_s, x_r\right)$ and $m = m\left(x_s, x_r\right)$. Then $mx_r \leq x_s \leq Mx_r$. Since the norm is monotonic, we have $mr \leq s \leq Mr$ so that $M \geq \frac{s}{r} > 1$. We now show that $m > 1$ also. Suppose on the contrary that $m \leq 1$. Then, since F is decreasing and p-convex,

$$m^p Fx_r \geq F\left(mx_r\right) \geq Fx_s \geq F\left(Mx_r\right) \geq M^p Fx_r.$$

This gives

$$\begin{cases} M\left(Fx_s, Fx_r\right) \leq m^p \\ m\left(Fx_s, Fx_r\right) \geq M^p \end{cases}$$

and so $d\left(Fx_s, Fx_r\right) \leq |p|\, d\left(x_s, x_r\right)$. It follows that

$$d\left(x_s, x_r\right) = d\left(Ax_s, Ax_r\right) \leq k(L) d\left(Fx_s, Fx_r\right) \leq k(L)\,|p|\, d\left(x_s, x_r\right).$$

But $k(L)\,|p| < 1$ and so $d\left(x_s, x_r\right) = 0$. As a result, $M = m$, which contradicts the fact that $M > 1$. Hence $m > 1$ and so $x_r \leq x_s$. Since A is decreasing, we have $\lambda_s x_s = Ax_s \leq Ax_r = \lambda_r x_r$ so that $x_s \leq \left(\lambda_r/\lambda_s\right) x_r$. This gives $\left(\lambda_r/\lambda_s\right) \geq M > 1$ and therefore $\lambda_r > \lambda_s$ as required.

The fact that there is exactly one r such that $\lambda_r = 1$ now follows from (4.3) and the intermediate value theorem. This completes the proof of the theorem. □

Remark 2. (a) The result of Theorem 4.2 is only proved in [6] if $-1 < p \leq 0$. (b) The argument in the proof of Theorem 4.2 cannot be applied directly to the case $f(x) = x^p,\ p < 0$, since $f(0)$ is undefined.

Finally, We consider the case where

(H6) $f:(0,\infty)\to(0,\infty)$ is a continuous increasing function and satisfies

$$f(\lambda x)\ge \lambda^p f(x)\quad \text{for all } x>0 \text{ and } 0<\lambda<1.$$

Obviously in (H6) it is necessary that $p\ge 0$. Then the operator F defined by (3.2) is increasing and p-concave. Since L is linear and positive, it follows that $A=L\circ F$ is increasing and p-concave. By Theorem 3.4 of [4], we have the following result.

Theorem 4.3. *Suppose that* (H1)*,* (H2)*,* (H3) *and* (H6) *hold, where* $0\le p<1$*. Then the equation* (3.1) *has a unique positive* ω*-periodic solution.*

Remark 3. (a) Theorem 4.3 is also proved in [6]. (b) The requirement $0\le p<1$ is critical for Theorem 4.3 to hold. Indeed, the function $f(x)=x$ satisfies condition (H6) for any $p\ge 1$. However, the example in Remark 1 shows that the conclusion of Theorem 4.3 is not true in this linear case.

Department of Mathematics
National Tsing Hua University
Hsinchu 30043
Taiwan
e-mail: mjhuang@math.nthu.edu.tw

References

[1] P. J. Bushell, *Hilbert's metric and positive contraction mappings in a Banach space*, Arch. Rational Mech. Anal. **52** (1973), 330–338.

[2] ______, *The Cayley-Hilbert metric and positive operators*, Linear Algebra Appl. **84** (1986), 271–280.

[3] S. S. Cheng and G. Zhang, *Existence of positive periodic solutions for non-autonomous functional differential equations*, Electron. J. Differential Equations **59** (2001), 1–8.

[4] M.-J. Huang, C.-Y. Huang and T.-M. Tsai, *Applications of Hilbert's projective metric to a class of positive nonlinear operators*, Linear Algebra Appl. **413** (2006), 202–211.

[5] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, Noordhoff, Groningen, 1964.

[6] X.-L. Liu and W.-T. Li, *Existence and uniqueness of positive periodic solutions of functional differential equations*, J. Math. Anal. Appl. **293** (2004), 28–39.

[7] A. J. B. Potter, *Applications of Hilbert's projective metric to certain classes of non-homogeneous operators*, Quart. J. Math. Oxford Ser. (2) **28** (1977), 93–99.

[8] H. Wang, *Positive periodic solutions of functional differential equations*, J. Differential Equations **202** (2004), 354–366.