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On T direction of algebroidal function

By

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Abstract

In this paper, by using the sphere characteristic function $T(r,w)$, we define and establish the existence of a new singular direction for the algebroidal function w, namely a T direction, for which the characteristic function $T(r,w)$ is used as a comparison function. This extended the previous result due to Guo, Zheng and Ng in [Bull. Austral. Math. Soc. 69(2004), 277-287].

1. Introduction and results

Let $w = w(z)$ be a $\nu-$valued algebroidal function defined by the irreducible equation

$$A_\nu(z)w^\nu + A_{\nu-1}(z)w^{\nu-1} + \cdots + A_1(z)w + A_0(z) = 0, \tag{1.1}$$

where $A_j(z)(j = 0, 2, \cdots, \nu)$ are entire functions without any common zeros. The studies of the singular direction for algebroidal function $w(z)$ due to Valiron [1] concerning the Borel directions and the Julia directions, were generalized in 1960's for algebroidal functions, see Toda [2]. Recently, L*ü* Yinian [3], [4] and L*ü*, Yinian, Gu Yongxing [5] proved some more precise versions than Valiron [1] and Toda [2] for the Julia directions and the Borel directions for algebroidal functions. A ray $\arg z = \theta$ is called a Borel direction of order $\rho(0 < \rho < \infty)$ for a $\nu-$valued algebroidal function $w(z)$, if it has the following property: for every $0 < \varepsilon < \pi$,

$$\limsup_{r\to\infty} \frac{\log n(r,\theta,\varepsilon,a)}{\log r} \geq \rho,$$

for all a in $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ with at most 2ν exceptions, where $n(r,\theta,\varepsilon,a)$ is the number of the solution of $w(z) = a$ in $\{z : \theta - \varepsilon < \arg z < \theta + \varepsilon\} \cap \{|z| < r\}$, counting with multiplicities. L*ü* Yinian and Gu Yongxing [5] proved the following.

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Theorem A. *Suppose that $w(z)$ is a $\nu-$valued algebroidal function of order $\rho(0<\rho<\infty)$ defined by* (1.1)*. Then there at least exists a Borel direction of order ρ of $w(z)$.*

Recalling the definition of the Borel direction, this characterization is effective only for the finite and positive order functions. When the order $\rho=0$ or ∞, it is not better to use the order to characterize the growth of w. In this case, Zheng Jianhua [6] considered the T direction which gives another singular direction for the meromorphic function. We follow Zheng's definition. Let $\{z:\theta-\varepsilon<\arg z<\theta+\varepsilon\}$ be a sector. A radial $\arg z=\theta$ is called a T direction of a meromorphic function $f(z)$ provided that for any given $b\in\mathbb{C}_\infty$, with the exceptional values at most two values, for arbitrary small $\varepsilon>0$,

$$\limsup_{r\to\infty}\frac{N(r,\theta,\varepsilon,a)}{T(r,f)}>0,$$

where $N(r,\theta,\varepsilon,a)$ is a integrated counting function which counts the zero points of $f(z)-b$ in $\{z:\theta-\varepsilon<\arg z<\theta+\varepsilon\}$.

Now we give an analogy to the definition of Zheng for the T direction of algebroidal function.

Definition 1.1. A ray $\arg z=\theta$ is called a T direction for a $\nu-$valued algebroidal function $w(z)$, provided that for any $0<\varepsilon<\pi/2$,

$$\limsup_{r\to\infty}\frac{N(r,\theta,\varepsilon,a)}{T(r,w)}>0,$$

holds for any given a in $\mathbb{C}_\infty$ with at most 2ν exceptions.

Recently the existence theorem of the T direction of a meromorphic function is established, see Guo H., Zheng J. H. and T. W., Ng [7]. It is shown that any meromorphic function $f(z)$ has at least one T direction provided that $\limsup_{r\to\infty} T(r,f)/(\log r)^2=+\infty$. Note that we have an example due to Ostrovskii [8]. Namely there is a transcendental meromorphic function such that $T(r,f)=O(\log^2 r)$ and which has no T directions (and no Julia direction). In this note, we consider a generalization of the T directions for an algebroidal function and state the main results here.

Theorem 1.1. *Let $w(z)$ be a $\nu-$valued algebroidal function on the whole complex plane defined by* (1.1)*. If*

$$\limsup_{r\to\infty}\frac{T(r,w)}{(\log r)^2}=+\infty, \tag{1.2}$$

then $w(z)$ has at least one T direction.

2. Notations and lemmas

Suppose that $w=w(z)$ is a $\nu-$valued algebroidal function defined by the expression (1.1) on the whole complex plane. Now, we give some standard notations and fundament results which can be found in [9]. The single valued domain of definition of $w(z)$ is a ν sheets covering of z plane, a Riemann surface, denoted by $\widetilde{R}_z$. It is denoted by $\widetilde{z}$ that the point in $\widetilde{R}_z$ whose projection in the z plane is z. The part of $\widetilde{R}_z$, which covers a disk $|z| < r$, is denoted by $|\widetilde{z}| < r$. Write

$$S(r,w) = \frac{1}{\pi}\int\int_{|\widetilde{z}|<r}\left(\frac{|w'(z)|}{1+|w(z)|^2}\right)^2 dw, \qquad T(r,w) = \frac{1}{\nu}\int_0^r \frac{S(t,w)}{t}dt.$$

$S(r,w)$ is called the mean covering number of $|\widetilde{z}| \leq r$ into w sphere under the mapping $w = w(z)$, $T(r,w)$ is called the characteristic function of $w(z)$. The order of algebroidal function $w(z)$ is denoted by

$$\rho = \limsup_{r\to\infty} \frac{\log T(r,w)}{\log r}.$$

Put

$$N(r,a) = \frac{1}{\nu}\int_0^r \frac{n(t,a)-n(0,a)}{t}dt + \frac{n(0,a)}{\nu}\log r,$$
$$m(r,w) = \frac{1}{2\pi\nu}\int_{|\widetilde{z}|=r} \log^+ |w(re^{i\theta})|d\theta, \quad z = re^{i\theta},$$

where $n(r,a)$ is the number of zeros, counted according to their multiplicities, of $w(z) - a$ in $|\widetilde{z}| \leq r$. We have

$$T(r,w) = m(r,w) + N(r,\infty) + O(1).$$

Let $n(r,\widetilde{R}_z)$ be the number of the branch points of $\widetilde{R}_z$ in $|\widetilde{z}| \leq r$, counted with the order of branch. Denote

$$N(r,\widetilde{R}_z) = \frac{1}{\nu}\int_0^r \frac{n(t,\widetilde{R}_z)-n(0,\widetilde{R}_z)}{t}dt + \frac{n(0,\widetilde{R}_z)}{\nu}\log r.$$

By [9],

$$N(r,\widetilde{R}_z) \leq 2(\nu-1)T(r,w) + O(1). \tag{2.1}$$

We define an angular domain $\triangle(\theta_0,\delta) = \{z||\arg z - \theta_0| < \delta\}, 0 \leq \theta_0 < 2\pi, 0 < \delta < \frac{\pi}{2}$. The part of $\widetilde{R}_z$ which lies over $\triangle(\theta_0,\delta)$ is denoted by $\widetilde{\triangle}(\theta_0,\delta)$. Let $n(r,\theta_0,\delta,a)$ be the number of $w(z) - a$ in $\widetilde{\triangle}(\theta_0,\delta)\bigcap\{|\widetilde{z}| \leq r\}$ and let $n(r,\triangle(\theta_0,\delta),\widetilde{R}_z)$ be the number of the branch points in the same region. Put

$$N(r,\theta_0,\delta,a) = \frac{1}{\nu}\int_0^r \frac{n(t,\theta_0,\delta,a)}{t}dt$$
$$N(r,\triangle(\theta_0,\delta),\widetilde{R}_z) = \frac{1}{\nu}\int_0^r \frac{n(r,\triangle(\theta_0,\delta),\widetilde{R}_z)}{t}dt$$

In order to prove the Theorem 1.1, we give some lemmas as following.

Lemma 2.1. *Let $S(r)$ be a positive continuous non-decreasing function of r in $[0,+\infty)$. Suppose that*

$$\liminf_{r\to\infty}\frac{\log S(r)}{\log r}=\mu<+\infty,$$
$$\limsup_{r\to\infty}\frac{S(r)}{\log^2 r}=+\infty.$$

Then for any $h>0$, there exist the sequences $\{r_n\}$ and $\{R_n\}$, $R_n^{1-o(1)}\le r_n\le R_n(n\to\infty)$, satisfying

$$\lim_{n\to\infty}\frac{S(r_n)}{\log^2 r_n}=+\infty,\quad S(e^hR_n)\le e^{h\mu}S(R_n)(1+o(1))(n\to\infty).$$

Lemma 2.1 can be found in [10] and the following Lemma 2.2 due to [3].

Lemma 2.2. *Suppose that $w(z)$ is a $\nu-$valued algebroidal function in an angular domain $\triangle_0=\{z:|\arg z-\theta|<\delta_0\}$. Let $\Omega=\{z:|\arg z-\theta|\le\delta\}$ be an angular domain, contained in $\triangle_0$, where $\theta\in[0,2\pi)$ and $0<\delta\le\delta_0$. The part of $\widetilde{R}_z$ which lies over Ω is denoted by $\widetilde{\Omega}$. Let*

$$S(r,\Omega,w)=\frac{1}{\pi}\int\int_{\widetilde{\Omega}}\left(\frac{|w'(z)|}{1+|w(z)|^2}\right)^2dw,$$

and $a_1,a_2,\cdots,a_q$ $(q>2)$ be q distinct points on w sphere $\mathbb{C}_\infty$. Then for arbitrarily constant $\lambda>1$ and positive integer α, we have

$$(q-2)S(r,\Omega,w)\le 2\sum_{j=1}^{q}n(\lambda^{2\alpha}r,\triangle_0,a_j)+(1+\frac{1}{\alpha})n(\lambda^{2\alpha}r,\triangle_0,\widetilde{R}_z)$$
$$+(q-2)S(\lambda^{2\alpha},\Omega,w)+\frac{2A}{(1-k)\log\lambda}\log^+ r.$$

where A is a constant depending only on $a_1,a_2,\cdots,a_q$, and $k(0<k<1)$ depending only on δ,δ_0,α and λ.

Lemma 2.3. *Let $B(r)$ be a positive and continuous function in $[0,+\infty)$ which satisfies $\limsup\limits_{r\to\infty}\frac{\log B(r)}{\log r}=\infty$. Then there exist continuously differentiable functions $\rho(r)$ and $U(r)$, which satisfy the following conditions.*

1. *$\rho(r)\downarrow 0$ and $\rho^{'}(r)$ monotone increasing.*
2. *$\lim\limits_{r\to\infty}r\rho^{'}(r)\log r\log\log r=0$.*
3. *For sufficient large r, we have $B(r)\ll U(r)=r^{exp(\frac{1}{\rho(r)})}$, where " $\ll$ " note that $B(r)\le U(r)$ and there is a sequence $\{r_n\}\to\infty$, such that $B(r_n)=U(r_n)$.*
4. *$U(R)<(1+o(1))U(r)$, where $R=r+\frac{r\log r}{\log U(r)\log^2\log U(r)}$.*

A proof of Lemma 2.3 can be found in [11] and the following Lemma 2.4 due to [2].

Lemma 2.4. *Let $w(z)$ be a $\nu-$valued algebroidal function defined by (1.1) in $|z|<1$ and $\{a_1,a_2,\cdots,a_q\}$ be $q(>2)$ distinct complex numbers. Put $\sum_{j=1}^{q} n(1,a_j)<\infty$, $n(1,\widetilde{R}_z)<\infty$. Then*

$$(q-2)S(r,w)\leq\sum_{j=1}^{q} n(1,a_j)+n(1,\widetilde{R}_z)+\frac{A}{1-r},$$

where $0<r<1$ and A is a constant depending only upon $\{a_1,a_2,\cdots,a_q\}$.

Lemma 2.5. *Suppose that $w(z)$ is a $\nu-$valued algebroidal function defined by (1.1) in the sector $\Omega(\psi_1,\psi_2)=\{z:\psi_1<\arg z<\psi_2\}(\psi_1<\psi_2)$, continuously differentiable functions $\rho(r)$ and $U(r)$ satisfy the condition 1, 2, 4 that stated in Lemma 2.3, $T(r,\Omega(\psi_1,\psi_2),w)\leq U(r)=r^{\exp(\frac{1}{\rho(r)})}$ and $\{a_1,a_2,\cdots,a_q\}$ be $q(\geq 2)$ distinct points. Then for arbitrary $\psi,\delta',\delta(0<\delta'<\delta,\psi_1<\psi-\delta<\psi-\delta'<\psi_2)$ we have*

$$\begin{aligned}(q-2)T(r,\Omega(\psi-\delta',\psi+\delta'),w)\leq&\sum_{j=1}^{q} N(R,\Omega(\psi-\delta,\psi+\delta),a_j)\\&+N(R,\Omega(\psi-\delta,\psi+\delta),\widetilde{R}_z)+o(U(r)).\end{aligned}$$

Proof. A proof of Lemma 2.3 can be found in [12]. For the sake of convenience, here we only give an outline of the proof. Put

$$f(z)=\frac{(ze^{-i\psi})^{\frac{\pi}{\delta}}+2(ze^{-i\psi})^{\frac{\pi}{2\delta}}R^{\frac{\pi}{2\delta}}-R^{\frac{\pi}{\delta}}}{(ze^{-i\psi})^{\frac{\pi}{\delta}}-2(ze^{-i\psi})^{\frac{\pi}{2\delta}}R^{\frac{\pi}{2\delta}}-R^{\frac{\pi}{\delta}}}.$$

It is easy to verify that $f(z)$ maps conformally the sector $E:\{|z|<R\}\cap\{|\arg z-\psi|\leq\delta\}$ into the unit disc $|f(z)|<1$.

Put $F:=\{r_0\leq|z|\leq r\}\cap\{|\arg z-\psi|\leq\delta'\}$, $M=\max\{\frac{1}{1-|f(z)|}:z\in F\}$. We can prove that M is bounded by using the argument adopted by Sun in Lemma 7 of [11], and hence we omit the details. Since M is bounded, we know that $f(z)$ maps conformally F into some region in unit disc. By applying Lemma 2.4,

$$\begin{aligned}&(q-2)S(r,\Omega(\psi-\delta',\psi+\delta'),w)\\&\leq\sum_{j=1}^{q} n(R,\Omega(\psi-\delta,\psi+\delta),a_j)+n(R,\Omega(\psi-\delta,\psi+\delta),\widetilde{R}_z)\\&\quad+O(R^{\frac{3\pi}{\delta}}\log^2 U(r)/\log r).\end{aligned}$$

Since $U(r)=r^{\exp(\frac{1}{\rho(r)})}$, $R=r+\frac{r\log r}{\log U(r)\log^2\log U(r)}$, we have

$$dR=\left[1+\frac{1+o(1)}{\exp(\frac{1}{\rho(r)})\log^2\log U(r)}-\frac{o(1)}{\exp(\frac{1}{\rho(r)})\log r}\right]dr,$$

$$\frac{dR}{R}=\left[1+\frac{o(1)}{\exp(\frac{1}{\rho(r)})}\right]\frac{dr}{r}=(1+o(1)\rho(r))\frac{dr}{r},$$

and

$$\begin{aligned}
&(q-2)\left(1-\rho\left(\frac{1}{2}r\right)\right)\int_{\frac{1}{2}r}^{r}\frac{S(r,\Omega(\psi-\delta^{'},\psi+\delta^{'}),w)}{r}dr\\
&\leq (q-2)\int_{\frac{1}{2}r}^{r}\frac{S(r,\Omega(\psi-\delta^{'},\psi+\delta^{'}),w)}{r}(1+o(1)\rho(r))dr\\
&\leq (q-2)\int_{0}^{r}\frac{S(r,\Omega(\psi-\delta^{'},\psi+\delta^{'}),w)}{R}dR\\
&<\sum_{j=1}^{q}\int_{0}^{R}\frac{n(R,\Omega(\psi-\delta,\psi+\delta),a_j)}{R}dR+\int_{0}^{R}\frac{n(R,\Omega(\psi-\delta,\psi+\delta),\widetilde{R}_z)}{R}dR\\
&\quad+O(r^{\frac{2\pi}{\delta}}\log^2 U(r))\\
&=\sum_{j=1}^{q}N(R,\Omega(\psi-\delta,\psi+\delta),a_j)+N(R,\Omega(\psi-\delta,\psi+\delta),\widetilde{R}_z)\\
&\quad+O(r^{\frac{2\pi}{\delta}}\log^2 U(r)).
\end{aligned}$$

Hence

$$\begin{aligned}
&(q-2)T(r,\Omega(\psi-\delta^{'},\psi+\delta^{'}),w)\\
&\quad\leq\sum_{j=1}^{q}N(R,\Omega(\psi-\delta,\psi+\delta),a_j)+N(R,\Omega(\psi-\delta,\psi+\delta),\widetilde{R}_z)\\
&\quad\quad+O(r^{\frac{2\pi}{\delta}}\log^2 U(r))+(q-2)\rho(\frac{r}{2})T(\frac{r}{2},\Omega(\psi-\delta^{'},\psi+\delta^{'}),w)\\
&\quad\leq\sum_{j=1}^{q}N(R,\Omega(\psi-\delta,\psi+\delta),a_j)\\
&\quad\quad+N(R,\Omega(\psi-\delta,\psi+\delta),\widetilde{R}_z)+o(U(r)+qU(\frac{r}{2})).
\end{aligned}$$

By $U(\frac{r}{2})\leq (1/2)^{\exp(\frac{1}{\rho(\frac{r}{2})})}U(r)=o(1)U(r)$. Combining the above two inequalities Lemma 2.5 follows. □

Lemma 2.6. *Let $w(z)$ be a $\nu-$valued algebroidal function defined by* (1.1) *on the whole complex plane and satisfies* (1.2). *Let $m(m\geq 4)$ be a positive integer, $\theta_0=0,\theta_1=\frac{2\pi}{m},\cdots,\theta_{m-1}=\frac{(m-1)2\pi}{m},\theta_m=\theta_0$, and $\triangle(\theta_i)=\{z||\arg z-\theta_i|<\frac{2\pi}{m}\}, i=0,1,\cdots,m-1;\triangle(\theta_m)=\triangle(\theta_0)$. Then among these m angular domains $\{\triangle(\theta_i)\}$, there is at least an angular domain $\triangle(\theta_i)$ such that the relative expression*

$$\limsup_{r\to\infty}\frac{N(r,\triangle(\theta_i),a)}{T(r,w)}>0,$$

holds for all $a\in\mathbb{C}_\infty$ with at most 2ν exceptions.

Proof. We need to consider two different cases.

Case 1. Suppose that $\liminf_{r\to\infty} \frac{\log T(r,w)}{\log r} = \mu < +\infty$. Suppose the lemma 2.6 does not hold. Then for any angular domain $\triangle(\theta_i)(1 \le i \le m-1)$, we have $q = 2\nu + 1$ distinct points $a_i^j (j = 1, 2, \cdots, q)$ in $\mathbb{C}_\infty$ such that

$$\sum_{i=0}^{m-1} \sum_{j=1}^{q} N(r, \triangle(\theta_{i+1}), a_{i+1}^j) = o(T(r, w)). \tag{2.2}$$

Let α be arbitrary positive integer. Put

$$\theta_{i,k} = \frac{2\pi i}{m} + \frac{2\pi k}{\alpha m}, 0 \le i \le m-1, 0 \le k \le \alpha - 1, \theta_{i,0} = \theta_i.$$

For sufficient large r, let

$$\triangle_{i,k} = \{z||z| < \lambda^{2\alpha} r, \theta_{i,k} \le \arg z < \theta_{i,k+1}\},$$

where $\lambda > 1$. Then

$$\{|z| < \lambda^{2\alpha} r\} = \sum_{k=0}^{\alpha-1} \sum_{i=0}^{m-1} \triangle_{i,k},$$

Hence there must be one $k_0 (0 \le k_0 \le \alpha - 1)$, such that

$$\sum_{i=0}^{m-1} n(\triangle_{i,k_0}, \widetilde{R}_z) \le \frac{1}{\alpha} n(\lambda^{2\alpha} r, \widetilde{R}_z).$$

Define the angular domains

$$\Omega_i = \left\{ z| \frac{\theta_{i,k_0} + \theta_{i,k_0+1}}{2} \le \arg z \le \frac{\theta_{i+1,k_0} + \theta_{i+1,k_0+1}}{2} \right\},$$
$$\triangle_i^0 = \{z|\theta_{i,k_0} < \arg z < \theta_{i+1,k_0+1}\} \subset \triangle(\theta_{i+1}).$$

Since $\triangle_i^0$ only covers $\triangle_{i,k_0}$ twice, we have

$$\sum_{i=0}^{m-1} n(\lambda^{2\alpha} r, \triangle_i^0; \widetilde{R}_z) \le \left(1 + \frac{1}{\alpha}\right) n(\lambda^{2\alpha} r, \widetilde{R}_z). \tag{2.3}$$

Applying Lemma 2.2 to $\triangle_i^0, \Omega_i$, we have

$$(q-2)S(r, \Omega_i, w) \le 2 \sum_{j=1}^{q} n(\lambda^{2\alpha} r, \triangle_i^0, a_{i+1}^j) + \left(1 + \frac{1}{\alpha}\right) n(\lambda^{2\alpha} r, \triangle_i^0, \widetilde{R}_z)$$
$$+ (q-2)S(\lambda^{2\alpha}, \Omega_i, w) + \frac{2A}{(1-k)\log \lambda} \log^+ r.$$

Since $S(r,w)=\sum\limits_{i=0}^{m-1} S(r,\Omega_i,w)$. Adding both sides of the above expression from i to $m-1$, we can obtain

$$(q-2)S(r,w) \\ \leq 2\sum_{i=0}^{m-1}\sum_{j=1}^{q} n(\lambda^{2\alpha}r,\triangle(\theta_{i+1}),a_{i+1}^{j}) + \left(1+\frac{1}{\alpha}\right)^2 n(\lambda^{2\alpha}r,\widetilde{R}_z)+O(\log r).$$

Divided both sides of the above expression by r, and then integrating both sides from 1 to r, thus we obtain

$$(q-2)T(r,w) \leq 2\sum_{i=0}^{m-1}\sum_{j=1}^{q} N(\lambda^{2\alpha}r,\triangle(\theta_{i+1}),a_{i+1}^{j}) \\ + \left(1+\frac{1}{\alpha}\right)^2 N(\lambda^{2\alpha}r,\widetilde{R}_z)+O(\log^2 r).$$

Applying (2.1) and (2.2) to the above inequality shows that

$$(2.4)\quad (q-2)T(r,w) \leq \left(2\left(1+\frac{1}{\alpha}\right)^2(\nu-1)+o(1)\right)T(\lambda^{2\alpha}r,w)+O(\log^2 r).$$

By the hypothesis and applying Lemma 2.1 to $T(r,w)$, we have

$$\lim_{n\to\infty}\frac{T(r_n,w)}{\log^2 r_n}=+\infty,\quad T(\lambda^{2\alpha}R_n,w)\leq \lambda^{2\alpha\mu}T(R_n,w),$$

and where $R_n^{1-o(1)} \leq r_n \leq R_n (n\to\infty)$. From this we can obtain

$$\lim_{n\to\infty}\frac{T(R_n,w)}{\log^2 R_n}=+\infty.$$

In (2.4), we let $r=R_n$ and obtain

$$(q-2)T(R_n,w) \leq \left(2\left(1+\frac{1}{\alpha}\right)^2(\nu-1)+o(1)\right)\lambda^{2\alpha\mu}T(R_n,w)+O(\log^2 R_n).$$

Hence

$$q-2 \leq 2(1+\frac{1}{\alpha})^2(\nu-1)\lambda^{2\alpha\mu}.$$

By a simple calculation, we can obtain that $q-2\leq 2(\nu-1)$. This contradicts $q=2\nu+1$.

Case 2. Suppose that $\liminf\limits_{r\to\infty}\frac{\log T(r,w)}{\log r}=+\infty$. That is to say $w(z)$ is an infinite order function. By Lemma 2.3, there exists $U(r)$ satisfying the conditions that stated in Lemma 2.3. We can assert that there is at least an angular domain $\triangle(\theta_i)$ such that the relative expression

$$\limsup_{r\to\infty}\frac{N(r,\triangle(\theta_i),a)}{U(r)}>0, \tag{2.5}$$

holds for all $a\in\mathbb{C}_\infty$ with at most 2ν exceptions. In fact, if this is not the case, then for any angular domain $\triangle(\theta_i)(1\le i\le m-1)$, we have $q=2\nu+1$ distinct points $a_i^j\in\mathbb{C}_\infty(j=1,2,\cdots,q)$ such that for any i and j,

$$N(r,\triangle(\theta_{i+1}),a_{i+1}^j)=o(U(r)). \tag{2.6}$$

Let α be arbitrary positive integral, put

$$\theta_{i,k}=\frac{2\pi i}{m}+\frac{2\pi k}{\alpha m},0\le i\le m-1,0\le k\le \alpha-1,\theta_{i,0}=\theta_i,$$

and $\triangle_{i,k}=\{z||z|<R,\theta_{i,k}\le \arg z<\theta_{i,k+1}\}$. Then $\{|z|<R\}=\sum\limits_{k=0}^{\alpha-1}\sum\limits_{i=0}^{m-1}\triangle_{i,k}$. Hence there must be one $k_0(0\le k_0\le\alpha-1)$, such that

$$\sum_{i=0}^{m-1}n(\triangle_{i,k_0},\widetilde{R}_z)\le\frac{1}{\alpha}n(R,\widetilde{R}_z).$$

Let the angular domain

$$\Omega_i=\left\{z|\frac{\theta_{i,k_0}+\theta_{i,k_0+1}}{2}\le\arg z\le\frac{\theta_{i+1,k_0}+\theta_{i+1,k_0+1}}{2}\right\},$$
$$\triangle_i^0=\{z|\theta_{i,k_0}<\arg z<\theta_{i+1,k_0+1}\}\subset\triangle(\theta_{i+1}).$$

Since $\triangle_i^0$ only cover $\triangle_{i,k_0}$ twice, we have

$$\sum_{i=0}^{m-1}n(R,\triangle_i^0,\widetilde{R}_z)\le\left(1+\frac{1}{\alpha}\right)n(R,\widetilde{R}_z),$$

by a simple calculation, we can obtain that

$$\sum_{i=0}^{m-1}N(R,\triangle_i^0,\widetilde{R}_z)\le\left(1+\frac{1}{\alpha}\right)N(R,\widetilde{R}_z)+O(1).$$

Applying Lemma 2.5 to $\triangle_i^0,\Omega_i$, we have

$$(q-2)T(r,\Omega_i,w)\le\sum_{j=1}^{q}N(R,\triangle_i^0,a_i^j)+N(R,\triangle_i^0,\widetilde{R}_z)+o(U(r)),$$

Adding both sides of the above expression from $i=0$ to $m-1$, applying (2.6), we can further obtain that

$$(q-2)T(r,w)\le\left(1+\frac{1}{\alpha}\right)N(R,\widetilde{R}_z)+o(U(r)).$$

From (2.1),

$$(q-2)T(r,w) \le 2(\nu-1)\left(1+\frac{1}{\alpha}\right)T(R,w)+o(U(r)).$$

Furthermore, we can have

$$\begin{aligned}(q-2)\frac{T(r,w)}{U(r)} &\le 2(\nu-1)\left(1+\frac{1}{\alpha}\right)\frac{T(R,w)}{U(r)}+\frac{o(U(r))}{U(r)}\\ &= 2(\nu-1)\left(1+\frac{1}{\alpha}\right)\frac{T(R,w)}{U(R)}\frac{U(R)}{U(r)}+\frac{o(U(r))}{U(r)}\end{aligned}$$

From Lemma 3, we have $q-2 \le 2(\nu-1)(1+\frac{1}{\alpha})$. Letting $\alpha \to \infty$, we have $q-2 \le 2(\nu-1)$. This contradicts $q = 2\nu+1$ and hence (2.5) follows. Furthermore, we have

$$\begin{aligned}\limsup_{r\to\infty}\frac{N(r,\triangle(\theta_i),a)}{T(r,w)} &= \limsup_{r\to\infty}\frac{N(r,\triangle(\theta_i),a)}{U(r)}\frac{U(r)}{T(r,w)}\\ &\ge \limsup_{r\to\infty}\frac{N(r,\triangle(\theta_i),a)}{U(r)}\liminf_{r\to\infty}\frac{U(r)}{T(r,w)} > 0\end{aligned}$$

and hence Lemma 2.6 holds. □

3. The proof of the Theorem 1.1

Proof. By Lemma 2.6, for arbitrary positive integer m, there exists an angular domain $\triangle(\theta_m) = \{z||\arg z-\theta_m| < \frac{2\pi}{m}\}$ such that for any a, we have

$$\limsup_{r\to\infty}\frac{N(r,\triangle(\theta_m),a)}{T(r,w)} > 0, \tag{3.1}$$

except for 2ν exceptions at most. Choosing subsequence of $\{\theta_m\}$, still denote it $\{\theta_m\}$, we assume that $\theta_m \to \theta_0$. Put $L : \arg z = \theta_0$. Then L is the T direction of Theorem 1.1.

In fact, for any $\delta(0 < \delta < \pi/2)$, when m is sufficiently large, we have $\triangle(\theta_m) \subset \triangle(\theta_0,\delta)$. By (3.1), we obtain

$$\limsup_{r\to\infty}\frac{N(r,\theta_0,\delta,a)}{T(r,w)} \ge \limsup_{r\to\infty}\frac{N(r,\triangle(\theta_m),a)}{T(r,w)} > 0$$

at most 2ν exceptions for a. Hence Theorem 1.1 holds in this case. □

4. Examples of T direction of algebroidal function

In order to give an example of T direction of algebroidal function, we firstly prove the following Theorem 4.1.

Theorem 4.1. *Let $w(z)$ be a $\nu-$valued algebroidal function on the whole complex plane defined by the following irreducible equation*

$$g(z)w^{\nu} - h(z) = 0, \tag{4.1}$$

where $g(z)(\not\equiv 0), h(z)$ are entire functions without any common zeros. Let $f(z) = \frac{h(z)}{g(z)}$. Suppose that $L : \arg z = \theta$ is a T direction of $f(z)$. Then L must be a T direction of $w(z)$.

Proof. We can follows from (4.1) that

$$N(r, w) = \frac{1}{\nu} N(r, g(z) = 0) = \frac{1}{\nu} N(r, f);$$

$$m(r, w) = \frac{1}{2\pi\nu} \sum_{i=1}^{\nu} \int_0^{2\pi} \log^+ \sqrt[\nu]{|f(re^{i\theta})|} d\theta = \frac{1}{\nu} m(r, f).$$

Hence $T(r, w) = \frac{1}{\nu} T(r, f)$. Suppose that $L : \arg z = \theta$ is a T direction of $f(z)$. Then for any $\varepsilon > 0$, we have

$$\limsup_{r\to\infty} \frac{N(r, \theta, \varepsilon, f = a)}{T(r, f)} > 0$$

holds for any value a, except for 2 exceptions at most. Since for any $b \in \mathbb{C}_\infty$, we have

$$N(r, \theta, \varepsilon, w^{\nu} = b) = N(r, \theta, \varepsilon, f = b).$$

Then

$$\limsup_{r\to\infty} \frac{N(r, \theta, \varepsilon, w^{\nu} = b)}{T(r, w)} > 0,$$

holds for any value b, except for 2 exceptions at most. Therefore,

$$\limsup_{r\to\infty} \frac{N(r, \theta, \varepsilon, w = \sqrt[\nu]{b})}{T(r, w)} > 0,$$

holds for any value b, except for 2 exceptions at most. Let $b_0 \in \mathbb{C}_\infty$ be an exception, then there exist ν values $x_i, i = 1, 2, \cdots, \nu$ such that $x_i^{\nu} = b_0$ and

$$\limsup_{r\to\infty} \frac{N(r, \theta, \varepsilon, w = x_i)}{T(r, w)} = 0, i = 1, 2, \cdots, \nu.$$

Therefore,

$$\limsup_{r\to\infty} \frac{N(r, \theta, \varepsilon, w = a)}{T(r, w)} > 0,$$

holds for any value a, except for 2ν exceptions at most i.e. L is a T direction of $w(z)$. □

We are now in the position to construct an example of T direction for algebroidal function.

Example 4.1. Let

$$f(z)=\frac{\prod_{n=0}^{+\infty}(1-\frac{z}{e^{\sqrt{n}}})}{\prod_{n=0}^{+\infty}(1+\frac{z}{e^{\sqrt{n}}})}=\prod_{n=0}^{+\infty}\frac{e^{\sqrt{n}}-z}{e^{\sqrt{n}}+z}.$$

Since the exponent of convergence of sequence $\{e^{\sqrt{n}}\}$ is zero (In fact, for any $\varepsilon>0$, $\Sigma(e^{\sqrt{n}})^{-\varepsilon}$ converges), and $\Sigma(e^{\sqrt{n}})^{-1}$ converges, using a theorem in J. K. Langley [15, p35], we have $\prod_{n=0}^{+\infty}(1-\frac{z}{e^{\sqrt{n}}})$ and $\prod_{n=0}^{+\infty}(1-\frac{z}{e^{\sqrt{n}}})$ converges. Hence $\prod_{n=0}^{+\infty}\frac{e^{\sqrt{n}}-z}{e^{\sqrt{n}}+z}$ converges. It follows from Sauer [13] that, $T(r,f(z))=(\frac{1}{3}+o(1))\log^3 r$. Suppose that $f(z)=\frac{h(z)}{g(z)}$, where $g(z),h(z)$ are entire functions, for any $\nu\geq 2$,

$$g(z)w^{\nu}-h(z)=0,$$

give a $\nu-$valued algebroidal function $w(z)$. By the proof of Theorem 4.1, we have $w(z)$ is of zero order of growth and satisfies (1.2). It follows from Zheng [14] that, $f(z)$ has exactly two T directions, the negative and positive imaginary axis. Using Theorem 4.1, the negative and positive imaginary axis are T directions of algebroidal function $w(z)$. In fact, we can derive that there is no T direction other than the two directions on imaginary axis. The Mobius transformations $\frac{e^{\sqrt{n}}-z}{e^{\sqrt{n}}+z}$ map the right half plane to the interior of the unit disk and the left half plane to the exterior of the unit disk. Hence there is no T direction of $f(z)$ other than the two directions on imaginary axis.

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