Deformation theory of rigid-analytic spaces

By

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Introduction

The purpose of this paper is to prove several basic results concerning a deformation theory of rigid analytic spaces. We shall give a cohomological description of infinitesimal deformations, and prove an existence of a formal versal family for a proper rigid analytic space. Moreover we shall comment on the meaning of deformation theory of rigid analytic spaces, to the moduli problems of schemes, though the presented result is still far from the expected applications to such problems.

The original idea of deformations goes back to Kodaira and Spencer. They developed the theory of deformations of complex manifolds. Their deformation theory of complex manifolds is of great importance and becomes a standard tool for the study of complex analytic spaces. Our study of deformations of rigid analytic spaces is motivated by analogy to the case of complex analytic spaces. More precisely, our interest in the development of such a theory comes from two sources. First, we want to construct an analytic moduli theory via rigid analytic stacks by generalizing the classical deformation theory due to Kodaira-Spencer, Kuranishi and Grauert to the non-Archimedean theory. This viewpoint will be discussed in Section 5. Secondly, we may hope that our theory is useful in arithmetic geometry no less than the complex-analytic deformation theory is very useful in the study of complex-analytic spaces. Actually, our deformation theory will be one of the key ingredients of the generalization of the theory of p-adic period mappings due to Rapoport-Zink (cf. [21]).

Now we discuss the contents of this paper.

In Section 1, we first recall the definition of deformations of rigid analytic spaces and the different aspect of deformations of them, to that of schemes, already pointed out in [14, 7.3]. We also study the stability of Grothendieck topology for deformations of rigid analytic spaces.

In Section 2, we shall construct cotangent complexes which fits in with deformations of rigid analytic spaces. We should remark that cotangent complexes for rigid analytic spaces were constructed in the framework of Huber's

theory (cf. [14]). However it seems more natural to develop such theory in the framework of Raynaud's viewpoint [5]–[7], where one can use the machinery of formal-algebraic geometry in more direct way. Therefore we adopt a slightly different formulation. In fact, in virtue of formal geometry, the proofs of some results on cotangent complexes presented here are more direct (and slightly clearer) than the proofs in [14].

In Section 3, we shall give a cohomological description of infinitesimal deformations.

In Section 4, we shall prove an existence of a formal versal family for a proper rigid analytic space.

In Section 5, we shall pose a conjecture that states a rigid analogue of Kuranishi's existence theorem for versal deformations of complex analytic spaces, which was our primary interest. After posing it, we explain why this is important and meaningful.

Finally, in the appendix, we give a convenient criterion for an existence of square-zero deformations of a ringed topos. This criterion may be quite useful also in the other situations.

Notations And Conventions

(1) Unless otherwise stated, K will be a complete non-Archimedean valued field. We denote by $K\langle X_1, \ldots, X_n \rangle$ the Tate algebra in n indeterminates.

(2) For the basic facts and definitions concerning rigid analytic spaces we refer to [3], [5]–[7].

(3) For an affinoid algebra A, we denote by $\mathrm{Sp}(A)$ the associated affinoid space (cf. [3, Chapter 7, 8, 9]).

(4) All rigid analytic spaces in this paper will be quasi-compact and quasi-separated rigid analytic spaces. Quasi-separatedness means that the diagonal morphism $X \to X \times X$ is quasi-compact.

(5) *Properness* means Kiehl's properness (cf. [3]).

(6) The *very weak topology* on an affinoid space means Grothendieck topology such that it has rational subdomains as the admissible open sets, and its admissible coverings are unions of finitely many rational subdomains. However, unless otherwise stated, we always equip rigid spaces with the *strong topology* in the sense of [3].

(7) By $|\bullet|_{sp}$ we mean the *spectral norm* (cf. [3, 3.2]).

(8) Let $\mathcal{A}$ be an admissible formal scheme (resp. an $\mathcal{O}_{\mathcal{X}}$-module on the admissible formal scheme $\mathcal{X}$,..etc.) in the sense of [5]. Then we denote by $\mathcal{A}^{\mathrm{rig}}$ the rigid analytic space (resp. the $\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}}$-module on the rigid analytic space $\mathcal{X}^{\mathrm{rig}}$,..etc) associated to $\mathcal{A}$, which is defined in [5, Section 4, 5]. We say that $\mathcal{A}$ is a formal model of $\mathcal{A}^{\mathrm{rig}}$. Note that, in Section 2, we will extend the definition of $\mathcal{A}^{\mathrm{rig}}$ for $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{A}$ which are not necessarily coherent.

(9) Let A be a ring and B an A-algebra. Let M be a B-module. A short exact sequence $0 \to M \xrightarrow{i} E \xrightarrow{\pi} B \to 0$ where E is an A-algebra and π is a surjective homomorphism of A-algebras with $i(M)^2 = 0$ in E is said to be a square-zero extension of B by M over A.

1. First properties of local deformations

In this section, we will define and prove first properties of local deformations.

Definition 1.1. Let $f : X \to S$ be a flat morphism (i.e., a flat morphism of the ringed sites) of rigid analytic spaces and $S \to S'$ be a closed immersion of rigid analytic spaces with the nilpotent kernel $\mathcal{I} := \mathrm{Ker}(\mathcal{O}_{S'} \to \mathcal{O}_S)$. We say that a pair $(f' : X' \to S', \phi : X' \times_{S'} S \xrightarrow{\sim} X)$ is a deformation of $f : X \to S$ to S' if the following properties are satisfied,

1. f' is a flat morphism of rigid analytic spaces,

2. ϕ is an isomorphism of rigid analytic spaces.
A morphism from $(f' : X' \to S', \phi : X' \times_{S'} S \xrightarrow{\sim} X)$ to $(f'' : X'' \to S', \psi : X'' \times_{S'} S \xrightarrow{\sim} X)$ is a S'-morphism $\alpha : X' \to X''$ of rigid analytic spaces such that $\psi \circ \alpha|_S \circ \phi^{-1} = \mathrm{id}_X$.

Example 1.2. (1) Let $S \to S'$ be a closed immersion of schemes locally of finite type over K with the nilpotent kernel $\mathcal{I} := \mathrm{Ker}(\mathcal{O}_{S'} \to \mathcal{O}_S)$. Let $f : X \to S$ be a flat morphism of K-schemes and suppose that X is of finite type over K. Let the pair $(X'/S', X' \times_{S'} S \cong X)$ be a flat deformation of X to S'. Then the analytification $(X'^{\,\mathrm{an}}/S'^{\,\mathrm{an}}, X'^{\,\mathrm{an}} \times_{S'^{\,\mathrm{an}}} S^{\mathrm{an}} \xrightarrow{\sim} X^{\mathrm{an}})$ is a flat deformation of X^{an} to $S'^{\,\mathrm{an}}$. Here for any scheme W locally of finite type over K, we denote by W^{an} the associated rigid analytic space (See [2] for the analytifications).

(2) Let T be a split K-torus and M a split lattice of rank $\dim(T)$ in the sense of [4]. Note that we can regard M as group K-scheme. If the closed immersion $i : M \to T$ defines a lattice of full rank in T, the quotient $A := T/M$ is the rigid analytic group (See [4, p.661]). Let B be an Artin local K-algebra. As we see later (Lemma 4.1), B is an affinoid K-algebra. Let $\tilde{i} : M \times_K B \to T \times_K B$ be a closed immersion which extends the morphism i. Then the rigid analytic group $(T \times_K B)/(M \times_K B)$ over $\mathrm{Sp}(B)$ defines a deformation of A/K to B.

Remark 1.3 (cf. [14] 7.3.28-38). It may happen that a square-zero extension of an affinoid algebra is not an affinoid algebra. Let $0 \to M \to E \to A \to 0$ be a square-zero extension of an affinoid algebra A over K, where M is a finitely generated A-module. By the fundamental theorem due to L. Illusie

(cf. [18, Chapter 3, 1.2.3]), the set of isomorphism classes of square-zero extensions of A by M is classified by the group $\mathrm{Ext}^1_A(\mathsf{L}_{A/K}, M)$ where $\mathsf{L}_{A/K}$ is the cotangent complex defined in [18, Chapter 2]. If E is an affinoid algebra and A is a Tate algebra $K\langle T_1, \dots T_r \rangle$, then there exists a splitting $A \to E$ of $E \to A$ and hence, the extension class of this extension in $\mathrm{Ext}^1_A(\mathsf{L}_{A/K}, M)$ is zero. On the other hand, by Theorem A.1, there exists a canonical injective map $\mathrm{Ext}^1_A(\Omega^1_{A/K}, M) \to \mathrm{Ext}^1_A(\mathsf{L}_{A/K}, M)$ where $\Omega^1_{A/K}$ is the usual Kählar differential module. Therefore, in order to see the existence of such an extension with E not being an affinoid algebra, it suffices to show the following statement.

Proposition 1.4.   *Let $A = \mathbb{Q}_p\langle T \rangle$ be a Tate algebra over $\mathbb{Q}_p$. Then there exists a finitely generated A-module M such that $\mathrm{Ext}^1_A(\Omega^1_{A/\mathbb{Q}_p}, M)$ is nonzero.*

Proof.   First we claim that, for any A-module M, we have $\mathrm{Ext}^1_A(\Omega^1_{A/\mathbb{Q}_p}, M)$ $= \mathrm{Ext}^1_A(\mathrm{Ker}(\pi), M)$, where $\mathrm{Ker}(\pi)$ is the kernel of the homomorphism

$$0 \to \mathrm{Ker}(\pi) \to \Omega^1_{A/\mathbb{Q}_p} \xrightarrow{\pi} \Omega^{\mathrm{rig}}_{A/\mathbb{Q}_p} \to 0.$$

Here $\Omega^{\mathrm{rig}}_{A/\mathbb{Q}_p}$ is the differential module defined in [6, Section 1]. To prove our claim, we look at the associated long exact sequence to the short exact sequence

$$\mathrm{Ext}^1_A(\Omega^{\mathrm{rig}}_{A/\mathbb{Q}_p}, M) \to \mathrm{Ext}^1_A(\Omega^1_{A/\mathbb{Q}_p}, M) \to \mathrm{Ext}^1_A(\mathrm{Ker}(\pi), M) \to \mathrm{Ext}^2_A(\Omega^{\mathrm{rig}}_{A/\mathbb{Q}_p}, M).$$

Then the claim follows easily from the fact that $\Omega^{\mathrm{rig}}_{A/\mathbb{Q}_p}$ is a free A-module.

Thus, what to prove is the existence of a finitely generated A-module M such that $\mathrm{Ext}^1_A(\mathrm{Ker}(\pi), M)$ is non-zero. Suppose we have such an M, which is not necessarily finitely generated, then we can actually find a finitely generated A-module M having the same property. Indeed, Since A is a principal ideal domain, we have an A-injective resolution, $0 \to A \to I \to I' \to 0$. From the long exact sequence arising from this resolution, we see that $\mathrm{Ext}^2_A(\Omega^1_{A/\mathbb{Q}_p}, A) = 0$. We have also an A-free resolution $0 \to F' \to F \to N \to 0$ because every A-submodule of a free A-module is free. Then we derive $\mathrm{Ext}^1_A(\Omega^1_{A/\mathbb{Q}_p}, A) \neq 0$ from the long exact sequence,

$$\mathrm{Ext}^1_A(\Omega^1_{A/\mathbb{Q}_p}, F') \to \mathrm{Ext}^1_A(\Omega^1_{A/\mathbb{Q}_p}, F) \to \mathrm{Ext}^1_A(\Omega^1_{A/\mathbb{Q}_p}, N) \to \mathrm{Ext}^2_A(\Omega^1_{A/\mathbb{Q}_p}, F')$$

and $\mathrm{Ext}^2_A(\Omega^1_{A/\mathbb{Q}_p}, F') = 0$.

The existence of a module M with $\mathrm{Ext}^1_A(\mathrm{Ker}(\pi), M) \neq 0$ (not necessarily finitely generated) follows from the following lemma.   □

Lemma 1.5. $\mathrm{Ker}(\pi)$ *is a non-zero injective A-module. In particular* $\mathrm{Ker}(\pi)$ *is not A-projective.*

Proof.   First, we will show $\mathrm{Ker}(\pi)$ is A-injective. To this aim, since A is a principal ideal domain, it suffices to prove that $\mathrm{Ker}(\pi)$ is a divisible A-module.

Let m and a be elements in $\mathrm{Ker}(\pi)$ and A respectively, and suppose that $a \neq 0$. We want to find an element n of $\mathrm{Ker}(\pi)$ such that $m = an$. By induction, we may assume that a is irreducible. However there exists the exact sequence

$$0 \to \mathrm{Hom}(\Omega^{\mathrm{rig}}_{A/\mathbb{Q}_p}, A/(a)) \xrightarrow{\xi} \mathrm{Hom}(\Omega^1_{A/\mathbb{Q}_p}, A/(a)) \to \mathrm{Hom}(\mathrm{Ker}(\pi), A/(a)) \to 0$$

where ξ is an isomorphism by

$$\mathrm{Hom}(\Omega^{\mathrm{rig}}_{A/\mathbb{Q}_p}, A/(a)) \cong \mathrm{Der}_{\mathbb{Q}_p}(A, A/(a)) \cong \mathrm{Hom}(\Omega^1_{A/K}, A/(a)).$$

Thus $\mathrm{Ker}(\pi) \otimes_A A/(a) = 0$. Finally $d(\exp(pT))$ lies in $\mathrm{Ker}(\pi)$ because $\exp(pT) = \Sigma^\infty_{n=0}(pT)^n/n!$ is transcendental over $\mathbb{Q}_p(T)$. $\qquad\square$

Next we investigate the stability of the Grothendieck topology of a rigid analytic space under deformations. First of all, we will consider a class of morphisms of rigid analytic spaces defined as follows. A morphism $f : X \to Y$ is said to be *of affinoid type* if f is the composite map $X \xrightarrow{g} Y \times_K \mathrm{Sp}(K\langle T_1, \ldots, T_r \rangle) \xrightarrow{pr_1} Y$ for some positive integer r and some closed immersion g.

Proposition 1.6. *Let $u : S \to S'$ be a closed immersion of rigid analytic spaces with nilpotent kernel $\mathrm{Ker}(u^* : \mathcal{O}_{S'} \to \mathcal{O}_S)$ and X a rigid analytic space. Suppose $f : X \to S$ be a morphism of affinoid type and $f' : \tilde{X} \to S'$ is a (possibly non-flat) deformation of f. Then, f' is a morphism of affinoid type.*

Proof. First we prove the case when $S := \mathrm{Sp}(A)$, $S' := \mathrm{Sp}(A')$ and $I := \mathrm{Ker}(u^* : A' \to A)$. Suppose that $I^n = 0$. By induction on n, it suffices to consider the case where $I^2 = 0$. Now we have an exact sequence

$$0 \longrightarrow I\mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Since $I^2 = 0$, $I\mathcal{O}_{\tilde{X}} = I\mathcal{O}_X$. Thus by Tate's acyclicity theorem ([3, 8.2]), we have

$$\check{H}^1(\tilde{\mathcal{U}}, I\mathcal{O}_{\tilde{X}}) = \check{H}^1(\mathcal{U}, I\mathcal{O}_X) = (0)$$

where $\tilde{\mathcal{U}} = \{\tilde{U}_i\}^r_{i=0}$ is a finite affinoid cover on $\tilde{X}$ and $\mathcal{U} = \{U_i\}^r_{i=0}$ is the reduction of $\tilde{\mathcal{U}}$ to X. By $\check{H}(\tilde{\mathcal{U}}, \bullet)$, we mean the Čech cohomology with respect to $\tilde{\mathcal{U}} = \{\tilde{U}_i\}$. By considering the long exact sequence of the Čech cohomology with respect to the very weak topology on $\tilde{X}$, we have an exact sequence

$$0 \longrightarrow H^0(X, I) \longrightarrow H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow 0.$$

We put $H^0(X, \mathcal{O}_X) = A\langle X_1, \ldots, X_n \rangle / J$. Let us construct a surjective map from $A'\langle T_1, \ldots T_r \rangle$ to $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$ which extends a natural surjective map $A\langle X_1, \ldots, X_n \rangle \to H^0(X, \mathcal{O}_X)$. Let $\xi_1, \ldots, \xi_n$ be elements of $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$ which are liftings of $X_1, \ldots, X_n$, respectively. Then, we have the following diagram

$$
\begin{array}{ccc}
A'[X_1, \ldots, X_n] & \longrightarrow & A\langle X_1, \ldots X_n \rangle \\
\downarrow{\scriptstyle \pi} & & \downarrow \\
H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^0(X, \mathcal{O}_X),
\end{array}
$$

where π is defined by $X_i \to \xi_i$. The right vertical arrow and the lower horizontal arrow are surjective. We want to see that π is continuous when we equip $K[X_1, \ldots, X_n]$ with the Gaussian norm. If necessary, we can choose $\xi_1, \ldots, \xi_n$ such that $\max_{1 \leq k \leq r} |\operatorname{res}_{\tilde{U}_k}(\xi_i)|_{sp} \leq 1$ for $1 \leq i \leq n$ where $\operatorname{res}_{\tilde{U}_k}(\xi_i)$ is the restriction of ξ_i to $H^0(\tilde{U}_k, \mathcal{O}_{\tilde{X}})$. Indeed, by [3, 3.8.2.2], $|X_i|_{sp} \leq |X_i| \leq 1$ on $A\langle X_1, \ldots, X_n\rangle / J$ and thus we have the same inequality on U_i for $1 \leq i \leq r$ by [3, 3.8.1.4]. Hence, we have $|\operatorname{res}_{\tilde{U}_k}(\xi_i)|_{sp} \leq 1$ for any i, k. This implies the composite map

$$A'[X_1, \ldots, X_n] \xrightarrow{\pi} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to \bigoplus_{i=0}^{r} H^0(\tilde{U}_i, \mathcal{O}_{\tilde{U}_i})$$

is a continuous map. Now by Tate's acyclicity theorem we have

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \operatorname{Ker}(\bigoplus_{i=0}^{r} H^0(\tilde{U}_i, \mathcal{O}_{\tilde{U}_i}) \rightrightarrows \bigoplus_{i<j} H^0(\tilde{U}_i \cap \tilde{U}_j, \mathcal{O}_{\tilde{U}_i \cap \tilde{U}_j})).$$

Note that the topology of $\bigoplus_{i=0}^{r} H^0(\tilde{U}_i, \mathcal{O}_{\tilde{U}_i})$ (i.e. direct sum of the topologies on affinoid algebras $H^0(\tilde{U}_i, \mathcal{O}_{\tilde{U}_i})$) induces the topology on $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$ which is complete. Thus we see π is also continuous. From this, there exists a unique homomorphism

$$\Pi : A'\langle X_1, \ldots, X_n\rangle \longrightarrow H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$$

which extends the homomorphism π. We claim that Π is surjective. Indeed, the kernel of the surjection $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^0(X, \mathcal{O}_X)$ is $IH^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$. On the other hand, it is clear that the image of π contains $IH^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$. Hence, we see that Π is a surjection. Therefore, there exists the commutative diagram of rigid analytic spaces

$$
\begin{array}{ccc}
X & \xrightarrow{\ i\ } & \tilde{X} \\
\alpha \downarrow & & \downarrow \beta \\
\mathbb{D}_A^n & \xrightarrow{\ j\ } & \mathbb{D}_{A'}^n
\end{array}
$$

where $\mathbb{D}_A^n$ and $\mathbb{D}_{A'}^n$ are $\operatorname{Sp}(A\langle X_1, \ldots, X_n\rangle)$ and $\operatorname{Sp}(A'\langle X_1, \ldots, X_n\rangle)$ respectively. i and j are maps which induce bijective maps of underlying sets. The morphism $j \circ \alpha$ induce an injective map of underlying sets. Hence β is bijective onto the image of $j \circ \alpha$ as sets. To prove the proposition, it suffices to show that homomorphism $\mathcal{O}_{\mathbb{D}_{A'}^n, \beta(x)} \xrightarrow{\beta^*} \mathcal{O}_{\tilde{X}, x}$ is surjection for any point x of $\tilde{X}$. Let a be an element of $\mathcal{O}_{\tilde{X}, x}$. There is an element b in $\mathcal{O}_{\mathbb{D}_A^n, \beta(x)}$ such that $\alpha^*(b) = i^*(a)$. Moreover there is an element c in $\mathcal{O}_{\mathbb{D}_{A'}^n, \beta(x)}$ such that $j^*(c) = b$. Then, $\beta^*(c) - a$ is in $\operatorname{Ker}(i^*) = I\mathcal{O}_{\tilde{X}, x}$. Since $I^2 = 0$, we have $I \operatorname{Image}(\beta^*) = I\mathcal{O}_{\tilde{X}, x}$. Thus we have an element d in $\mathcal{O}_{\mathbb{D}_{A'}^n, \beta(x)}$ such that $\beta^*(d + c) = a$, since I is contained by $\operatorname{Image}(\beta^*)$.

Finally the assertion for the general case follows from the proof of the local case. $\square$

Proposition 1.7.   *Let $u : A' \to A$ be a surjective homomorphism of affinoid K-algebras with the nilpotent kernel and X be a K-affinoid space over*

$\mathrm{Sp}(A)$. *Let U be a rational subdomain of X. Suppose $\tilde{X} \to \mathrm{Sp}(A')$ is a (possibly non-flat) deformation of X and $\tilde{X}$ is an affinoid spaces. Then, the lifting $\tilde{U}$ of U in $\tilde{X}$ is a rational subdomain of $\tilde{X}$.*

Proof. We put

$$
\begin{aligned}
U &= X(f_0, \ldots, f_r) \\
&= \{x \in X \mid |f_1(x)| \le |f_0(x)|, \ldots, |f_r(x)| \le |f_0(x)|\} \\
&= \mathrm{Sp}(H^0(X, \mathcal{O}_X)\langle T_1, \ldots, T_r\rangle/(f_1 - T_1 f_0, \ldots, f_r - T_r f_0))
\end{aligned}
$$

where $f_0, \ldots, f_r$ generates the unit ideal. We choose elements $\tilde{f}_0, \ldots, \tilde{f}_r \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})$ which are the liftings of $f_0, \ldots, f_r$ respectively. Since $X \to \tilde{X}$ is a nilpotent thickening, we have

$$
\begin{aligned}
\tilde{U} &= \tilde{X}(\tilde{f}_0, \ldots, \tilde{f}_r) \\
&= \{x \in \tilde{X} \mid |\tilde{f}_1(x)| \le |\tilde{f}_0(x)|, \ldots, |\tilde{f}_r(x)| \le |\tilde{f}_0(x)|\} \\
&= \mathrm{Sp}(H^0(\tilde{X}, \mathcal{O}_{\tilde{X}})\langle T_1, \ldots, T_r\rangle/(\tilde{f}_1 - T_1 \tilde{f}_0, \ldots, \tilde{f}_r - T_r \tilde{f}_0)).
\end{aligned}
$$

This implies the proposition. □

Theorem 1.8. *Let $u : S \to S'$ be a closed immersion of rigid analytic spaces with nilpotent kernel $\mathrm{Ker}(u^* : \mathcal{O}_{S'} \to \mathcal{O}_S)$ and $f : X \to S$ a morphism of rigid analytic spaces. Let $f' : \tilde{X} \to S'$ be a (possibly non-flat) deformation of f. Suppose U is an admissible open set with respect to the strong topology on X. Then, the lifting $\tilde{U}$ of U in $\tilde{X}$ is the admissible open set of $\tilde{X}$. Similarly, an admissible covering of X lifts to admissible covering of $\tilde{X}$. In particular the Grothendieck topology of a rigid analytic space is stable under nilpotent deformations.*

Proof. By Proposition 1.6 and [3, 9.1.3.2], we may assume that X, $\tilde{X}$, S and S' are affinoid spaces. Next, we note that the strongest topology among the topologies which are slightly finer (See for the definition [3, 9.1.2.1]) than very weak topology coincides with the strong topology by the theorem of Gerritzen-Grauert [3, 7.3.5.3].

Thus to prove the assertion, it suffices to check that (cf. [3, 9.1.4.2]):

1. The set $\tilde{U}$ admits a covering $\{\tilde{U}_i\}_i$ by affinoid subdomains $\tilde{U}_i \subset \tilde{X}$ such that, for any affinoid morphism $\phi : Y \to \tilde{X}$ with $\phi(Y) \subset \tilde{U}$, the covering $\{\phi^{-1}(\tilde{U}_i)\}_i$ of Y has a finite rational subdomain covering which refines it.

2. Let $\{\tilde{V}_j\}_j$ be a covering which are the liftings of an admissible covering of the admissible open set $V \subset X$. Note that if the first half of our claim is verified, $\tilde{V}_i$ are admissible. Let $\tilde{V} \subset \tilde{X}$ be the lifting of V. Then, for any affinoid morphism $\phi : Y \to \tilde{X}$ with $\phi(Y) \subset \tilde{V}$, the covering $\{\phi^{-1}(\tilde{V}_i)\}_i$ of Y has a finite rational subdomain covering which refines it.

Since $|X| = |\tilde{X}|$, we can easily check these conditions by using Proposition 1.7. □

2. Rigid cotangent complex

In this section, we construct cotangent complexes for deformations of rigid analytic spaces, and prove some basic results in the framework of [5]–[7].

2.1. Analytic cotangent complex of formal schemes

First of all, following [14], we will recall the analytic cotangent complexes for formal schemes locally of topologically finite presentation over $\mathrm{Spf}\, R$, where R is the ring of integer of K. Let π be an element in the maximal ideal of R and A a complete R-algebra topologically of finite presentation. Consider the π-adic completion functor

$$(A\text{-Mod}) \longrightarrow (A\text{-Mod})$$
$$L \mapsto L^{\wedge}.$$

This functor induces the derived functor

$$\mathsf{D}^-(A\text{-Mod}) \longrightarrow \mathsf{D}^-(A\text{-Mod}).$$

Indeed, for any quasi-isomorphism of complexes of flat A-modules $K^{\bullet} \to L^{\bullet}$, the induced homomorphism $(K^{\bullet})^{\wedge} \to (L^{\bullet})^{\wedge}$ ((complex)$^{\wedge}$ denotes the termwise completion) is a quasi-isomorphism (cf. [14, 7.1.11]). Moreover, the derived category $\mathsf{D}^-(R\text{-Mod})$ is naturally identified with the localization of the homotopy category $K^-(A\text{-flat Mod})$ up to quasi-isomorphisms. Thus π-adic completion functor induces the derived functor.

Let $\phi : A \to B$ be a homomorphism of complete R-algebras of topologically finite presentation. The B-module of analytic differentials relative to ϕ is defined as $\Omega_{B/A}^{\mathrm{an}} := \Omega_{B/A}^{1\wedge}$. The analytic cotangent complex of ϕ is the complex $\mathsf{L}_{B/A}^{\mathrm{an}} := \mathsf{L}_{B/A}^{\wedge}$. Here $\mathsf{L}_{B/A}$ is the usual cotangent complex of ϕ (cf. [18]). There exists a natural map $\mathsf{L}_{B/A}^{\mathrm{an}} \to \Omega_{B/A}^{\mathrm{an}}$ and an isomorphism $H_0(\mathsf{L}_{B/A}^{\mathrm{an}}) \cong \Omega_{B/A}^{\mathrm{an}}$. If ϕ is smooth, then there is a natural quasi-isomorphism $\mathsf{L}_{B/A}^{\mathrm{an}} \cong \Omega_{B/A}^{\mathrm{an}}[0]$.

Next we define the analytic cotangent complex for formal schemes locally of topologically finite presentation over $\mathrm{Spf}\, R$ by gluing the complexes constructed as above. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of formal schemes locally of topologically finite presentation over $\mathrm{Spf}\, R$ (cf. [5]). We suppose that $\mathcal{Y}$ is affine for a while. For an affine open set $\mathcal{U}$ in $\mathcal{X}$, the small category $F_{\mathcal{U}}$ of all affine open sets $\mathcal{V}$ in $\mathcal{Y}$ with $f(\mathcal{U}) \subset \mathcal{V}$ is a cofiltered family under inclusion. For every $\mathcal{V} \in F_{\mathcal{U}}$, $\mathcal{O}_{\mathcal{Y}}(\mathcal{V})$ is a complete R-algebra of topologically finite presentation. The cofiltered family of maps $\mathcal{O}_{\mathcal{Y}}(\mathcal{V}) \to \mathcal{O}_{\mathcal{X}}(\mathcal{U})$ gives rise to the correspondence

$$\mathcal{U} \longrightarrow \mathsf{L}(\mathcal{U}/\mathcal{Y}) := \operatorname*{colim}_{\mathcal{V} \in F_{\mathcal{U}}} \mathsf{L}_{\mathcal{O}_{\mathcal{X}}(\mathcal{U})/\mathcal{O}_{\mathcal{Y}}(\mathcal{V})}^{\mathrm{an}}.$$

Now note that every usual cotangent complex is constructed as a complex of free modules in a functorial fashion. Hence for any homomorphism $A \to B$, we can construct the analytic cotangent complex of $A \to B$ as a complex of flat B-modules in a functorial fashion. (The flatness follows from the fact that

for any B-flat module F, the completion $F^\wedge$ is also B-flat (cf. for example [14, 7.1.6 (1)])). Therefore, by [10, Chapter 0, 3.2.1], we can extend the above correspondence to the complex of presheaves on $\mathcal{X}$. The analytic cotangent complex is defined as the complex of termwise the associated sheaves. Finally for a general formal scheme $\mathcal{Y}$, we can construct the complex by gluing the complex constructed locally on $\mathcal{Y}$.

Proposition 2.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Z} \to \mathcal{Y}$ be morphisms of formal schemes locally of topologically finite presentation over* $\operatorname{Spf} R$. *Let $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ be the fibre product of $\mathcal{X}$ and $\mathcal{Z}$ over $\mathcal{Y}$ in the category of formal R-schemes. Then there exists a natural quasi-isomorphism*

$$L\operatorname{pr}_1^* \mathsf{L}^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}} \xrightarrow{\sim} \mathsf{L}^{\mathrm{an}}_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{Z}/\mathcal{Z}}$$

where $\operatorname{pr}_1 : \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X}$ *is the first projection.*

Proof. Since our assertion is a local issue, we may suppose that $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are affine. Set $\mathcal{X} = \operatorname{Spf} B$, $\mathcal{Y} = \operatorname{Spf} A$ and $\mathcal{Z} = \operatorname{Spf} C$. First we shall show that there exists a natural quasi-isomorphism $\phi : (B\hat{\otimes}_A C) \otimes_B \mathsf{L}^{\wedge}_{B/A} \to \mathsf{L}^{\wedge}_{(B\hat{\otimes}_A C)/C}$. We remark that $\operatorname{Spf} B\hat{\otimes}_A C$ is the fibre product of $\operatorname{Spf} B$ and $\operatorname{Spf} C$ over $\operatorname{Spf} A$ in the category of formal R-schemes. In this proof, unless otherwise stated, we view complexes as just complexes (not objects in the derived category) and we denote by $\xrightarrow{\mathrm{qis}}$ a quasi-isomorphism of complexes. Actually, usual cotangent complexes are constructed as complexes of flat modules via standard resolutions and their completions are also flat. There exist natural quasi-isomorphisms $((B\hat{\otimes}_A C) \otimes_{(B\otimes_A C)} (B \otimes_A C) \otimes_B \mathsf{L}_{B/A})^\wedge \xrightarrow{\sim} ((B\hat{\otimes}_A C) \otimes_B \mathsf{L}_{B/A})^\wedge \xrightarrow{\mathrm{qis}} ((B\hat{\otimes}_A C) \otimes_B \mathsf{L}^{\wedge}_{B/A})^\wedge$. The second quasi-isomorphism follows from [14, Lemma 7.1.25]. On the other hand, by the base change theorem of usual cotangent complexes [18, Chapter 2, 2.2], we have a natural quasi-isomorphism $((B\hat{\otimes}_A C) \otimes_{(B\otimes_A C)} (B \otimes_A C) \otimes_B \mathsf{L}_{B/A})^\wedge \xrightarrow{\mathrm{qis}} ((B\hat{\otimes}_A C) \otimes_{(B\otimes_A C)} \mathsf{L}_{(B\otimes_A C)/C})^\wedge$. The next claim implies a quasi-isomorphism $(B\hat{\otimes}_A C) \otimes_B \mathsf{L}^{\wedge}_{B/A} \xrightarrow{\mathrm{qis}} ((B\hat{\otimes}_A C) \otimes_B \mathsf{L}^{\wedge}_{B/A})^\wedge$ and thus we have a natural quasi-isomorphism $(B\hat{\otimes}_A C) \otimes_B \mathsf{L}^{\wedge}_{B/A} \xrightarrow{\sim} ((B\hat{\otimes}_A C) \otimes_{(B\otimes_A C)} \mathsf{L}_{(B\otimes_A C)/C})^\wedge$ in the derived category.

Claim 2.1.1. Let $A \to B \to C$ be homomorphisms of admissible R-algebras and $K^\bullet := \mathsf{L}^{\mathrm{an}}_{B/A}$ the analytic cotangent complex. Then there is a natural quasi-isomorphism

$$K^\bullet \overset{\mathsf{L}}{\otimes}_B C \xrightarrow{\sim} (K^\bullet \overset{\mathsf{L}}{\otimes}_B C)^\wedge$$

in the derived category $\mathsf{D}^-(C\text{-Mod})$.

Proof. It suffices to show that for any positive integer n, the truncation

$$\tau_{\lceil -n}(K^\bullet \overset{\mathsf{L}}{\otimes}_B C) \xrightarrow{\sim} \tau_{\lceil -n}(K^\bullet \overset{\mathsf{L}}{\otimes}_B C)^\wedge$$

is a quasi-isomorphism. Due to [14, 7.1.15, 7.1.33 (1)] and the fact that C is coherent, we can assume that $\tau_{[-n}K^\bullet$ is a complex of free B-modules of finite type. Therefore it suffices to show that for a free B-module of finite type F, the natural homomorphism $F \otimes_B C \to (F \otimes_A B)^\wedge$ is an isomorphism. However it is clear. $\qquad\square$

Therefore, to see that ϕ is a quasi-isomorphism, it suffices to show that the natural morphism $\psi : ((B\hat{\otimes}_A C) \otimes_{B\otimes_A C} \mathsf{L}_{(B\otimes_A C)/C})^\wedge \to \mathsf{L}^\wedge_{(B\hat{\otimes}_A C)/C}$ is a quasi-isomorphism. Note that, by the base-change theorem of usual cotangent complexes [18, Chapter 2, 2.2], $((B\hat{\otimes}_A C) \otimes_{B\otimes_A C} \mathsf{L}_{(B\otimes_A C)/C})^\wedge_n := ((B\hat{\otimes}_A C) \otimes_{B\otimes_A C} \mathsf{L}_{(B\otimes_A C)/C})^\wedge \otimes_R (R/\pi^n R) \xrightarrow{\mathrm{qis}} \mathsf{L}_{(B_n \otimes_{A_n} C_n)/C_n}$, $(\mathsf{L}^\wedge_{(B\hat{\otimes}_A C)/C})_n := \mathsf{L}^\wedge_{(B\hat{\otimes}_A C)/C} \otimes_R (R/\pi^n R) \xrightarrow{\mathrm{qis}} \mathsf{L}_{(B_n \otimes_{A_n} C_n)/C_n}$ where $B_n = B \otimes_R (R/\pi^n R)$, $A_n = A \otimes_R (R/\pi^n R)$ and $C_n = C \otimes_R (R/\pi^n R)$. Thus the reduction $\psi \otimes_R (R/\pi^n R)$ is a quasi-isomorphism. Now consider the right derived functor

$$R\lim : \mathsf{D}((R\text{-Mod})^{\mathbb{N}})) \to \mathsf{D}(R\text{-Mod})$$

where $(R\text{-Mod})^{\mathbb{N}}$ is the projective system of R-modules and $\lim : (R\text{-Mod})^{\mathbb{N}} \to R\text{-Mod}, \{M_i, d_i : M_i \to M_{i-1}\}_{i \geq 1} \mapsto \mathrm{proj.}\lim_i M_i$ is the inverse limit functor. The projective systems of complexes $((B\hat{\otimes}_A C) \otimes_{B\otimes_A C} \mathsf{L}_{(B\otimes_A C)/C}^\wedge{}_n)_{n\geq 1}$ and $((\mathsf{L}^\wedge_{(B\hat{\otimes}_A C)/C})_n)_{n\geq 1}$ are acyclic for the functor $\lim$ because they are consisting of surjections. Thus we have that

$$R\lim((B\hat{\otimes}_A C) \otimes_{B\otimes_A C} \mathsf{L}_{(B\otimes_A C)/C}^\wedge{}_n)_{n\geq 1} = ((B\hat{\otimes}_A C) \otimes_{B\otimes_A C} \mathsf{L}_{(B\otimes_A C)/C})^\wedge$$

and

$$R\lim((\mathsf{L}^\wedge_{(B\hat{\otimes}_A C)/C})_n)_{n\geq 1} = \mathsf{L}^\wedge_{(B\hat{\otimes}_A C)/C}.$$

Since there exists a quasi-isomorphism

$$R\lim((B\hat{\otimes}_A C) \otimes_{B\otimes_A C} \mathsf{L}_{(B\otimes_A C)/C}^\wedge{}_n)_{n\geq 1} \xrightarrow{\mathrm{qis}} R\lim((\mathsf{L}^\wedge_{(B\hat{\otimes}_A C)/C})_n)_{n\geq 1},$$

we see that ψ is a quasi-isomorphism and thus ϕ is a quasi-isomorphism. Furthermore, a natural homomorphism $(B\hat{\otimes}_A C) \overset{\mathsf{L}}{\otimes}_B \mathsf{L}^\wedge_{B/A} \to \mathsf{L}^\wedge_{(B\hat{\otimes}_A C)/C}$ is a quasi-isomorphism (in the derived category) because $\mathsf{L}^\wedge_{B/A}$ consists of flat B-modules.

Next note that $(B\hat{\otimes}_A C) \overset{\mathsf{L}}{\otimes}_B \mathsf{L}^\wedge_{B/A}$ and $\mathsf{L}^\wedge_{(B\hat{\otimes}_A C)/C}$ are pseudo-coherent [*1]. Let L^Δ_i (resp. M^Δ_i) be the sheaf of the coherent $\mathcal{O}_{\mathrm{Spf}(B\hat{\otimes}_A C)}$-module associated to the coherent $B\hat{\otimes}_A C$-module $L_i := H_i((B\hat{\otimes}_A C) \overset{\mathsf{L}}{\otimes}_B \mathsf{L}^\wedge_{B/A})$ (resp. $M_i := H_i(\mathsf{L}^\wedge_{(B\hat{\otimes}_A C)/C}))$. To complete the proof of the proposition, it suffices to show that following claim.

[*1]Let n be a integer. We say that a complex of R-module $K^\bullet$ is n-pseudo-coherent if there exists a quasi isomorphism $C^\bullet \to K^\bullet$ where $C^\bullet$ is a complex bounded above and C^k is a finitely generated free R-module for every $k \geq n$. We say that $K^\bullet$ is pseudo-coherent if $K^\bullet$ is n-pseudo-coherent for every integer n. n-pseudo-coherence is stable under quasi-isomorphisms.

Claim 2.1.2.   Under the same assumption as above, there exists natural isomorphisms

$$L_i^\Delta \cong H_i(L\,\mathrm{pr}_1{}^*\, \mathsf{L}^{\mathrm{an}}_{\mathrm{Spf}\,B/\,\mathrm{Spf}\,A})$$

and

$$M_i^\Delta \cong H_i(\mathsf{L}^{\mathrm{an}}_{(\mathrm{Spf}\,B\hat\otimes_A C)/\,\mathrm{Spf}\,C}).$$

Proof.   The second assertion can be shown by the same way of the proof of the first assertion. Hence we will prove the first quasi-isomorphism. What to prove is that the natural isomorphism $\xi\,:\,L_i^\Delta \to H_i(L\,\mathrm{pr}_1{}^*\,\mathsf{L}^{\mathrm{an}}_{\mathrm{Spf}\,B/\,\mathrm{Spf}\,A})$ induces an isomorphism on each stalk. Let $U := \mathrm{Spf}\,R(U)$, $V := \mathrm{Spf}\,R(V)$, and $W := \mathrm{Spf}\,R(W)$ be affine open sets of $\mathrm{Spf}\,B\hat\otimes_A C$, $\mathrm{Spf}\,B$, and $\mathrm{Spf}\,A$ respectively and suppose that $\mathrm{pr}_1(U) \subset V$ and $f(V) \subset W$. By the transitivity of analytic cotangent complexes [14, 7.1.33 (2)], we easily see that there exists a natural isomorphism $H_i(R(U)\otimes_{R(V)} \mathsf{L}^\wedge_{R(V)/R(W)}) \cong H_i((R(U)\otimes_B \mathsf{L}^\wedge_{B/A})$. Therefore we have natural isomorphisms

$$H_i(L\,\mathrm{pr}_1{}^*\,\mathsf{L}^{\mathrm{an}}_{\mathrm{Spf}\,B/\,\mathrm{Spf}\,A})_x$$

$$\cong \operatorname*{colim}_{\substack{x\in U \\ \text{affine open}}} \operatorname*{colim}_{\substack{\mathrm{pr}_1(U)\subset V \\ \text{affine open}}} \operatorname*{colim}_{\substack{f(V)\subset W \\ \text{affine open}}} H_i(R(U)\otimes_{R(V)} \mathsf{L}^\wedge_{R(V)/R(W)})$$

$$\cong \operatorname*{colim}_{\substack{x\in U \\ \text{affine open}}} H_i(R(U)\otimes_B \mathsf{L}^\wedge_{B/A})$$

$$\cong \operatorname*{colim}_{\substack{x\in D(c) \\ c\in B\hat\otimes_A C}} H_i(R(D(c))\otimes_B \mathsf{L}^\wedge_{B/A})$$

$$\cong L_i \otimes_{(B\hat\otimes_A C)} \mathcal{O}_{\mathrm{Spf}(B\hat\otimes_A C),x}$$

where $D(c) = \{x\in \mathrm{Spf}(B\hat\otimes_A C), c\notin \mathfrak{m}_x\}$ and $R(D(c)) := H^0(D(c),\mathcal{O}_{\mathrm{Spf}(B\hat\otimes_A C)})$. The final isomorphism follows from the next lemma.

Lemma 2.2.   *Let A be a complete R-algebra of topologically finite presentation. For any $c\in A$, let $D(c) := \{x\in \mathrm{Spf}\,A; c\notin \mathfrak{m}_x\}$. Then the natural map $A \to H^0(D(c),\mathcal{O}_{\mathrm{Spf}\,A})$ is flat.*

Proof.   Note that $H^0(D(c),\mathcal{O}_{\mathrm{Spf}\,A})$ is the π-adic completion of A_c. Then our claim follows from the fact that the π-adic completion of a flat A-module is also A-flat. $\square$

In the rest of this section we often use the terminologies in [5]–[7]. For the definition of admissible formal schemes, admissible blow-up, smoothness, etc., we refer to [5]–[7]. Let $\mathcal{X}$ be an admissible formal scheme over R, and let X be the associated rigid analytic space. We define a functor

$$\mathrm{Rig}: \mathsf{D}^-(\mathcal{O}_{\mathcal{X}}\text{-Mod}) \to \mathsf{D}^-(\mathcal{O}_X\text{-Mod})$$

as follows. Consider the projective limit of local ringed spaces $\langle\mathcal{X}\rangle := \varprojlim_{\mathcal{X}'\to\mathcal{X}} \mathcal{X}'$, where the projective limit here is taken over all admissible blow-ups $\mathcal{X}' \to \mathcal{X}$

of $\mathcal{X}$ along admissible ideals as in [13, 4.1.3]. The projective limit exists in the category of local ringed spaces. We have the natural morphism of local ringed spaces

$$\pi : \langle \mathcal{X} \rangle \to \mathcal{X}.$$

Then there exists the natural composite of functors

$$(\mathcal{O}_{\mathcal{X}} - \text{modules}) \xrightarrow{\pi^*} (\mathcal{O}_{\langle \mathcal{X} \rangle} - \text{modules}) \xrightarrow{\otimes K} (\mathcal{O}_X - \text{modules}),$$

which is right exact (the topology on $\langle \mathcal{X} \rangle$ induces the Grothendieck topology on X). For coherent $\mathcal{O}_{\mathcal{X}}$-modules, the functor Rig defined here coincides with the functor Rig defined in [5, p.315]. This functor induces a derived functor $\text{Rig} : \mathsf{D}^-(\mathcal{O}_{\mathcal{X}}\text{-Mod}) \to \mathsf{D}^-(\mathcal{O}_X\text{-Mod})$.

Lemma 2.3.
(1) *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of morphisms of rigid analytic spaces. Then there is a natural quasi-isomorphism*

$$L(g \circ f)^* \xrightarrow{\sim} Lf^* \circ Lg^*.$$

(2) *Let $\mathcal{X} \xrightarrow{\tilde{f}} \mathcal{Y} \xrightarrow{\tilde{g}} \mathcal{Z}$ be a sequence of admissible formal R-schemes which is a formal model of $X \xrightarrow{f} Y \xrightarrow{g} Z$ (Actually we can choose such a sequence of formal schemes for any sequence of rigid analytic spaces by [5, Theorem 4.1]). Then there is a natural quasi-isomorphism*

$$(L\tilde{f}^*(\mathsf{L}_{\mathcal{Y}/\mathcal{Z}}^{\mathrm{an}}))^{\mathrm{rig}} \xrightarrow{\sim} Lf^*(\mathsf{L}_{\mathcal{Y}/\mathcal{Z}}^{\mathrm{an}})^{\mathrm{rig}}.$$

Proof. (1) Our claim follows from Grothendieck spectral sequence (cf. [19, 1.8.7]).

(2) First, note that the assertion is local on $\mathcal{X}$. Thus we suppose that $\mathcal{X}$ and $\mathcal{Y}$ are affine. Put $A = H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $B = H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. Moreover, we may replace $\mathsf{L}_{\mathcal{Y}/\mathcal{Z}}^{\mathrm{an}}$ by $\mathsf{L}(n) := \tau_{[-n} \mathsf{L}_{\mathcal{Y}/\mathcal{Z}}^{\mathrm{an}}$ and prove our assertion for the latter complexes for every integer n. Take a complex of finite presented B-flat modules $\mathsf{L}_0^{\bullet}$ which represents the complex $\mathsf{L}(n)$. Then we have $(L\tilde{f}^*(\mathsf{L}(n)))^{\mathrm{rig}} = (\mathsf{L}_0^{\bullet} \otimes_B A \otimes_R K)^{\sim}$ (By $(\bullet)^{\sim}$ we denote the associated sheaf). On the other hand, $\mathsf{L}_0^{\bullet} \otimes_R K$ is a complex of finitely presented $B \otimes_R K$-flat module which represents $\mathsf{L}(n)^{\mathrm{rig}}$. Thus we have a quasi-isomorphism $Lf^*(\mathsf{L}(n)^{\mathrm{rig}}) \cong ((\mathsf{L}_0^{\bullet} \otimes_R K) \otimes_{(B \otimes K)} (A \otimes K))^{\sim}$. Therefore, we have a natural quasi-isomorphism $(\mathsf{L}(n) \overset{\mathsf{L}}{\otimes}_B A)^{\mathrm{rig}} \xrightarrow{\sim} \mathsf{L}(n)^{\mathrm{rig}} \overset{\mathsf{L}}{\otimes}_{(B \otimes_R K)} (A \otimes_R K)$. $\square$

Let $f : X \to Y$ be a morphism of rigid analytic spaces over K. There exists a morphism of admissible formal R-schemes

$$\tilde{f} : \mathcal{X} \longrightarrow \mathcal{Y}$$

which is a formal model of f (cf. [5, Theorem 4.1]).

Proposition 2.4. *The complex $(\mathsf{L}^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}})^{\mathrm{rig}}$ is independent of the choice of the formal model $\tilde{f} \colon \mathcal{X} \to \mathcal{Y}$ of $f \colon X \to Y$, i.e., depends only on the morphism $f : X \to Y$.*

Proof. Let $\tilde{f}' : \mathcal{X}' \to \mathcal{Y}'$ be another formal model of $f : X \to Y$. By an easy application of the theorem of Raynaud [5, Theorem 4.1], we can find the following commutative diagram

$$
\begin{array}{ccccc}
\mathcal{X} & \xleftarrow{\ \alpha\ } & \mathcal{X}'' & \xrightarrow{\ s\ } & \mathcal{X}' \\
\downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \tilde{f}''} & & \downarrow{\scriptstyle \tilde{f}'} \\
\mathcal{Y} & \xleftarrow{\ \beta\ } & \mathcal{Y}'' & \xrightarrow{\ t\ } & \mathcal{Y}
\end{array}
$$

such that $\tilde{f}, \tilde{f}', \tilde{f}''$ are formal models of $f : X \to Y$ and α, β, s, t are admissible blowing ups. It suffices to show that there are natural isomorphisms $(\mathsf{L}^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}})^{\mathrm{rig}} \cong (\mathsf{L}^{\mathrm{an}}_{\mathcal{X}''/\mathcal{Y}''})^{\mathrm{rig}} \cong (\mathsf{L}^{\mathrm{an}}_{\mathcal{X}'/\mathcal{Y}'})^{\mathrm{rig}}$. We will show an existence of the first isomorphism. The second one follows from the proof of the first one. By the transitivity to the sequences $\mathcal{X}'' \to \mathcal{X} \to \mathcal{Y}$ and $\mathcal{X}'' \to \mathcal{Y}'' \to \mathcal{Y}$ ([14, 7.2.13]), we have distinguished triangles

$$
L\alpha^* \mathsf{L}^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}} \to \mathsf{L}^{\mathrm{an}}_{\mathcal{X}''/\mathcal{Y}} \to \mathsf{L}^{\mathrm{an}}_{\mathcal{X}''/\mathcal{X}} \to L\alpha^* \mathsf{L}^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}}[1]
$$

and

$$
L\tilde{f}''^* \mathsf{L}^{\mathrm{an}}_{\mathcal{Y}''/\mathcal{Y}} \to \mathsf{L}^{\mathrm{an}}_{\mathcal{X}''/\mathcal{Y}} \to \mathsf{L}^{\mathrm{an}}_{\mathcal{X}''/\mathcal{Y}''} \to L\tilde{f}''^* \mathsf{L}^{\mathrm{an}}_{\mathcal{Y}''/\mathcal{Y}}[1].
$$

According to [14, 7.2.42 (i)], after applying the derived functor

$$
\mathrm{Rig} : \mathsf{D}^-(\mathcal{O}_{\mathcal{X}''}\text{-Mod}) \to \mathsf{D}^-(\mathcal{O}_X\text{-Mod})
$$

to $L\tilde{f}''^* \mathsf{L}^{\mathrm{an}}_{\mathcal{Y}''/\mathcal{Y}}$ and $\mathsf{L}^{\mathrm{an}}_{\mathcal{X}''/\mathcal{X}}$, we have quasi-isomorphisms $(L\tilde{f}''^* \mathsf{L}^{\mathrm{an}}_{\mathcal{Y}''/\mathcal{Y}})^{\mathrm{rig}} \cong 0$ and $(\mathsf{L}^{\mathrm{an}}_{\mathcal{X}''/\mathcal{X}})^{\mathrm{rig}} \cong 0$. Therefore, by the above two triangles and Lemma 2.3 (2), we have a quasi-isomorphism $(\mathsf{L}^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}})^{\mathrm{rig}} \cong (\mathsf{L}^{\mathrm{an}}_{\mathcal{X}''/\mathcal{Y}})^{\mathrm{rig}} \cong (\mathsf{L}^{\mathrm{an}}_{\mathcal{X}''/\mathcal{Y}''})^{\mathrm{rig}}$. $\qquad\square$

We define a cotangent complex $\mathsf{L}^{\mathrm{rig}}_{X/Y}$ in $\mathsf{D}^-(\mathcal{O}_X\text{-Mod})$ of $f : X \to Y$ by $(\mathsf{L}^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}})^{\mathrm{rig}}$ for some formal model $\mathcal{X} \to \mathcal{Y}$. We shall refer to this complex as the *rigid cotangent complex*.

For every morphism $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ of admissible formal schemes, there is a natural morphism: $\mathsf{L}^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}} \to \Omega^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}}$ which induces an isomorphism $H_0(\mathsf{L}^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}}) \cong \Omega^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}}$ (cf. [14, 7.2.8]). Thus we have a natural morphism

$$
\mathsf{L}^{\mathrm{rig}}_{X/Y} \to \Omega^{\mathrm{rig}}_{X/Y}
$$

which induces an isomorphism

$$
H_0(\mathsf{L}^{\mathrm{rig}}_{X/Y}) \xrightarrow{\sim} \Omega^{\mathrm{rig}}_{X/Y}
$$

where $X := \mathcal{X}^{\mathrm{rig}}$ and $Y := \mathcal{Y}^{\mathrm{rig}}$ and $\Omega^{\mathrm{rig}}_{X/Y}$ is the differential module defined in [7, Section 1].

Remark 2.5.   Our construction also works in the case of relative rigid spaces introduced and studied by Bosch, Lütkebohmert and Raynaud (cf. [5]–[8]). However, in present paper, we concentrate our attentions on the classical case.

In virtue of the theorem of Raynaud [5, 4.1] and formal flattening theorem (See [6, 5.2]), as we will see below, the study of analytic cotangent complexes of rigid analytic spaces is reduced to the study of the formal case. Thus Proposition 2.4 is important.

Proposition 2.6.   *Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of rigid analytic spaces. Then there is a natural quasi-isomorphism*

$$L pr_1^* \mathsf{L}_{X/Y}^{\mathrm{rig}} \xrightarrow{\sim} \mathsf{L}_{X \times_Y Z/Z}^{\mathrm{rig}}$$

where pr_1 is the first projection $X \times_Y Z \to X$.

Proof.   Take formal models $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ and $\tilde{g} : \mathcal{Z} \to \mathcal{Y}$ of f and g respectively. Note that the fibre product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ in the category of formal R-schemes is the formal model of $X \times_Y Z$ by [5, 4.6]. Now our assertion follows from Proposition 2.1.   □

Proposition 2.7 (cf. [14] 7.2.42 (ii)).   *If $f : X \to Y$ is a smooth morphism, there is a natural quasi-isomorphism*

$$\mathsf{L}_{X/Y}^{\mathrm{rig}} \xrightarrow{\sim} \Omega_{X/Y}^{\mathrm{rig}}[0].$$

Proof.   Take a formal model $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ of $f : X \to Y$. Then, we derive our assertion by applying [14, 7.2.42 (i)] to $\tilde{f} : \mathcal{X} \to \mathcal{Y}$.   □

Proposition 2.8 (cf. [14] 7.2.39).   *Let $X \xrightarrow{f} Y \to Z$ be a sequence of morphisms of rigid analytic spaces. Then, there is a natural distinguished triangle in $\mathsf{D}^-(\mathcal{O}_X\text{-Mod})$*

$$Lf^* \mathsf{L}_{Y/Z}^{\mathrm{rig}} \to \mathsf{L}_{X/Z}^{\mathrm{rig}} \to \mathsf{L}_{X/Y}^{\mathrm{rig}} \to Lf^* \mathsf{L}_{Y/Z}^{\mathrm{rig}}[1].$$

Proof.   By the theorem of Raynaud [5, 4.1], there is a formal model of $X \to Y \to Z$

$$\mathcal{X} \xrightarrow{\tilde{f}} \mathcal{Y} \longrightarrow \mathcal{Z}.$$

From the transitivity of this sequence [14, 7.2.13], we have

$$L\tilde{f}^* \mathsf{L}_{\mathcal{Y}/\mathcal{Z}}^{\mathrm{an}} \to \mathsf{L}_{\mathcal{X}/\mathcal{Z}}^{\mathrm{an}} \to \mathsf{L}_{\mathcal{X}/\mathcal{Y}}^{\mathrm{an}} \to L\tilde{f}^* \mathsf{L}_{\mathcal{Y}/\mathcal{Z}}^{\mathrm{an}}[1].$$

By applying the functor of derived categories

$$\mathrm{Rig} : \mathsf{D}^-(\mathcal{O}_{\mathcal{X}}\text{-Mod}) \to \mathsf{D}^-(\mathcal{O}_X\text{-Mod}),$$

we obtain the triangle

$$(L\tilde{f}^* \mathsf{L}_{\mathcal{Y}/\mathcal{Z}}^{\mathrm{an}})^{\mathrm{rig}} \to (\mathsf{L}_{\mathcal{X}/\mathcal{Z}}^{\mathrm{an}})^{\mathrm{rig}} \to (\mathsf{L}_{\mathcal{X}/\mathcal{Y}}^{\mathrm{an}})^{\mathrm{rig}} \to (L\tilde{f}^* \mathsf{L}_{\mathcal{Y}/\mathcal{Z}}^{\mathrm{an}}[1])^{\mathrm{rig}}.$$

Thus we have the required triangle by Lemma 2.3 (2).   □

Theorem 2.9 (cf. [14] 7.2.46)**.**
(1) *Let $i : Y \to X$ is a closed immersion of rigid analytic spaces. Then the natural morphism*

$$\mathsf{L}_{Y/X} \longrightarrow \mathsf{L}_{Y/X}^{\mathrm{rig}}$$

is a quasi-isomorphism. Here $\mathsf{L}_{Y/X}$ is the usual cotangent complex associated to the morphism of ringed topoi $i : Y \to X$.
(2) *Let $f : X \to Y$ be a morphism of rigid analytic spaces. Then $\mathsf{L}_{X/Y}^{\mathrm{rig}}$ is a pseudo-coherent complex of $\mathcal{O}_X$- modules.*

Proof. (1) First of all, take a formal model $\tilde{i} : \mathcal{Y} \to \mathcal{X}$ of $i : Y \to X$. Note that there is a natural isomorphism $\mathsf{L}_{Y/X} \cong (\mathsf{L}_{\mathcal{Y}/\mathcal{X}})^{\mathrm{rig}}$ by [18, 2.2.3]. Thanks to the formal flattening theorem [6, 5.4 (b)], we can modify $\tilde{i} : \mathcal{Y} \to \mathcal{X}$ to be a closed immersion. Then the claim is reduced to [14, 7.2.10 (2)].
(2) Let $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ be a formal model of $f : X \to Y$. Then the claim is reduced to the claim for $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ [14, 7.2.10 (1)]. $\qquad\square$

Proposition 2.10 (cf. [14] 7.2.48)**.** *Let $f : X \to Y$ be a closed immersion of rigid analytic spaces and $g : Y \to Z$ a smooth morphism of rigid analytic spaces. Let $\mathcal{I}$ be a coherent ideal of $\mathcal{O}_Y$ which defines X. Then there is a quasi-isomorphism*

$$\tau_{[-1}\mathsf{L}_{X/Y}^{\mathrm{rig}} \xrightarrow{\sim} [0 \to f^*(\mathcal{I}/\mathcal{I}^2) \xrightarrow{d} f^*\Omega_{Y/Z}^{\mathrm{rig}} \to 0].$$

Proof. First, we remark that by [18, Chapter 3, 1.2.8.1] and Theorem 2.9 (1), there exists a natural quasi-isomorphisms, $\tau_{[-1}\mathsf{L}_{X/Y}^{\mathrm{rig}} \cong \tau_{[-1}\mathsf{L}_{X/Y} \cong f^*(\mathcal{I}/\mathcal{I}^2)[1]$. Then, we can prove our assertion in the same way as the proof of [18, Chapter 3, 1.2.9.1] by using Propositions 2.7 and 2.8. $\qquad\square$

Let $X \to Y$ be a morphism of rigid analytic spaces and $\mathcal{S}$ a coherent $\mathcal{O}_X$-module. Let us consider a triple $(i : X \to X', \mathcal{S}, \phi)$ where i is a closed immersion of rigid analytic spaces over Y with the square-zero kernel $\mathcal{I} = \mathrm{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X)$ and an isomorphism of coherent $\mathcal{O}_X$-modules $\phi : i^*\mathcal{I} \to \mathcal{S}$. Then there exists a natural quasi-isomorphism $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}_{X/X'}^{\mathrm{rig}}, \mathcal{S}) \cong \mathrm{Hom}_{\mathcal{O}_X}(i^*\mathcal{I}, \mathcal{S})$, since we have a natural isomorphism $\tau_{[-1}\mathsf{L}_{X/X'}^{\mathrm{rig}} \cong i^*\mathcal{I}[1]$ by Proposition 2.10. On the other hand, the distinguished triangle (Proposition 2.8) associated to the sequence $X \to X' \to Y$ induces a homomorphism $p : \mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}_{X/X'}^{\mathrm{rig}}, \mathcal{S}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}_{X/Y}^{\mathrm{rig}}, \mathcal{S})$. Thus it gives rise to a map

$$e : \mathrm{EX}_Y(X, \mathcal{S}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}_{X/Y}^{\mathrm{rig}}, \mathcal{S}),$$
$$(i : X \to X', \mathcal{S}, \phi) \mapsto p(\phi),$$

where $\mathrm{EX}_Y(X, \mathcal{S})$ is the set of isomorphism classes of a triple $(i : X \to X', \mathcal{S}, \phi)$. Note that the set $\mathrm{EX}_Y(X, \mathcal{S})$ is the subset of the set $\mathrm{EXal}_Y(X, \mathcal{S})$ of isomorphism classes of the extensions of X by $\mathcal{S}$ over K as locally ringed spaces.

Indeed for elements $\alpha, \beta \in \mathrm{EX}_Y(X, \mathcal{S})$ and an isomorphism $\phi : \alpha \xrightarrow{\sim} \beta$ as locally ringed spaces over K which induces the identity on X, ϕ is actually the isomorphism as rigid analytic spaces by Proposition 1.6 and [3, 6.1.3.1]. Hence, taking [18, Chapter 3, 1.2.3] into account, e is injective because the composite map $\mathrm{EX}_Y(X, \mathcal{S}) \xrightarrow{e} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/Y}, \mathcal{S}) \to \mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}_{X/Y}, \mathcal{S})$ is equal to the composite map $\mathrm{EX}_Y(X, \mathcal{S}) \to \mathrm{EXal}_Y(X, \mathcal{S}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}_{X/Y}, \mathcal{S})$.

Theorem 2.11 (cf. [14] 7.3.22)**.**   *The map e is a bijection.*

Proof.   What we need to show is that elements of the image $\mathrm{Ext}^1(\mathsf{L}^{\mathrm{rig}}_{X/Y}, \mathcal{S})$ in $\mathrm{Ext}^1(\mathsf{L}_{X/Y}, \mathcal{S}) \cong \mathrm{EXal}_Y(X, \mathcal{S})$ represent the sheaves of rigid analytic spaces which are extensions of $\mathcal{O}_X$ by $\mathcal{S}$. Therefore the problem is local on X. Set $X := \mathrm{Sp}(B)$ and $Y := \mathrm{Sp}(A)$. We choose a closed immersion $i : X \cong \mathrm{Sp}(C/I) \to Z := \mathrm{Sp}(C)$ where $C = A\langle X_1, \ldots, X_r \rangle$. By Proposition 2.10, we have

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/Y}, \mathcal{S}) \cong \mathrm{Hom}(I/I^2, \mathcal{S})/\mathrm{Image}(d^*).$$

However a map $\phi : I/I^2 \to \mathcal{S}$ and the canonical immersion $j : I/I^2 \to C/I^2$ define an extension $(C/I^2 \oplus \mathcal{S})/\mathrm{Image}((j, \phi))^\sim$ of $\mathcal{O}_X$ by $\mathcal{S}$ which is a sheaf arising from an affinoid algebra. It is easy to see that this extension corresponds to the element ϕ in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/Y}, \mathcal{S})$. $\qquad\square$

Remark 2.12.   For readers who know Huber's theory of adic spaces and [14, Section 7], we shall give some comments on the relation of cotangent complexes defined here with cotangent complexes defined [14, Section 7]. Let $f : X \to Y$ be a morphism of rigid analytic spaces and let $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ be a formal model of f. Then we obtain the following commutative diagram of ringed spaces (moreover adic spaces):

$$
\begin{array}{ccc}
\langle \mathcal{X} \rangle & \xrightarrow{\ \pi\ } & \mathcal{X} \\
{\scriptstyle \langle \tilde{f} \rangle} \downarrow & & \downarrow {\scriptstyle \tilde{f}} \\
\langle \mathcal{Y} \rangle & \longrightarrow & \mathcal{Y}.
\end{array}
$$

We regard $\langle \mathcal{X} \rangle$ (resp. $\langle \mathcal{Y} \rangle$) as an adic space $(\langle \mathcal{X} \rangle, \mathcal{O}_{\langle \mathcal{X} \rangle} \otimes_R K, \mathcal{O}_{\langle \mathcal{X} \rangle})$ (resp. $(\langle \mathcal{Y} \rangle, \mathcal{O}_{\langle \mathcal{Y} \rangle} \otimes_R K, \mathcal{O}_{\langle \mathcal{Y} \rangle})$), and denote by $\mathsf{L}^+_{\langle \mathcal{X} \rangle / \langle \mathcal{Y} \rangle}$ the complex constructed in (cf. [14, 7.2.32]). By inspecting the constructions in [14, Section 7] (it is a straightforword argument), we can see that there exists a natural morphism of complexes

$$\pi^* \mathsf{L}^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}} \to \mathsf{L}^+_{\langle \mathcal{X} \rangle / \langle \mathcal{Y} \rangle}$$

and furthermore after tensoring K the morphism

$$(*) \qquad\qquad \pi^* \mathsf{L}^{\mathrm{an}}_{\mathcal{X}/\mathcal{Y}} \otimes_R K \to \mathsf{L}^{\mathrm{an}}_{\langle \mathcal{X} \rangle / \langle \mathcal{Y} \rangle} := \mathsf{L}^+_{\langle \mathcal{X} \rangle / \langle \mathcal{Y} \rangle} \otimes_R K$$

is a quasi-isomorphism if the next claim holds:

Claim 2.12.1. Consider the commutative diagram of admissible formal schemes:

$$\begin{array}{ccc} \operatorname{Spf} D & \longrightarrow & \operatorname{Spf} B \\ \downarrow & & \downarrow \\ \operatorname{Spf} C & \longrightarrow & \operatorname{Spf} A. \end{array}$$

Assume that the associated morphisms of rigid analytic spaces

$$\operatorname{Sp} D \otimes_R K \to \operatorname{Sp} B \otimes_R K$$

and

$$\operatorname{Sp} C \otimes_R K \to \operatorname{Sp} A \otimes_R K$$

are open immersions. Then the natural morphism

(**) $$\mathsf{L}^{\mathrm{an}}_{B/A} \otimes_B D \otimes_R K \to \mathsf{L}^{\mathrm{an}}_{D/B} \otimes_R K$$

is a quasi-isomorphism.

We remark that the morphism $(**)$ is an algebraic counterpart of $(*)$.

Proof of Claim. By the transitivity, we have two distinguished triangles

$$\mathsf{L}^{\mathrm{an}}_{B/A} \otimes_B D \otimes_R K \to \mathsf{L}^{\mathrm{an}}_{D/A} \otimes_R K \to \mathsf{L}^{\mathrm{an}}_{D/B} \otimes_R K \to \mathsf{L}^{\mathrm{an}}_{B/A} \otimes_B D \otimes_R K[1]$$

and

$$\mathsf{L}^{\mathrm{an}}_{C/A} \otimes_C D \otimes_R K \to \mathsf{L}^{\mathrm{an}}_{D/A} \otimes_R K \to \mathsf{L}^{\mathrm{an}}_{D/B} \otimes_R K \to \mathsf{L}^{\mathrm{an}}_{C/A} \otimes_C D \otimes_R K[1].$$

By our assumption and [14, 7.2.42 (i)], we obtain quasi-isomorphisms $\mathsf{L}^{\mathrm{an}}_{D/B} \otimes_R K \cong 0$ and $\mathsf{L}^{\mathrm{an}}_{C/A} \otimes_C D \otimes_R K \cong 0$. Therefore we have a quasi-isomorphism

$$\mathsf{L}^{\mathrm{an}}_{B/A} \otimes_B D \otimes_R K \cong \mathsf{L}^{\mathrm{an}}_{D/A} \otimes_R K \cong \mathsf{L}^{\mathrm{an}}_{D/B} \otimes_R K$$

and this completes the proof. □

From the quasi-isomorphism $(*)$, we see that, by the functor of restriction

$$\mathsf{D}^-(\mathcal{O}_{\langle \mathcal{X} \rangle} \otimes_R K \text{-Mod}) \to \mathsf{D}^-(\mathcal{O}_X \text{-Mod}),$$

$\mathsf{L}^{\mathrm{an}}_{\langle \mathcal{X} \rangle / \langle \mathcal{Y} \rangle}$ is sent to our cotangent complex $\mathsf{L}^{\mathrm{an}}_{X/Y}$. Moreover, if we use the functor

$$\mathsf{D}^-(\mathcal{O}_{\mathcal{X}} \text{-Mod}) \to \mathsf{D}^-(\mathcal{O}_{\langle \mathcal{X} \rangle} \otimes_R K \text{-Mod}); \quad \mathcal{M} \mapsto \pi^* \mathcal{M} \otimes_R K$$

instead of the functor Rig, we can prove Proposition 2.6–2.10 for Huber's adic spaces, which are already proved in [14].

3.   Cohomological descriptions of Local deformations

Now by applying the results in Section 2, we will prove the following theorem.

Theorem 3.1.   *Let $u : S \to S'$ be a closed immersion of rigid analytic spaces with the nilpotent kernel $\mathcal{I}$ of $u^* : \mathcal{O}_{S'} \to \mathcal{O}_S$ such that $\mathcal{I}^2 = 0$. Let $f : X \to S$ be a flat morphism of rigid analytic spaces over K.*
(1) The obstruction for the existence of the lifting of f to S' lies in $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/S}, f^*\mathcal{I})$. Here by the lifting we mean a flat morphism $f' : X' \to S'$ such that $X' \times_{S'} S \cong X$ over S.
(2) If the obstruction o is zero, the set of isomorphism classes of the liftings of f to S' forms a torsor under $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/S}, f^*\mathcal{I})$.
(3) Let $\tilde{f} : \tilde{X} \to S'$ be a flat deformation of X. Then, the automorphism group of the lifting $\tilde{X}$ is canonically isomorphic to $\mathrm{Ext}^0_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/S}, f^*\mathcal{I})$.

Proof.   By applying Proposition 2.8 to the sequence $X \to S \to S'$, we have a distinguished triangle

$$Lf^*\mathsf{L}^{\mathrm{rig}}_{S/S'} \to \mathsf{L}^{\mathrm{rig}}_{X/S} \to \mathsf{L}^{\mathrm{rig}}_{X/S'} \to Lf^*\mathsf{L}^{\mathrm{rig}}_{S/S'}[1].$$

On the other hand, by Proposition 2.10, we have $\tau_{[-1}\mathsf{L}^{\mathrm{rig}}_{S/S'} \cong u^*\mathcal{I}[1]$. Thus there exists a natural isomorphism, $\mathrm{Ext}^1_{\mathcal{O}_X}(Lf^*\mathsf{L}^{\mathrm{rig}}_{S/S'}, \mathcal{I}\mathcal{O}_X) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}\mathcal{O}_X, \mathcal{I}\mathcal{O}_X)$. Since $\Omega^{\mathrm{rig}}_{S/S'} = 0$, we have $\mathrm{Ext}^0_{\mathcal{O}_X}(Lf^*\mathsf{L}^{\mathrm{rig}}_{S/S'}, \mathcal{I}\mathcal{O}_X) \cong \mathrm{Hom}_{\mathcal{O}_X}(f^*\Omega^{\mathrm{rig}}_{S/S'}, \mathcal{I}\mathcal{O}_X) = 0$. Thus there is a long exact sequence

$$
\begin{aligned}
0 &\longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/S}, \mathcal{I}\mathcal{O}_X) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/S'}, \mathcal{I}\mathcal{O}_X) \\
&\xrightarrow{\ \xi\ } \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}\mathcal{O}_X, \mathcal{I}\mathcal{O}_X) \xrightarrow{\ \delta\ } \mathrm{Ext}^2_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/S}, \mathcal{I}\mathcal{O}_X).
\end{aligned}
$$

An element in $\mathrm{EX}_{S'}(X, \mathcal{I}\mathcal{O}_X)$ canonically induces an element which lies in $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}\mathcal{O}_X, \mathcal{I}\mathcal{O}_X)$. Thus we have a map

$$\alpha : \mathrm{EX}_{S'}(X, \mathcal{I}\mathcal{O}_X) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}\mathcal{O}_X, \mathcal{I}\mathcal{O}_X).$$

There is a natural isomorphism $\mathrm{EX}_{S'}(X, \mathcal{I}\mathcal{O}_X) \xrightarrow{\sim} \mathrm{Ext}^1(\mathsf{L}^{\mathrm{rig}}_{X/S'}, \mathcal{I}\mathcal{O}_X)$ by Theorem 2.11 and this isomorphism identifies ξ with α. Let u be the element in $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}\mathcal{O}_X, \mathcal{I}\mathcal{O}_X)$ which is induced by f. The existence of a flat deformation of X over S' is equivalent to the existence of an element of $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/S'}, \mathcal{I}\mathcal{O}_X)$ which induces u by ξ, so (1) follows. Now it is clear that if $\delta u = 0$, the isomorphism classes of flat deformations is $\xi^{-1}(u) \cong \mathrm{Ext}^1_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/S}, \mathcal{I}\mathcal{O}_X)$. This shows (2).   Finally the isomorphism $H_0(\mathsf{L}^{\mathrm{rig}}_{X/Y}) \cong \Omega^{\mathrm{rig}}_{X/Y}$ implies that $\mathrm{Ext}^0_{\mathcal{O}_X}(\mathsf{L}^{\mathrm{rig}}_{X/S}, \mathcal{I}\mathcal{O}_X) \cong \mathrm{Hom}(\Omega^{\mathrm{rig}}_{X/S}, \mathcal{I}\mathcal{O}_X)$. It is well-known that for every flat deformation of X to S', its automorphism group is isomorphic to the right-hand

group. (We can show this by the completely same argument as the scheme-case
(cf. [15])). $\qquad\square$

4. Formal versal deformation

4.1. Preliminaries

In this section, we will consider the deformations of rigid analytic K-spaces
to the spectrums of local Artin K-algebras. Since every local Artin K-algebras
is a finite K-vector space, it is the topological ring whose topology are induced
by the topology of K. We call this topology *the canonical topology*.

Lemma 4.1. *Let K be a complete non-Archimedean valued field. Then,
an Artin local K-algebra with the canonical topology is a K-affinoid algebra.*

Proof. Since a local Artin K-algebra A is of finite type over K, we have
the representation of A by the quotient ring of a polynomial ring

$$A \cong K[X_1, \ldots, X_n]/I.$$

If we equip A with canonical topology and $K[X_1, \ldots, X_n]/I$ with Gauss norm,
this is an isomorphism of complete topological rings. The Tate algebra
$K\langle X_1, \ldots, X_n \rangle$ is flat over $K[X_1, \ldots, X_n]$. Thus there exists a canonical iso-
morphism

$$K[X_1, \ldots, X_n]/I \cong K\langle X_1, \ldots, X_n \rangle/IK\langle X_1, \ldots, X_n \rangle.$$

This completes the proof. $\qquad\square$

Remark 4.2. By the above lemma, we can attach an affinoid space
$\mathrm{Sp}(A)$ to any Artin local K-algebra A. Since every homomorphism of affinoid
algebras as K-algebras is automatically a continuous homomorphism, we can
view the category of Artin local K-algebras as the full subcategory of the
category of K-affinoid algebras.

Definition 4.3. For a rigid analytic space X over K, a *local deforma-
tion functor* $\mathbb{D}_X$ of X is the functor defined as follows:

$$\mathbb{D}_X : \left(\text{Artin local } K\text{-algebras with residue field } K \right) \longrightarrow \left(\text{Sets} \right)$$

Let A be an Artin local K-algebra with the maximal ideal $\mathfrak{m}_A$ and the residue
field K. Then the set $\mathbb{D}_X(A)$ is the isomorphism classes of pairs $(f : \tilde{X} \to
\mathrm{Sp}(A), \alpha : \tilde{X} \times_A A/\mathfrak{m}_A \xrightarrow{\sim} X)$ of a flat morphism f of rigid analytic spaces
over K and α is an isomorphism of rigid analytic spaces over K.

4.2. Schlessinger's theory

Let X be a rigid analytic spaces over $\mathrm{Sp}(K)$.

Definition 4.4. Let $\mathcal{O}$ be a complete local noetherian K-algebra with
maximal ideal $\mathfrak{m}$ with residue field K. Let $\{X_n\}_{n \geq 0}$ be a family of deforma-
tions of X to $\mathrm{Sp}(\mathcal{O}/\mathfrak{m}^{n+1})$ such that X_n is a flat deformation to $\mathrm{Sp}(\mathcal{O}/\mathfrak{m}^{n+1})$

and $X_n \times_{\mathrm{Sp}(\mathcal{O}/\mathfrak{m}^{n+1})} \mathrm{Sp}(\mathcal{O}/\mathfrak{m}^{m+1}) \cong X_m$ for $m \leq n$. We say that a pair $(\mathcal{O}, \{X_n\}_{n \geq 0})$ is *a formal versal deformation* of X if it satisfies the following conditions.

(1) Suppose that A is an Artin local K-algebra with the maximal ideal $\mathfrak{m}_A$ such that $\mathfrak{m}_A^{n+1} = 0$. If $\tilde{X}$ is a deformation of X to $\mathrm{Sp}(A)$, there exists a local homomorphism $f : \mathcal{O}/\mathfrak{m}^{n+1} \to A$ such that $\tilde{X}$ is isomorphic to $X_n \times_{\mathrm{Sp}(\mathcal{O}/\mathfrak{m}^{n+1})} \mathrm{Sp}(A)$.

(2) If $A = K[\epsilon]/(\epsilon^2)$, such f is unique.

The existence of a formal versal deformation of X is equivalent to the existence of a prorepresentable hull of the functor $\mathbb{D}_X$, in the sense of Schlessinger (cf. [22]).

We have a convenient criterion for the existence of a prorepresentable hull.

Theorem 4.5 (Schlessinger [22]). *Let $A' \to A$ and $A'' \to A$ be morphisms of Artin local K-algebras. Consider the natural map*

$$F : \mathbb{D}_X(A' \times_A A'') \to \mathbb{D}_X(A') \times_{\mathbb{D}_X(A)} \mathbb{D}_X(A'').$$

The $\mathbb{D}_X$ has a prorepresentable hull if and only if the following conditions are satisfied:

(H1) *If $p : A'' \to A$ is a small surjection, F is surjective. Here we say that $A'' \to A$ is a small surjection if it is surjective and $\mathfrak{m}_{A''} \cdot \mathrm{Ker}(A'' \to A) = 0$, where $\mathfrak{m}_{A''}$ is the maximal ideal of A''.*

(H2) *F is bijective when $A = K$, $A'' = K[\epsilon]/(\epsilon^2)$.*

(H3) $\dim_K \mathbb{D}_X(K[\epsilon]/(\epsilon^2)) < \infty.$

4.3. Existence of a formal versal deformation
Theorem 4.6. *Let X be a proper rigid analytic space over K. Then, there exists a formal versal deformation of X.*

First we will show (H1).

Claim 4.6.1. The functor $\mathbb{D}_X$ satisfies (H1).

Proof. Let (ξ', ξ'') be an element of $\mathbb{D}_X(A') \times_{\mathbb{D}_X(A)} \mathbb{D}_X(A'')$. We put

$$\xi := (\tilde{X}/\mathrm{Sp}(A), \phi : \tilde{X} \times_A K \cong X),$$
$$\xi' := (\tilde{X}'/\mathrm{Sp}(A'), \phi' : \tilde{X}' \times_A K \cong X),$$
$$\xi'' := (\tilde{X}''/\mathrm{Sp}(A''), \phi : \tilde{X}'' \times_A K \cong X),$$

such that $\alpha^* \xi' = \xi$ and $\beta^* \xi'' = \xi$ where $A' \xrightarrow{\alpha} A$, $A'' \xrightarrow{\beta} A$. Fix an affinoid open set $\mathrm{Sp}(R)$ of $\tilde{X}$. Note that $|X| = |\tilde{X}| = |\tilde{X}'| = |\tilde{X}''|$. Here $| \bullet |$ means the underlying set. The subspace $\tilde{X}'|_{\mathrm{Sp}(R)}$ (resp. $\tilde{X}''|_{\mathrm{Sp}(R)}$) of $\tilde{X}'$ (resp. $\tilde{X}''$)

which is the lifting of $\mathrm{Sp}(R)$ is an affinoid space by Proposition 1.6. We set $\mathrm{Sp}(R') = \tilde{X}'|_{\mathrm{Sp}(R)}$, $\mathrm{Sp}(R'') = \tilde{X}''|_{\mathrm{Sp}(R)}$ and $s : R' \to R$, $t : R'' \to R$. To show our claim, it suffices to prove that $R' \times_R R''$ is an affinoid algebra. First we define a norm on $R' \times R''$ by the direct sum of the norms of R' and R''. This norm induces the topology on $R' \times_R R''$. This topology is complete. Indeed, if $\{(a_n, b_n)\}_{n \leq 1}$ is a Cauchy sequence of $R' \times_R R''$, $\{s(a_n) = t(b_n)\}_{n \leq 1}$ is also a Cauchy sequence of R since $|t(b_n) - t(b_m)| \leq |b_n - b_m|$. Now what to do is to construct a continuous surjective map form an affinoid algebra to $R' \times_R R''$. To this end, take a generator $\{\xi_1, \ldots, \xi_r\}$ of $\mathrm{Ker}(t)$ such that $|\xi_i|_{sp} \leq 1$ for $1 \leq i \leq r$. Next put $R' = K\langle X_1, \ldots, X_n \rangle / I$. We can choose elements $\eta_1, \ldots, \eta_n$ in R'' such that $s(X_i) = t(\eta_i)$ and $|\eta_i|_{sp} \leq 1$ for $1 \leq i \leq n$. Indeed, since t is a small surjection, $|r|_{sp} = |t(r)|_{sp}$ for $r \in R''$ and $|s(X_i)|_{sp} \leq |X_i|_{sp} \leq |X_i| \leq 1$ for $1 \leq i \leq n$. Therefore, we have the following homomorphism

$$K[S_1, \ldots, S_n, T_1, \ldots, T_r] \to R' \times_R R''$$

defined by $S_i \to (X_i, \eta_i)$ and $T_j \to (0, \xi_j)$. Since $|X_i|_{sp} \leq 1$, $|\eta_i|_{sp} \leq 1$ and $|\xi_j|_{sp} \leq 1$ for all i and j, this homomorphism is uniquely extended to the continuous homomorphism (See [12, 3.4.7])

$$K\langle S_1, \ldots, S_n, T_1, \cdots, T_r \rangle \to R' \times_R R''.$$

Furthermore, from the construction, it is clear that this continuous homomorphism is surjective. Hence we see (H1). $\qquad \square$

Next we show (H2) and (H3).

Claim 4.6.2. The functor $\mathbb{D}_X$ satisfies (H2).

Proof. We can show this by the completely same argument as the scheme-case (cf. [22]). $\qquad \square$

Claim 4.6.3. The functor $\mathbb{D}_X$ satisfies (H3).

Proof. Let $f : X \to \mathrm{Sp}(K)$ be a structure morphism. Then, the isomorphism of derived functors

$$Rf_* R\mathcal{H}om(-, \mathcal{O}_X) \cong R\,\mathrm{Hom}(-, \mathcal{O}_X)$$

induces the spectral sequence

$$E_2^{p,q} = R^q f_* \mathcal{E}xt^p(\mathsf{L}_{X/K}^{\mathrm{rig}}, \mathcal{O}_X) \Rightarrow \mathrm{Ext}^{p+q}(\mathsf{L}_{X/K}^{\mathrm{rig}}, \mathcal{O}_X).$$

Then it suffices to show that $R^q f_* \mathcal{E}xt^p(\mathsf{L}_{X/K}^{\mathrm{rig}}, \mathcal{O}_X)$ is coherent for all p and q. Note that by Theorem 2.9 (2), $\mathsf{L}_{X/K}^{\mathrm{rig}}$ is pseudo-coherent. Thus $R\mathcal{H}om(\mathsf{L}_{X/K}^{\mathrm{rig}}, \mathcal{O}_X)$ is pseudo-coherent by [11, Chapter 0, 12.3.3] and [17, Chapter 1, 7.3]. By Kiehl's finiteness theorem, $R^q f_* \mathcal{E}xt^p(\mathsf{L}_{X/K}^{\mathrm{rig}}, \mathcal{O}_X)$ is a finite K-vector space for all p and q. $\qquad \square$

By the above three claims, we can complete the proof of Theorem 4.6 by Schlessinger's criterion.

5.   Towards the global moduli theory via Rigid geometry

In Section 4, we proved the existence of a versal family in the formal sense for deformations of rigid analytic spaces. However the analogy of rigid geometry with complex analytic geometry gives us a deeper question. When one compares the deformation theory of complex analytic spaces with one of the algebraic categories, no one doubts that the most important advantage is the existence of versal family of deformations proven by Kuranishi and Grauert (cf. [16], [20]). Formal deformations of algebraic varieties are not necessarily algebraic. Thus there is no algebraic analogue of the theorem of Kuranishi and Grauert. As we know, there is no logical relation between complex geometry and rigid geometry. However one can conjecture the following.

Conjecture.   Let X be a proper rigid analytic spaces over K. Then there exists a flat morphism of rigid analytic spaces $F : \mathfrak{X} \longrightarrow \mathfrak{S}$ and a K-rational point p of $\mathfrak{S}$ such that the completion of F at p is isomorphic to the formal versal deformation of X.

This assertion can be viewed as a fairly precise non-Archimedean analogue of the existence theorem of versal families for deformations of complex analytic spaces due to Kuranishi and Grauert. Let $\mathcal{S}$ be a rigid analytic torus, i.e., a quotient space T^{an}/Γ where T is a split K-torus and Γ is a torsion-free lattice of rank $\dim T$. Then, we can prove that the conjecture holds for $\mathcal{S}$ by using p-adic uniformization theory due to Bosch-Lütkebohmert-Raynaud. Unfortunately, at the time of writing this paper, the author do not have a proof of this conjecture for general rigid analytic spaces. But the author expects that this conjecture is true and propose it.

Let us give one sufficient condition which implies the conjecture. Let K be a discrete valuation field and R its ring of integers with residue field k. Let X be a proper rigid analytic K-space. To prove the conjecture for X, it suffices to show the existence of a locally noetherian adic formal scheme over $\mathrm{Spf}\,R$ which satisfies the followings (cf. [2] section 0.2).

1. Its reduction is a scheme locally of finite type over $\mathrm{Spec}\,k$.

2. It is a formal model of the formal deformation family of the rigid analytic space X.

However it seems difficult to prove the conjecture in general.

Let us explain why this problem is important. Suppose that we want to construct a moduli space of interesting geometric objects. From the stack theoretic viewpoint, versal spaces for deformations of them (here "versal space" is not in formal sense but has a geometric structure such as a scheme (resp. complex analytic or rigid analytic etc...)) are local components of the smooth cover of an algebraic (resp. complex analytic, rigid analytic) stack (cf. [1]). Thus, roughly speaking, this conjecture says that in rigid geometry, the existence of

local deformation theory implies the global moduli stack which is represented by a rigid analytic stack.

Furthermore this conjecture will be important to the geometry of algebraic schemes over an arbitrary field. Let X be an algebraic scheme over an arbitrary field k. By considering the constant deformation $\hat{X} \to \mathrm{Spf}(k[\![T]\!])$. one can associate a rigid analytic space $\hat{X}^{\mathrm{rig}} \to \mathrm{Spf}(k[\![T]\!])^{\mathrm{rig}}$. By this technique the deformation of X to a family of algebraic schemes is contained in the rigid analytic versal family if it exists. Therefore the above conjecture is of prime importance not only in rigid geometry but also in algebraic geometry.

Appendix

In this appendix, we prove a convenient criterion of the existence of a non-trivial square-zero extension of a ringed topos.

Let A be a ring and B an A-algebra. Let M be a B-module and $\mathsf{L}_{B/A}$ the cotangent complex of the structure homomorphism $A \to B$. By $\mathrm{EXal}_A(B, M)$, we denote the set of isomorphism classes of square-zero extension of B by M over A. By the fundamental theorem due to L. Illusie ([18, Chapter 3, 1.2.3]), there exists a natural bijection

$$\phi : \mathrm{Ext}^1_A(\mathsf{L}_{B/A}, M) \xrightarrow{\sim} \mathrm{EXal}_A(B, M).$$

On the other hand, there exists a natural homomorphism

$$\pi : \mathrm{Ext}^1_A(\Omega^1_{B/A}, M) \to \mathrm{Ext}^1_A(\mathsf{L}_{B/A}, M)$$

which is induced by $\mathsf{L}_{B/A} \to \Omega^1_{B/A}$ (cf. [18, Chapter 2, 1.2.4]).

Theorem A.1. *The natural map* $\pi : \mathrm{Ext}^1_A(\Omega^1_{B/A}, M) \to \mathrm{Ext}^1_A(\mathsf{L}_{B/A}, M)$ *is injective. In particular, if* $\mathrm{Ext}^1_A(\Omega^1_{B/A}, M) \neq 0$*, there exists a non-trivial square-zero extension of B by M over A.*

Proof. First, note that it suffices to show that the map $\psi := \phi \circ \pi : \mathrm{Ext}^1_A(\Omega^1_{B/A}, M) \to \mathrm{EXal}_A(B, M)$ is injective. Let us construct explicitly the map ψ (cf. [18, Chapter 3, 1.1.8]). For an element ξ in $\mathrm{Ext}^1_A(\Omega^1_{B/A}, M)$, let $(0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} \Omega^1_{B/A} \to 0)$ be the corresponding short exact sequence. We define an A-algebra structure of $B \oplus N$ by $(b, n) + (b', n') := (b + b', n + n')$ and $(b, n) \cdot (b', n') := (b \cdot b', b \cdot n' + b' \cdot n)$. We also define an A-algebra structure of $B \oplus \Omega^1_{B/A}$ by the same way. Consider the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & C & \xrightarrow{\mathrm{pr}_2} & B & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle (\mathrm{Id}, d_{B/A})} & & \\
0 & \longrightarrow & M & \xrightarrow{(0,\alpha)} & B \oplus N & \xrightarrow{(\mathrm{Id},\beta)} & B \oplus \Omega^1_{B/A} & \longrightarrow & 0,
\end{array}
$$

where C is the fibre product $(B \oplus N) \times_{(B \oplus \Omega^1_{B/A})} B$. Then we define $\psi(\xi)$ by $(0 \to M \to C \to B \to 0)$. Since $\mathrm{EXal}_A(B, M)$ has a group structure by ϕ, it suffices to prove the following claim.

Claim A.1.1. Under the same assumption as above, if the exact sequence $(0 \to M \to C \to B \to 0)$ has a splitting $a : B \to C$, the exact sequence $(0 \to M \to N \to \Omega^1_{B/A} \to 0)$ also has a splitting $\Omega^1_{B/A} \to N$.

Proof. Let $\mathrm{Der}_A(B, N)$ be the set of A-derivations of B to N. Then note that we have a natural isomorphisms $\mathrm{Hom}_B(\Omega^1_{B/A}, N) \xrightarrow{\sim} \mathrm{Der}_A(B, N)$ and $\mathrm{Der}_A(B, N) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-alg}/\mathrm{B}}(B, B \oplus N)$ (cf. [18, Chapter 2, 1.1.1.4 and 1.1.2.6]). Let $\delta : \Omega^1_{B/A} \to N$ be a homomorphism that corresponds to the element $\mathrm{pr}_1 \circ a$ in $\mathrm{Hom}_{A\text{-alg}/\mathrm{B}}(B, B \oplus N)$ by the above isomorphisms. Then it is easy to see that δ is a splitting of $\beta : N \to \Omega^1_{B/A} \to 0$. $\qquad \square$

Thus we completes the proof of the theorem. $\qquad \square$

Let $f : X \to Y$ be a morphism of ringed topoi. Let $\Omega^1_{X/Y}$ be a Kählar differential module of $f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$, and $\mathsf{L}_{X/Y}$ the cotangent complex (cf. [18, chapter 2]). By $\mathrm{EXal}_Y(X, \mathcal{M})$, we denote the set of isomorphism classes of square-zero extension of X by a $\mathcal{O}_X$-module $\mathcal{M}$ over Y. By the same procedure with the above, we have a natural bijection $\phi : \mathrm{Ext}^1_Y(\mathsf{L}_{X/Y}, \mathcal{M}) \xrightarrow{\sim} \mathrm{EXal}_Y(X, \mathcal{M})$ and a homomorphism $\pi : \mathrm{Ext}^1_Y(\Omega^1_{X/Y}, \mathcal{M}) \to \mathrm{Ext}^1_Y(\mathsf{L}_{X/Y}, \mathcal{M})$.

Corollary A.2. *Let $X \to Y$ be a morphism of ringed topoi and $\mathcal{M}$ a $\mathcal{O}_X$-module. The natural map $\pi : \mathrm{Ext}^1_Y(\Omega^1_{X/Y}, \mathcal{M}) \to \mathrm{Ext}^1_A(\mathsf{L}_{X/Y}, \mathcal{M})$ is injective. In particular, if $\mathrm{Ext}^1_Y(\Omega^1_{X/Y}, \mathcal{M}) \neq 0$, there exists a non-trivial square-zero extension of X by $\mathcal{M}$ over Y.*

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