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On a compactification of Green spaces. Dirichlet problem and theorems of Riesz type

By

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Introduction

In this paper motivated by a recent paper of S. Mori [9] we shall study a compactification of Green spaces under the use of L. Naim's results [10] on Martin spaces and discuss the Dirichlet problem and some applications to the functiontheory.

We consider, as the basic space, a Green space R and define in §2 a compactification R^* of R, maximal ideal space of a normed ring $\mathfrak{S}$. Every non-negative continuous superharmonic function on R can be extended continuously onto R^* (Lemma 3). The ideal boundary R^*-R has a compact subset with remarkable properties, which is called, after H. L. Royden, the harmonic boundary (sec. 4).

In §3 we treat Dirichlet problems for functions given on the harmonic boundary. Since it is shown that R^* is not metrisable (Theorem 2), the usual Perron's approach must be somewhat modified, in particular for the discussion of solvability, but the results are quite similar.

As applications we show finally in §4 a theorem (Theorem 11) of Riesz type for several complex variables and refer to a Constantinescu-Cornea's theorem [4] on open Riemann surfaces which, in case of the unit circle, reduces exactly to the theorem of Riesz-Lusin-Privaloff-Frostman-Nevanlinna.

§ 1. Preliminaries

1. We shall state briefly some definitions and L. Naim's results [10] which will be used in the sequel. As the basic space R we consider a Green space introduced by M. Brelot and G. Choquet [2]. Bounded domains in the euclidean space of dimension $\geqq 2$ and Riemann surfaces of hyperbolic type are the most typical examples of Green spaces. Let q_0 be a fixed point of R and

$$K(p, q) = \begin{cases} G(p, q)/G(p, q_0) \\ 1 \qquad (p = q = q_0) \end{cases}$$

where G denotes the Green function on R, then there exists a compact metric space $\hat{R}$, Martin space, such that R is dense on $\hat{R}$ and for any sequence of points $\{p_n\}$ $(p_n \in R)$ tending to a point $s \in \hat{R} - R$, $\{K(p_n, q)\}$ converges uniformly on every compact subset of R to a uniquely determined harmonic function, say $K(s, q) \equiv K_s(q)$. We denote by Δ the ideal boundary $\hat{R} - R$ and by Δ_1 the set of points $s \in \Delta$ for which K_s are minimal in the class of positive harmonic functions on R. According to R. S. Martin [8] every positive harmonic function h on R can be represented by a canonical measure μ_h on Δ_1 as

$$h(p) = \int_{\Delta_1} K_s(p) \, d\mu_h(s) , \qquad p \in R$$

In case of $h \equiv 1$, we write $\mu_1 = \chi$.

Every K-potential carried with a positive measure on R

$$U(p) = \int_R K(p, q) \, dm(q)$$

is extended onto $\hat{R}$ as a lower semi-continuous function. Now a set $E \subset R$ is called to be *thin* ("effilé") at a point $s \in \Delta \cap \bar{E}$ (bar means the closure taken on $\hat{R}$) if there exists a K-potential U with a property

$$U(s) < \liminf_{p \to s,\, p \in E} U(p) .$$

E is also said to be thin at all points of Δ not belonging to $\bar{E}$. As the elementary properties

a) The union of two sets both thin at s is thin at s.
b) Every polar set is thin at each point of Δ.
c) A boundary point $s\in\Delta$ belongs to Δ_1 if and only if R is not thin at s.

The sets on R whose complements in R are thin at $s\in\Delta_1$ make a filter $\mathfrak{F}_s$ by a). Every set of $\mathfrak{F}_s$ is not thin at $s\in\Delta_1$ on account of a) and c). The limit of a function along $\mathfrak{F}_s$ is called the *pseudo-limit* (or *fine limit*) at s and denoted by $\lim_{\mathfrak{F}_s}$. The results of Naim used in the following are:

(1°) Every Green potential with a positive measure possesses a pseudo-limit zero on Δ_1, $[\chi]$. Here $[\chi]$ means "except a set of χ-measure zero".

(2°) *Maximum principle* (I). Let u be a subharmonic function on R bounded above. If for every $s\in\Delta_1$, $[\chi]$, there exists a set E_s which is not thin at s and

$$\limsup_{p\to s,\, p\in E_s} u(p)\leqq 0 \tag{1}$$

then $u\leqq 0$ throughout R.
In particular, if the pseudo-limits of u are $\leqq 0$, on Δ_1, $[\chi]$, then $u\leqq 0$ throughout R.

§ 2. Normed ring associated with R

2. Let $\mathfrak{S}$ be a family of functions on R with the following properties:

(i) $f\in\mathfrak{S}$ is bounded and continuous on R
(ii) for every $f\in\mathfrak{S}$ there exists a function $T(f)=T\cdot f\in HB$ (bounded harmonic on R) such that the difference $\varphi=f-T\cdot f$ possesses a pseudo-limit zero on Δ_1, $[\chi]$.

Note that the representation

$$f=T\cdot f+\varphi\,,\qquad f\in\mathfrak{S} \tag{2}$$

is unique on account of Maximum principle (I). We introduce $\mathfrak{S}$ the following norm

$$\|f\|=\sup_{p\in R}|f(p)| \tag{3}$$

LEMMA 1. $||f|| \geqq ||T \cdot f||$, $f \in \mathfrak{S}$

PROOF. Given $\varepsilon > 0$, there exists a set $E_s \in \mathfrak{F}_s$ at each $s \in \Delta_1$, $[\chi]$ on which $|f - T \cdot f| < \varepsilon$. Hence we have by Maximum principle (I)

$$||T \cdot f|| = \sup [\limsup_{p \to s,\, p \in E_s} T \cdot f(p)]$$
$$\leqq \sup [\limsup_{p \to s,\, p \in E_s} f(p)] + \varepsilon \leqq ||f|| + \varepsilon ,$$

where sup is taken for $s \in \Delta_1$, $[\chi]$, q.e.d.

Now the following functions are contained in the class $\mathfrak{S}$;

(α) *HB-functions on R.* In fact, for $f \in HB$ we have merely to take $T \cdot f = f$.

(β) *Bounded continuous super- and sub-harmonic functions on R.* For instance, if f is superharmonic and we take as $T \cdot f$ the greatest harmonic minorant of f, then $\varphi = f - T \cdot f$ is a Green potential by Riesz' decomposition (cf. [2]), hence $f \in \mathfrak{S}$ by (1°).

(γ) *Continuous functions on $\hat{R}$ restricted to R.* Indeed, for a continuous function f on R we consider as $T \cdot f$ the solution of the Dirichlet problem for f restricted to Δ, then $T \cdot f$ possesses the pseudo-limit f on Δ, $[\chi]$ (cf. [10]).

LEMMA 2. *$\mathfrak{S}$ makes a ring with unit* 1 *under the usual addition and multiplication.*

PROOF. Let $f_i = T \cdot f_i + \varphi_i$ $(i = 1, 2)$ be any two elements of $\mathfrak{S}$. Clearly, $f \equiv f_1 + f_1 = T \cdot f + \varphi$ $(\varphi = \varphi_1 + \varphi_1)$ belongs to $\mathfrak{S}$ and T is linear over $\mathfrak{S}$;

$$(4) \qquad T(f_1 + f_2) = T \cdot f_1 + T \cdot f_2 .$$

As for the product, one can write it, on account of the boundedness of f_i, such as

$$f_1 f_2 = (T \cdot f_1)(T \cdot f_2) + \psi$$

where $\lim_{\mathfrak{F}_s} \psi = 0$, $s \in \Delta_1$, $[\chi]$. While for any HB-functions u and v

$$uv = [(u+v)^2 - (u-v)^2]/4 \in \mathfrak{S} ,$$

because each term in the right hand side is subharmonic. Hence $(T \cdot f_1)(T \cdot f_2) = w + \phi$, $w \in HB$ and $\lim_{\mathfrak{F}_s} \phi = 0$, $s \in \Delta_1$, $[\chi]$. Thus we know $f_1 f_2 = w + \varphi$ $(\varphi = \psi + \phi)$ belongs to $\mathfrak{S}$ and $w = T(f_1 f_2)$, i.e.

$$(5) \qquad T(f_1 f_2) = T((T\cdot f_1)\cdot(T\cdot f_2)) .$$

THEOREM 1. *The space* $\mathfrak{S}$ *is a normed ring with respect to the norm* (3).

It remains to prove the completeness. Let $\|f_m - f_n\| \to 0$ $(m, n \to \infty)$ where $f_n = T\cdot f_n + \varphi_n$, then $\{f_n\}$ converges uniformly to a bounded continuous function f on R. Since by Lemma 1 and (4)

$$\| T\cdot f_n - T\cdot f_n \| \leqq \| f_m - f_n \| ,$$

the sequence $\{T\cdot f_n\}$ also converges uniformly to an HB-function on R. Writting this limit function as $T\cdot f$,

$$f = Tf + \varphi , \quad \varphi = \lim_{n\to\infty} \varphi_n$$

where the covergence is uniform on R. There exists a set $e \subset \Delta_1$ of χ-measure zero, outside of which all φ_n possess a pseudo-limit zero. Given $\varepsilon > 0$, $|\varphi - \varphi_n| < \varepsilon/2$ on R for a large number n and there is a set $E_s^n \in \mathfrak{F}_s$, $s \in \Delta_1 - e$ on which $|\varphi_n| < \varepsilon/2$. Hence $|\varphi| < \varepsilon$ on E_s^n which implies $\lim_{\mathfrak{F}_s} \varphi = 0$, $s \in \Delta_1$, $[\chi]$. That is, $f \in \mathfrak{S}$.

3. Let $\mathfrak{S}_0$ be the subclass of $\mathfrak{S}$ such as

$$\mathfrak{S}_0 = \{\varphi \in \mathfrak{S},\ \lim_{\mathfrak{F}_s} \varphi = 0,\ s \in \Delta_1, [\chi]\} ,$$

which is complete with respect to the norm. The set of bounded continuous functions whose carriers are compact on R makes an ideal $\mathfrak{S}_c \subset \mathfrak{S}_0$. Let $\mathfrak{M}$ be the set of all maximal ideals of $\mathfrak{S}$. Introducing $\mathfrak{M}$ the closure topology due to Gelfand-Silov (cf. [5], [7]), then $\mathfrak{M}$ becomes a compact Hausdorff space, say R^*, on which the homeomorphic image of R is open and dense. We denote it again by R. Every function of $\mathfrak{S}$ can be continuously extended to R^*. The closed set

$$\Delta^* = R^* - R$$

is called the ideal boundary of R. Δ^* consists of maximal ideals containing $\mathfrak{S}_c$. We note that, besides the elements of $\mathfrak{S}$, following functions also can be continuously extended to R^*. Hereafter we say that a function f is continuous in the wide sense if f is a continuous mapping into the extended real line.

Lemma 3. *Let f be a positive functions continuous in the wide sense on R. If $f_n(p)=\min(f(p), n)\in\mathfrak{S}$ $(n=1, 2, \cdots)$, then at each $q\in\Delta^*$ f has a finite or infinite limit*[1] *equal to $\lim_{n\to\infty} f_n(q)$ and f defined by these limits is continuous on R^* in the wide sense.*

In particular, positive superharmonic functions continuous in the wide sense on R have this property.

Proof is immediate and omitted.

In the following we shall denote the function f extended on R^* as above by f again.

Theorem 2. *Our compact Hausdorff space R^* is not metrisable.*

Proof. Suppose that R^* is metrisable, then there would exist a countable number of points $q_\nu\in\Delta^*$ $(\nu=1, 2, \cdots)$ which are dense in a compact subset Δ^* of R^*. Further for each q_ν we can take a sequence of points p_ν^μ $(\mu=1, 2, \cdots)$ on R tending to q_ν. We rearrange them as $\{p_n\}$ $(n=1, 2, \cdots)$. If constants $c_n>0$ are chosen such that $\sum_{n=1}^{\infty} c_n G(p_0, p_n)<\infty$ where $p_0(\neq p_n)\in R$, then the function

$$G(p)=\sum_{n=1}^{\infty} c_n G(p, p_n)$$

is non-constant positive and harmonic on R except p_n $(n=1, 2, \cdots)$. For a positive number $M>G(p_0)$

$$G^M(p)=\min(G(p), M)\leq M$$

is a continuous superharmonic function $\in\mathfrak{S}$, hence by the minimum principle there is a point $q^*\in\Delta^*$ such that

$$G^M(q^*)=\inf_{p\in R} G^M(p)<M$$

Since $\{q_\nu\}$ is dense in Δ^* and G^M is continuous on R^*, for sufficiently small $\varepsilon>0$ there exists a point q_j such that

$$G^M(q_j)\leq M-\varepsilon, \qquad q_j\in\Delta^*$$

While $G(p_j^\mu)=+\infty$ $(\mu=1, 2, \cdots)$, therefore we have $G^M(p_j^\mu)=M$ and $G^M(q_j)=M$ for $\mu\to\infty$. This is a contradiction.

1) Limit taken over the filter of neighborhoods of q.

4. Let Δ_1^* be the set of maximal ideals containing an ideal $\mathfrak{S}_0$. Δ_1^* is a compact subset of Δ^*, which is called the *harmonic boundary* of R and plays an important role in our theory.

THEOREM 3 (*Maximum principle* $(\mathrm{II})_1$). *Every sub (super)-harmonic function on R which is bounded above (below) and continuous in the wide sense attains its maximum (minimum) on the harmonic boundary Δ_1^* of R.*

PROOF. It suffices to prove for bounded functions. Let u be a bounded continuous superharmonic function on R. Let $\inf_R u=\lambda$ and $\mathfrak{J}$ be the principal ideal generated by $\tilde{u}=u-\lambda\ (\geqq 0)\in\mathfrak{S}$. Then $\mathfrak{J}\cup\mathfrak{S}_0$ becomes a (proper) ideal of $\mathfrak{S}$. To see this, suppose $\mathfrak{J}\cup\mathfrak{S}_0=\mathfrak{S}$, then there exist functions f and φ_0 such that

$$\tilde{u}f+\varphi_0=1\,,\qquad f\in\mathfrak{S},\ \ \varphi_0\in\mathfrak{S}_0\,.$$

While

$$\tilde{u}f=w+\varphi_1\,,\quad w=T(\tilde{u}\cdot f)\,,\qquad \varphi_1\in\mathfrak{S}_0\,.$$

Hence by Maximum principle (I) we have

$$w\equiv 1\,.$$

On the other hand

$$w\leqq\tilde{u}M-\varphi_1\,,\qquad M=\max(0,\ \sup_R f)$$

Since $w-\tilde{u}M$ is subharmonic, it follows by Maximum principle (I)

$$w\leqq\tilde{u}M\ \text{ on } R,\ \text{hence}\ \inf_R w\leqq 0$$

which is a contradiction. Thus we know $\mathfrak{J}\cup\mathfrak{S}_0$ is an ideal containing $\tilde{u}$, which implies $\tilde{u}$ vanishes at some point of Δ_1^*, q.e.d.

For the following purposes we prepare two lemmas.

LEMMA 4. *Let q be any point Δ_1^* and E a compact subset of Δ_1^* disjoint with q. Then there exists a positive HB-function u on R such that $u(q)=0$ and $u=1$ on E.*

PROOF. Since q does not belong the closure $\bar{E}$, there exists an $f\in\mathfrak{S}$ such that f belongs to ideal $\bigcap_{N\in E}N$, but not to maximal ideal

q. That is, $f=0$ on E and $f(q)\neq 0$. Now from the decomposition

$$f^2 = v+\varphi\,, \quad v = T\cdot f^2\,, \qquad \varphi\in\mathfrak{S}_0$$

and Maximum principle $(\mathrm{II})_1$ we find that v is a non-negative HB-function such that v vanishes on E and $v(q)=c\neq 0$. Now $v_1(p)=\min(v(p), c)\in\mathfrak{S}$, and

$$u = 1-(T\cdot v_1)/c$$

is the required.

LEMMA 5. *Let u_1 and u_2 be any superharmonic functions continuous in the wide sense on R, then*

$$(u_1\wedge u_2)(q) = \min(u_1(q), u_2(q))\,, \qquad q\in\Delta_1^*$$

where $u_1\wedge u_2$ means the greatest harmonic minorant of u_1 and u_2.

This is an immediate consequence of the Riesz' decomposition

$$\min(u_1, u_2) = u_1\wedge u_2+g\,,$$

g being a Green potential. Since $g_n=\min(g, n)\in\mathfrak{S}_0$, g_n hence g vanish on Δ_1^* (Lemma 3).

§ 3. Dirichlet problems

5. Let f be a real-valued function given on the harmonic boundary Δ_1^* of a Green space R and $\bar{U}_f$ resp. $\underline{U}_f$ the families of continuous (in the wide sense) superharmonic resp. subharmonic functions u resp. v on R satisfying the boundary conditions;

$$(6) \qquad \varliminf_{p\to q} u(p)\geqq f(q)\,, \quad \text{resp.} \quad \varlimsup_{p\to q} v(p)\leqq f(q)\,, \qquad q\in\Delta_1^*\,.$$

Let $\bar{H}_f$ resp. $\underline{H}_f$ be the lower resp. upper envelopes of functions belonging to $\bar{U}_f$ resp. $\underline{U}_f$. $\bar{H}_f$ and $\underline{H}_f$ are harmonic or $\pm\infty$ on R. If they are coincident we write it H_f and say that f is *solvable*, provided that H_f is finite. In this section we treat the case that f is bounded. In this case it should be noted that the limits exist in (6), moreover there exist the decreasing resp. increasing sequences of harmonic functions u_n resp. v_n such that

$$(7) \qquad \lim_{n\to\infty} u_n = \bar{H}_f \quad \text{resp.} \quad \lim_{n\to\infty} v_n = \underline{H}_f\,.$$

Indeed, for any bounded $u \in \bar{U}_f$, $T \cdot u \leqq u$ and $T \cdot u = u$ on Δ_1^*, hence $T \cdot u \in \bar{U}_f$, moreover, by Lemma 5 $T \cdot u \wedge T \cdot v$ also belongs to $\bar{U}_f$, provided that u and v belong to $\bar{U}_f$. Therefore we can find the sequences $\{u_n\}$ and $\{v_n\}$ in (7) (cf. [7]). Now we can immediately solve the following Dirichlet problem as in the classical case.

THEOREM 4. *Let f be a bounded function on the harmonic boundary Δ_1^* of R, then we have for any $q \in \Delta_1^*$*

$$(8) \qquad \lim_{p \to q,\, p \in R} \bar{H}_f(p) = \bar{H}_f(q) \leqq \limsup_{r \to q,\, r \in \Delta_1^*} f(r)\,.^{1)}$$

In particular, if f is continuous on Δ_1^*, it is solvable and

$$(9) \qquad H_f(p) \to H_f(q) = f(q)\,, \quad p \to q \in \Delta_1^*\,,$$

moreover for each point $p \in R$ there exists a regular (Borel) measure μ^p on Δ_1^* such that

$$(10) \qquad H_f(p) = \int_{\Delta_1^*} f(q)\, d\mu^p(q)\,.^{2)}$$

PROOF. Let $q \in \Delta_1^*$, then for any $\varepsilon > 0$ there exists a neighborhood V of q such that for $s \in V \cap \Delta_1^*$

$$f(s) \leqq \lambda + \varepsilon\,, \quad \lambda = \limsup_{r \to q,\, r \in \Delta_1^*} f(r)$$

Let $E = \Delta_1^* - V$, then by Lemma 4 there exists a positive HB-function W such that $W(q) = 0$ and $W = 1$ on E. It follows by Maximum principle $(\mathrm{II})_1$ that we have, for sufficiently large positive constant C

$$\bar{H}_f \leqq CW + \lambda + \varepsilon \qquad \text{in } R\,,$$

hence $\bar{H}_f(p) \to \bar{H}_f(q) \leqq \lambda + \varepsilon$, which implies (8). If f is continuous, $\bar{H}_f = \underline{H}_f$ by (8) and Maximum principle $(\mathrm{II})_1$. Thus H_f gives a positive linear functional (for each point $p \in R$) over the space of continuous functions on the compact Hausdorff space Δ_1^*, hence there exists a regular measure μ^p on Δ_1^* by which $H_f(p)$ can be expressed as (10).

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1) This means $\inf_{U_q} (\sup_{U_q \cap \Delta_1^*} f)$ where U_q denotes any neighborhood of q.

2) Hereafter we write μ instead of μ^p, in particular if some relations hold for any $p \in R$.

As the immediate consequences, we know at first that for any $f \in \mathfrak{S}$ $T \cdot f$ can be expressed as

$$T \cdot f = \int_{\Delta_1^*} f d\mu ,$$

because every function of $\mathfrak{S}$ vanishing on Δ_1^* belongs to $\mathfrak{S}_0$.

THEOREM 5 (*Maximum principle* $(\mathrm{II})_2$). *Let u be a subharmonic function which is bounded above and continuous in the wide sense on R. If $u \leqq 0$ on Δ_1^* except a set of μ-measure zero, then $u \leqq 0$ throughout R.*[1]

PROOF. It suffices to prove for a bounded u. By Riesz' decomposition we know $u \leqq T \cdot u$ on R and $u = T \cdot u$ on Δ_1^*. It follows that

$$u \leqq T \cdot u = \int_{\Delta_1^*} T \cdot u \, d\mu = \int_{\Delta_1^*} u \, d\mu \leqq 0 .$$

6. To treat the Dirichlet problem for non-continuous functions we start from

LEMMA 6. *Let $\{u_n\}$ be a monotone sequence of HB-functions on R and $\lim_{n \to \infty} u_n = u$ be bounded, then*

$$u(q) = \lim_{n \to \infty} u_n(q) , \qquad \textit{for } q \in \Delta_1^* - e, \ \mu(e) = 0 .$$

PROOF. Suppose $\{u_n\}$ is increasing, then we have

$$\lim_{n \to \infty} u_n(q) \leqq u(q) , \quad q \in \Delta_1^* \tag{11}$$

and

$$u = \lim_{n \to \infty} u_n = \lim_{n \to \infty} \int_{\Delta_1^*} u_n d\mu = \int_{\Delta_1^*} \lim_{n \to \infty} u_n d\mu , \qquad \text{on } R.$$

While since $u \in HB$, $u = \int_{\Delta_1^*} u \, d\mu$,

$$\int_{\Delta_1^*} (u(q) - \lim_{n \to \infty} (u_n(q)) \, d\mu(q) = 0 ,$$

from which the conclusion is obtained under (11).

1) Cf. the remark after Theorem 6.

LEMMA 7. *Let E be a closed subset of Δ_1^*. Then the characteristic function χ_E of E is solvable and*

$$H_{\chi_E} = \int_{\Delta_1^*} \chi_E d\mu = \mu(E)\,, \qquad \text{on } R\,,$$

moreover there exists a simultaneously open and closed set $E_0 \subset E$ such that $\mu(E-E_0)=0$ and H_{χ_E} vanishes on $\Delta_1^*-E_0$, $=1$ on E_0, moreover $H_{\chi_E}=H_{\chi_{E_0}}$.

PROOF. Take a decreasing sequence of HB-functions $\{u_n\}$ (cf. (7)) such that

$$\lim_{n\to\infty} u_n = \bar{H}_{\chi_E}\,, \qquad u_n \in \bar{U}_{\chi_E}\,.$$

where we may assume $u_n=1$ on E, because it is enough to consider $u_n \wedge 1$. By (8) $\bar{H}_{\chi_E}=0$ on an open set Δ_1^*-E on Δ_1^* and since $u_n=1$ on E, by Lemma 6 we have

$$\bar{H}_{\chi_E} = \chi_E \qquad \text{on} \quad \Delta_1^*\,,\ [\mu]\,.$$

Since $\bar{H}_{\chi_E} \in HB$, the set $E_0=\{\bar{H}_{\chi_E}=1\}$ $(\subset E)$ is closed and $\mu(E-E_0)=0$. Hence it is easily seen by Maximum principle $(\mathrm{II})_2$ that $\bar{H}_{\chi_E}=\bar{H}_{\chi_{E_0}}$. This implies that $\bar{H}_{\chi_E}$ vanishes on $\Delta_1^*-E_0$. While $\bar{H}_{\chi_E}$ is continuous, hence we find that E_0 must be open and immediately $\bar{H}_{\chi_E}=\underline{H}_{\chi_E}$, q.e.d.

The following lemma is valid also in our space.

LEMMA 8. *Let $\{\varphi_n\}$ be a monotone sequence of solvable functions on Δ_1^*, which converge to φ, then*

$$H_\varphi = \lim_{n\to\infty} H_{\varphi_n}\,.$$

THEOREM 6. *A measurable function f on Δ_1^* is solvable if and only if it is integrable with respect to μ. And then*

$$H_f = \int_{\Delta_1^*} f d\mu\,, \tag{12}$$

moreover

$$H_f = f \qquad on \quad \Delta_1^*,\ [\mu]\,. \tag{13}$$

PROOF. As is seen little later, for any simple (step) function theorem is valid. Now we may suppose a given f is non-negative.

Let $\{f_n\}$ be an increasing sequence of simple functions tending to f, then by Lemma 8

$$H_f = \lim_{n\to\infty} H_{f_n} = \lim_{n\to\infty} \int_{\Delta_1^*} f_n d\mu = \int_{\Delta_1^*} \lim_{n\to\infty} f_n d\mu = \int_{\Delta_1^*} f d\mu$$

therefore the first assertion of the theorem holds. H_f is defined on Δ_1^* by Lemma 3. Since $H_{f_n} = f_n$ on Δ_1^*, $[\mu]$, we have $H_f \geqq f_n$, successively $H_f \geqq f$ on Δ_1^*, $[\mu]$. Hence if $H_f < \infty$, $H_f = f$ on Δ_1^*, $[\mu]$ by (12).

To complete the proof, we consider the characteristic function χ_E of a μ-measurable set $E \subset \Delta_1^*$. There exist an increasing sequence of closed sets $E_n (\subset E)$ such that $\mu^p(E-E_0)=0$, $E_0 = \lim_{n\to\infty} E_n$. Since $\chi_{E_n} \to \chi_{E_0}$, using Lemma 7 we have by the same argument as above

$$H_{\chi_{E_0}}(p) = \int_{\Delta_1^*} \chi_{E_0} d\mu^p = \int_{\Delta_1^*} \chi_E d\mu^p\,, \qquad \text{for any } p\,,$$

and $H_{\chi_{E_0}} = \chi_E$ on Δ_1^*, $[\mu]$. While, $H_{\chi_{E_0}} \leqq \underline{H}_{\chi_E} \leqq \bar{H}_{\chi_E} \leqq H_{\chi_{E_0}} + \bar{H}_{\chi_{E-E_0}}$. Here it is proved that $\bar{H}_{\chi_{E-E_0}} = 0$, hence $H_{\chi_E} = H_{\chi_{E_0}}$. Indeed, if we take, for any $\varepsilon > 0$, an open set $e \supset E - E_0$ whose μ^p-measure $< \varepsilon$, then it is easily seen that

$$0 \leqq \bar{H}_{\chi_{E-E_0}}(p) = \int_{\Delta_1^*} \bar{H}_{\chi_{E-E_0}} d\mu^p \leqq \int_{\Delta_1^*} H_{\chi_e} d\mu^p < \varepsilon\,.$$

Since ε is arbitrary, $\bar{H}_{\chi_{E-E_0}}(p) = 0$ i.e. $\bar{H}_{\chi_{E-E_0}} \equiv 0$. Thus we know that Theorem is valid for simple functions.

REMRAK. For any Borel set $E \subset \Delta_1^*$, $H_{\chi_E} = \mu(E)$. Hence if $\mu^p(E) = 0$ for some point $p \in R$, $\mu^p(E) = 0$ for any $p \in R$ by means of the usual minimum principle. Therefore the Maximum principle $(\text{II})_2$ still holds if $u \leqq 0$ on $\Delta_1^* - E$, $\mu^p(E) = 0$ for some $p \in R$.

As an application,

THEOREM 7. *Let u be a superharmonic function which is bounded below and continuous in the wide sense, then the function u on Δ_1^* is integrable with respect to μ and is expressed as*

$$u = \int_{\Delta_1^*} u\, d\mu + w \qquad \text{in } R\,,$$

where w is a Green potential. In particular, if u is positive and harmonic on R, then w is singular and the integral term is quasi-bounded.

PROOF. Let $u_n(p)=\min(u(p), n)\in\mathfrak{S}$ $(n=1, 2, \cdots)$, then since by Theorem 4 and Lemma 3

$$u\geqq u_n\geqq\int_{\Delta_1^*}u_n d\mu \quad \text{in } R, \quad \text{hence} \quad u\geqq\int_{\Delta_1^*}u\, d\mu,$$

we find u is integrable on Δ_1^*. Moreover by (13) $w=0$ on Δ_1^*, $[\mu]$, therefore any HB-function ω $(\geqq 0)$ majorized by w must be identically zero by Maximum principle $(\text{II})_2$, q.e.d.

7. THEOREM 8. *A positive harmonic function w on R is singular if and only if w vanishes on Δ_1^*.*

PROOF. Let w be singular and $v=\min(w, k)$ $(k>0)$. Since v is a superharmonic function $\in\mathfrak{S}$, $0\leqq T\cdot v\leqq v\leqq w$, hence $T\cdot v\equiv 0$, that is, v reduces to a Green potential. It follows that $v=w=0$ on Δ_1^* by Lemma 3. The converse is trivial.

THEOREM 9. *If a single point $q\in\Delta_1^*$ has a positive measure with respect to μ, then $\mu(\{q\})$ is minimal in class HB. Conversely, any minimal function ω $(\sup_R\omega=1)$ in HB is identical with the μ-measure of an isolated point of Δ_1^*.*

PROOF. The first part of the theorem is evident, as $\{q\}$ is a closed set. To prove the converse, let $e=\{p\in\Delta_1^*, \omega(p)=1\}$. Since e is closed, $0\leqq H_{\chi_e}\leqq\omega$. It follows that $H_{\chi_e}=c\cdot\omega$ (c : const.>0), hence $\omega=0$ on Δ_1^*-e. If e contains at least two points q_1 and q_2, there exists by Lemma 4 an HB-function u $(0\leqq u\leqq 1)$ such that $u(q_1)=1$ and $u(q_2)=0$. Since $v\equiv u\wedge\omega\leqq\omega$, v is proportional with ω. While $v(q_2)=\min(u(q_2), \omega(q_2))=0$, which is absurd, q.e.d.

From above two theorems we have the following

COROLLARY. *Every positive minimal function Ω on R vanishes on Δ_1^*, provided that it is unbounded. If Ω is bounded, Ω vanishes on Δ_1^* except an isolated point q^* on Δ_1^* where $\Omega(q^*)=\sup_{p\in R}\Omega(p)$. Moreover the set of bounded minimal functions on R is countable.*

We note here that for an unbounded positive minimal function w the set

$$E_\infty = \{q \in \Delta^*,\ w(q) = +\infty\} \subset \Delta^* - \Delta_1^*$$

is *closed and connected* (cf. [6]), which is proved as follows. E_∞ is evidently closed. Let E_∞ consist of two disjoint components E_1 and E_2, then there exists a continuous function $f \geqq 0$ on R^* such that $f=0$ on E_1 and $=1$ on E_2. Let $U=\{f>1/2\}$, $m=\max_{\partial U} w\ (<\infty)$ and w_1 be a superharmonic function such that

$$w_1 = \begin{cases} w & \text{in } R-U \\ \min(w, m) & \text{in } U \end{cases}$$

Let u_1 be the greatest harmonic minorants of w_1, then u_1 are not identically zero. Indeed,

$$w_1 \geqq u \equiv \begin{cases} w-m & \text{in } D = \{p \in R-U\,;\, w(p) > m\} \\ 0 & \text{in } R-D \end{cases}$$

u is a non-constant subharmonic function, hence $u_1>0$. Since w is minimal and $u_1<w_1<w$, $u_1=cw$ (c: const.>0) which is absurd, because $u_1<w_1$ is bounded ($\leqq m$) on U.

§ 4. Applications

8. First of all we state some remarks on harmonic measures. Let A be a Borel set on Δ_1 and ω_A the harmonic measure of A in $\hat{R}$, i.e.

$$\omega_A(p) = \int_A K_s(p)\, d\chi(s)\,,$$

which is characterized by the property

$$\omega_A \wedge (1-\omega_A) = 0$$

(cf. [4]). Considering ω_A as an element of $\mathfrak{S}$ we have by Lemma 5

$$\min(\omega_A(q),\ 1-\omega_A(q)) = 0 \qquad \text{for} \quad q \in \Delta_1^*\,.$$

that is, the values of ω_A on Δ_1^* are either 1 or 0. Hence there exists a set E such that

$$\omega_A = \mu(E) = H_{\chi_E}\,, \quad E = \{q \in \Delta_1^*,\ \omega_A(q) = 1\}\,.$$

E is simultaneously open and closed on Δ_1^* by Lemma 7. It should be noted that if A is a closed set, ω_A possesses the pseudo-limit zero on $\Delta_1 - A$, [χ]. To see this, let G_n $(n=1, 2, \cdots)$ be sets of points of R whose distances from A are not greater than $1/n$, then by Martin [8]

$$\omega_A(p) = \lim_{n\to\infty} 1^*_{G_n}(p).$$

In general, for a positive superharmonic function u and a closed set G on R $u^*_G(p)$ stands for a superharmonic function which is equal u on G except a set of capacity zero and equal $H^{R-G}_{\underset{*}{u}}$ on $R-G$. $H^{R-G}_{\underset{*}{u}}$ is the solution of Dirichlet problem on $R-G$ with the boundary function $\underset{*}{u}$ where $\underset{*}{u}$ means the function u extended by 0 onto an Alexandroff point of R. Let σ^r_s denotes a sphere with diameter r whose center is $s \in \Delta_1$. Since for $s \in \Delta_1 - G_n$ and $r = 1/n$ $\sigma^r_s \subset \hat{R} - G_{2n}$, we have

$$0 \leqq \omega_A(p) \leqq 1^*_{G_{2n}}(p) \leqq H^{\sigma^r_s \cap R}_{\underset{*}{1}}(p), \qquad p \in \sigma^r_s \cap R.$$

While, by Naim [10] $\lim_{\mathfrak{F}_s} H^{\sigma^r_s \cap R}_{\underset{*}{1}}(p) = 0$ for any $r > 0$ and $s \in \Delta_1$, [χ], from which we can immediately get our conclusion.

Now from our point of view we shall prove some theorems valid on the Martin spaces.

THEOREM 10. *Let R be a Green space and Δ the Martin boundary of R. Let e be a set on Δ whose χ-measure (harmonic measure) is positive. If a positive superharmonic function u continuous in the wide sense on R possesses at each point of e a pseudo limit $+\infty$, then u is identically $+\infty$.*

PROOF. There exists a closed set $F \subset e$ such that $\omega_F > 0$, i.e. $\omega_F(q_0) = \chi(F) > 0$ (cf. § 1). Let

$$W(p) = \omega_F(p)/(u(p)+1).$$

$1/(u+1)$ is a non-negative bounded continuous subharmonic function. This is seen by considering the approximation of u by smooth superharmonic functions in the local. Hence $W \in \mathfrak{S}$. From above remark we know that the pseudo-limit of W is equal zero on Δ_1, [χ], hence

$$W \in \mathfrak{S}_0 ,$$

and that there exists a closed set E on Δ_1^* such that

$$\omega_F = H_{\chi_E} = \mu(E) > 0 \qquad \text{on} \quad R^* .$$

Since $\mu(E)=1$ on E and $W=0$ on Δ_1^*, it follows that

$$u = +\infty \qquad \text{on} \quad E .$$

While by Theorem 7 u is integrable on Δ_1^* with respect to μ, hence the set $\{q \in \Delta_1^* ; u(q) = +\infty\}$ must be of μ-measure zero, which is a contradiction.

As applications of this theorem we get theorems of Riesz type.

THEOREM 11. *Let D be a domain in $2n$-dimensional euclidean space R^{2n} admitting a Green function and Δ the Martin boundary of D. Let*

$$\varphi(z_1, \cdots, z_n) = (\varphi_1(z_1, \cdots, z_n), \cdots, \varphi_m(z_1, \cdots, z_n)) , \qquad (m \geqq 1)$$

be an analytic transformation of D into R^{2m} and $\varphi(D)$ denotes the image of D in R^{2m}. Suppose E is a set in R^{2m} such that there exists a positive continuous pluri-superharmonic function[1] *Ω on R^{2m} (resp. $\varphi(D)$) which becomes $+\infty$ on E (resp. $\varphi(D) \frown E$). If φ possesses a pseudo-limit $\in E$ at each point of the set $e \subset \Delta$ whose χ-measure (harmonic measure) is positive, then the mapping φ degenerates so that $\varphi(D) \subset E$.*

In fact, $\Omega(\varphi(z_1, \cdots, z_n))$ becomes a positive pluri-superharmonic, hence superharmonic function continuous in the wide sense on D and possesses a pseudo-limit $+\infty$ at each point of e. Hence the theorem follows from the preceeding one.

If the boundary surface of D is sufficiently smooth, the Martin boundary of D is identical with the euclidean boundary. In the special case that φ is bounded and E the $2(m-\lambda)$-dimensional subspace

$$w_1 = a_1, \cdots, w_\lambda = a_\lambda \qquad (1 \leqq \lambda \leqq m) ,$$

it suffices to take

$$\Omega = -\log |(w_1 - a_1) \cdots (w_\lambda - a_\lambda)| + K \qquad (K:\ \text{const.} > 0) .$$

---

1) As for pluri-subharmonic functions see e.g. H. L. Bremermann [3].

In case of one variable we can choose as D any open Riemann surfaces possessing a Green function and get under the following remark a theorem of Constantinescu-Cornea [4] (in slightly restricted form) which reduced exactly to the classical Riesz-Lusin-Privaloff-Frostman-R. Nevanlinna's theorem.

Given a superharmonic function $v>0$ on R, the extremisation of v with respect to a set $G\subset R$ is, by definition, a positive superharmonic function on R which is the lower envelope of positive superharmonic functions on R majorizing v on G. We denote it by $E_G v$. If $R-G$ is open,

$$E_G v(p) = H_{\underset{*}{v}}^{R-G}(p), \qquad p\in R-G.$$

On the other hand, every harmonic function $u>0$ on R can be decomposed such as

$$u(p) = I_G u(p) + H_{\underset{*}{u}}^{G}(p), \qquad p\in G$$

where G is a domain (cf. [4]). Hence, in particular, $I_G K_s>0$ on G if and only if $E_G K_s$ is not identical with K_s, therefore by Naim's criterion, if and only if $R-G$ is thin at s. Thus we know that for $s\in\mathfrak{F}(\varphi)$, $\hat{M}(s)$ in [4] is a pseudo-limit of φ at s.

9. Finally we refer to a continuous mapping of our compact space R^* onto the Martin space $\hat{R}$ (cf. [5]). Let $C(\hat{R})$ be a ring of continuous functions on $\hat{R}$, which is a subring of $\mathfrak{S}$ by sec. 2 (γ). For each point (maximal ideal) $M\in R^*$ $\hat{M}'=\{f\in M\cap C(\hat{R})\}$ is an ideal of $C(\hat{R})$, moreover a maximal ideal, because an ideal $(\hat{M}', g)$ $(g\in C(\hat{R}),\ g\notin\hat{M}')$ would contain a non-vanishing constant $g(M)=g(p)-(g(p)-g(M))$. For $\hat{M}'$ there exists a unique point $\hat{M}\in\hat{R}$ such that $f(\hat{M})=0$ for every $f\in\hat{M}'$, otherwise for each point $p\in\hat{R}$ we have a function $f_p\in\hat{M}'$ which does not vanish at p. Hence from the compactness of $\hat{R}$ there exists a function $g=\sum_{i=1}^{N} f_{p_i}^2\in\hat{M}'$ such that $g\neq 0$ on $\hat{R}$. Since $1/g\in C(\hat{R})$, it follows that $1=g\cdot\frac{1}{g}\in\hat{M}'$ which is absurd. The mapping

$$\tau: M\rightarrow\hat{M}$$

gives a continuous mapping of R^* onto $\hat{R}$, which leaves each point of R invariant. Since R is dense in $\hat{R}$, $\tau(R^*)=R$ and $\tau(\Delta^*)=\Delta$.

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Added in Proof (July 10, 1962). Prof. M. Brelot has kindly informed me that he had given in his paper (Ann. Acad. Sci. Fenn. 250) a sharper result than Th. 10. By his theorem Th. 11 is improved correspondingly, that is, the continuity of Ω is unnecessary and at each point s of $e\,\varphi$ has merely to possess a limit $\in E$ (more generally, to approach E) along a non-thin set at s.