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Mixed problem for some semi-linear wave equation

By

Sigeru MIZOHATA and Masaya YAMAGUTI

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1. Introduction

Let us consider the equation of the form:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}+f(u)\frac{\partial u}{\partial t}+g(u) = 0 . \tag{1.1}$$

Our problem is the so-called initial-boundary problem to this equation. Namely, we want to show that, under some conditions on $f(u)$ and $g(u)$ which will be given later, there exists always a unique, genuine, global solution $u(x, t)$, $(x\geq 0, t\geq 0)$, for any couple of the initial $u(x, 0)\in C^2$, $u_t(x, 0)\in C^1$ $(x\geq 0)$ and the boundary $u(0, t)=\psi(t)\in C^2$ $(t\geq 0)$ data. Of course, we assume the compatibility conditions among these data up to order 2. Namely, $u(0, 0)=\psi(0)$, $u_t(0, 0)=\psi'(0)$ and

$$\psi''(0) = u_{xx}(0, 0)-f(u(0, 0))u_t(0, 0)-g(u(0, 0)) .$$

Our first step is to obtain an a priori estimate of the solution u and its first derivatives in the maximum norm. It is evident that without any condition on the behaviors of f and g, we cannot expect to have such an estimate. We tried to make this condition less stringent. However, we remark here that from the first we restricted the type of equation to the form (1.1). We could extend our reasoning to another types of equations. However, we don't insist on this matter.

Our second step is to show the local existence theorem with respect to both the Cauchy data and the Goursat data. This problem is already classical. However, to make this theorem immediately available to the step-by-step continuation of the solution, we discussed it in detail in section 4.

Now, we state our condition on f and g.

$$(1.2)\begin{cases} 1^\circ.\ \text{Differentiability condition.} \\ \qquad f(u),\ g(u) \in C^2 \quad \text{for} \quad -\infty < u < +\infty\,; \\ 2^\circ.\ f(u) \text{ satisfies an inequality of the following form:} \\ \qquad -L_0 < f(u) < C(u^2 + F(u)^2) + C_1\,, \quad \text{for} \quad -\infty < u < +\infty\,, \\ \text{where } F(u) = \int_0^u f(v)dv,\ L_0,\ C, \text{ and } C_1 \text{ are positive constants;} \\ 3^\circ.\ g(u) \text{ can be put in the form} \\ \qquad g(u) = g_1(u) + g_2(u)\,, \\ \text{where } |g_1(u)| < C|u| \quad (C: \text{constant}), \\ \text{sign}\, u.g_2(u) \geq 0 \quad \text{for } |u| \text{ large, and } |ug_2(u)| \leq CG_2(u) \\ \hfill \text{for } |u| \text{ large.}^{1)} \\ \text{where } G_2(u) = \int_0^u g_2(v)dv,\ C \text{ is a positive constant.} \end{cases}$$

Now we state our

Main Theorem.

Let us assume the condition (1.2). *Given any initial and boundary data satisfying the differentiability condition and the compatibility condition stated above, then there exists a unique solution* $u(x, t)$, $0 \leq x < \infty$, $0 \leq t < \infty$, *of class* C^2.

2. A priori estimate of solution.

In this section, we assume that the solution $u(x, t) \in C^2$ exists for $0 \leq t \leq h$, we fix h. Since the equation has the wave operator as its principal part, it suffices to assume the Cauchy data has its support limited to the right, say, for $x \leq L$. Then the solution, as far as it exists, has its support in $x \leq L + t$. Next, let us recall that $u(x, t)$ takes the value $\psi(t)$ for $x = 0$. We transform u into a new unknown function v in such a way that it have the boundary value $0 : v(0, t) = 0$. For this purpose, we take a function $\psi(x, t) \in C^2$,

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1) If we assume, in addition to the condition 2°, $\lim_{u \to \pm\infty} F(u) = \pm\infty$, we can relax the condition 3°, by

3°)′ $0 \leq \text{sign}\, u.g_2(u) \leq C(u^2 + G_2(u))$, for $|u|$ large.

defined in $0 \leq t \leq h$, $0 \leq x < +\infty$ satisfying

1°. $\psi(0, t) = \psi(t)$,

2°. $\psi(x, t)$ has its support in $x \leq L$.

Now set

$$v(x, t) = u(x, t) - \psi(x, t)\,, \tag{2.1}$$

then (1.1) becomes

$$v_{tt} - v_{xx} = -f(v+\psi)(v_t + \psi_t) - g(v+\psi) + (\psi_{xx} - \psi_{tt})\,. \tag{2.2}$$

Now, we want to derive an a priori estimate of v.
Let us proceed in a heuristic way. In the case of wave operatar, the energy $E(t) = \int_0^x v_t^2 + v_x^2 dx$ is constant. Taking account of this fact, we define an energy form $E_1(t)$ as follows:

$$E_1(t) = \int_0^\infty \frac{1}{2}(v_t^2 + v_x^2) + G(v+\psi) + C(v^2+1) dx\,, \tag{2.3}$$

where $G(u) = \int_0^u g(u) du$.
Here we take the above positive constant C in such a way that

$$G(v+\psi) + C(v^2+1) > \frac{C}{2}(v^2+1)\,.$$

This is always possible by virtue of the condition 3° of (1.2). Next, recalling that v $(= u + \psi)$ has its support in $x \leq L + h$, *this upper limit is taken always* $L+h$. Although we write the upper limit of the integral as ∞. Let us remark, in fact, that (2.3) would have no sense in we take ∞ as upper limit.
Now

$$E_1'(t) = \int_0^\infty v_t v_{tt} + v_x v_{xt} + g(v+\psi)(v_t + \psi_t) + 2Cvv_t dx\,.$$

Taking into account of

$$\int_0^\infty v_x v_{xt} dx = [v_x \cdot v_t] - \int_0^\infty v_{xx} \cdot v_t dx = -\int_0^\infty v_t \cdot v_{xx} dx\,,$$

$$E_1'(t) = \int_0^\infty v_t(v_{tt} - v_{xx}) + g(v+\psi)(v_t + \psi_t) + 2Cvv_t dx$$

$$= \int_0^\infty v_t\{-f(v+\psi)(v_t + \psi_t) - g(v+\psi) + \psi_1\} + g(v+\psi)(v_t + \psi_t) + 2Cvv_t dx\,,$$

where $\psi_1=\psi_{xx}-\psi_{tt}$,

$$=\int_0^\infty -f(v+\psi)v_t(v_t+\psi_t)+g(v+\psi)\psi_t+v_t\psi_1+2Cvv_t dx$$

$$(2.4)\qquad =\int_0^\infty -f(v+\psi)\left(v_t+\frac{\psi_t}{2}\right)^2 dx+\int_0^\infty f(v+\psi)\frac{\psi_t^2}{4}+g(v+\psi)\psi_t+v_t\psi_1 +2Cvv_t dx\,.$$

By the condition 2° of (1.2), i.e. by the lower boundedness of $f(u)$, it sufficeces to estimate the second integral of (2.4).
By a rough observation, we see that, if the functions f and g are all of order 2 namely if $f(u)$, $g(u)$ are of $O(u^2)$, the last integral is estimated by const. $E_1(t)$. However, this condition is relaxed considerably as we shall show it.
Define the second energy form:

$$(2.5)\quad E_2(t)=\int_0^\infty \frac{1}{2}\{v_t+F(v+\psi)-F(\psi)\}^2+\frac{1}{2}v_x^2+G(v+\psi)+C(v^2+1)dx.$$

with the same convention on the upper limit, where $F(u)=\int_0^u f(u)du$.
Now,

$$E_2'(t)=\int_0^\infty \{v_t+F(v+\psi)-F(\psi)\}\{v_{tt}+f(v+\psi)(v_t+\psi_t)-f(\psi)\psi_t\} +v_xv_{xt}+g(v+\psi)(v_t+\psi_t)+2Cvv_t dx\,.$$

Since $v_{tt}+f(v+\psi)(v_t+\psi_t)=v_{xx}-g(v+\psi)+\psi_1$,
and since

$$\int_0^\infty \{F(v+\psi)-F(\psi)\}v_{xx}=-\int_0^\infty v_x\{f(v+\psi)(v_x+\psi_x)-f(\psi)\psi_x\}dx$$
$$=-\int_0^\infty f(v+\psi)\left(v_x+\frac{\psi_x}{2}\right)^2 dx+\int_0^\infty f(v+\psi)\frac{\psi_x^2}{4}-f(\psi)\psi_x v_x dx\,,$$

we see that

$$E_2'(t)=\int_0^\infty \{F(v+\psi)-F(\psi)\}\{-g(v+\psi)+\psi_1-f(\psi)\psi_t\}dx$$
$$-\int_0^\infty f(v+\psi)\left(v_x+\frac{\psi_x}{2}\right)^2 dx+\int_0^\infty v_t\{-g(v+\psi)+\psi_1-f(\psi)\psi_t\}+$$
$$+g(v+\psi)(v_t+\psi_t)+2Cvv_t dx+\int_0^\infty \left\{f(v+\psi)\frac{\psi_x^2}{2}-f(\psi)\psi_x v_x\right\}dx\,.$$

Now the third integral is

$$\int_0^\infty v_t(\psi_1-f(\psi)\psi_t)+g(v+\psi)\psi_t+2Cvv_t dx\,.$$

Now consider, in the first integral, the term

$$-\int_0^\infty \{F(v+\psi)-F(\psi)\}g(v+\psi)dx\,.$$

Let us consider the integrand:

$$\{F(v+\psi(-F(\psi)\}\,\{g_1(v+\psi)+g_2(v+\psi)\} = \{F(v+\psi)-F(\psi)\}g_1(v+\psi) + \{F(v+\psi)-F(\psi)\}g_2(x+\psi)\,.$$

Now the first term in the right-hand side is majorized by

$$\{F(v+\psi)-F(\psi)\}^2+c'v^2\,.$$

Next, consider the second term. If v is large, $F(v+\psi)-F(\psi)\geq -2L_0(v+\psi)$, $g_2(v+\psi)\geq 0$, and if v tend to $-\infty$, $F(v+\psi)-F(\psi)\leq 2L_0v$, $g_2(v+\psi)\leq 0$. We see that $-\{F(v+\psi)-F(\psi)\}g_2(v+\psi)\leq 2L_0(v+\psi)g_2(v+\psi)$. for $|v|$ large.
Let us remark that, if we can assume that $\lim_{u\to+\infty}F(u)=\pm\infty$, we can deduce that

$$-\{F(v+\psi)-F(\psi)\}\cdot g_2(v+\psi)\leq 0\,.$$

Summing up these estimates, we see that

$$E_2'(t)\leq \text{const.}\int_0^\infty \{F(v+\psi)-F(\psi)\}^2+v_t^2+v_x^2+v^2 + |v+\psi|\,|g_2(v+\psi)|+\text{const.}\,dx\,. \tag{2.6}$$

where constants do not depend on v.
Now, we consider

$$E(t) = E_1(t)+E_2(t)\,. \tag{2.7}$$

Taking into account of

$$v_t^2+(v_t+F(v+\psi)-F(\psi))^2\geq \frac{1}{4}\{v_t^2+(F(v+\psi)-F(\psi))^2\}\,,$$

and of the condition (1.2), we see that

$$E'(t)\leq cE(t)\,, \tag{2.8}$$

where c is a positive constant not depending on v.
From this, we have

$$E(t)\leq E(0)e^{ct}\,.$$

Up to now, we assumed that $u(x, t)$, therefore $v(x, t)$ exists up to $t=h$. However the reasoning shows the following: As far as the solution is continued—supposing this upper limit of time less than h—$E(t)$ is majorized by $E(0)e^{ch}$.

The Sobolev's lemma implies that max $|v(x, t)|$ is majorized by const. $\int v^2+v_x^2dx$, we see finally the

Theorem 2.1. *Assume the condition* (1.2). *Then, there exists a positive constant* c *such that*

$$|u(x, t)|<c, \quad 0\leq t\leq h \tag{2.9}$$

as far as the solution is continued.

Remark 1. The above constant c may depend on the boundary and initial data and h. Here, its dependence is not the question. What the theorem says is the existence of such a finite constant for each data and h.

Remark 2. If we consider only the Cauchy problem, the above estimate is much simplified. In fact, we need only to take the energy form $E_1(t)$ of (2.3). Of course, in this case we need not change u to v. According to (2.4), (here ψ, $\psi_1\equiv 0$), denoting

$$E_1(t) = \int_{-\infty}^{\infty} \frac{1}{2}(u_t^2+u_x^2)+G(u)+C(u^2+1)dx\,.$$

with the same convention with respect to the integration, namely, the integration is taken over only the fixed interval in x containing the support of $u(x, t)$.

We have

$$E_1'(t) = \int_{-\infty}^{\infty} -f(u)u_t^2+2Cu_tu\,dx\leq cE_1(t)\,.$$

Here we need only to assume that

$$(2.10)\quad \begin{aligned} &1^\circ.\ \ f(u)>-L_0 \\ &2^\circ.\ \ g(u)=g_1(u)+g_2(u),\ |g_1(u)|<c|u| \\ &\qquad\qquad \operatorname{sign} u\cdot g_2(u)\geq 0. \quad \text{for } |u| \text{ large.} \end{aligned}$$

Under this condition, we can conclude (2.9).

Remark 3. The equation (1. 1) does not depend on x, t explicitly. One could extend our estimate to these cases under some conditions. A trivial extension is to add the first order linear terms. However, we don't try here these generalizations.

3. A priori estimate of first derivatives of u.

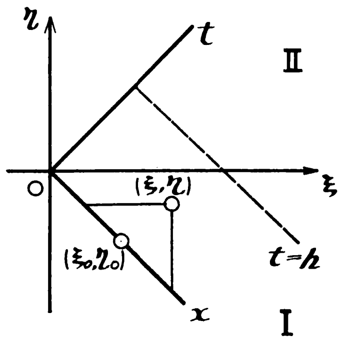

Fig. 1

In this section, we want to show that the first derivatives also remain bounded as far as the solution exists. For this proof, we use only the fact that u itself remain bounded. For this purpose we divide the domain $x \geq 0$, $t \geq 0$, in two domains.

(I): the domain of (x, t) such that $x \geq t$;

(II): the domain of (x, t) such that $t \geq x$. (See Fig. 1).

In the domain (I), $u(x, t)$ is determined only by the Cauchy data. Therefore, the boundedness of the first derivatives can be derived by Haar's Lemma. However, in the domain (II), the direct use of Haar's Lemma could not be expected.

Here, we want to obtain our results in both domains by the same principle.

As usual, we take the new independent variables (ξ, η) by:

$$\xi = x+t$$
$$\eta = t-x$$

Then (1. 1) takes the form

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = a(u)\frac{\partial u}{\partial\xi}+b(u)\frac{\partial u}{\partial\eta}+c(u) \tag{3. 1}$$

A) Now we argue about the domain (I).

We change the unknown function u to v in such a way that v have 0 Cauchy data.

For this purpose, let us remark that, from the Cauchy data we can deduce the 0 data. In fact, from the Cauchy data we can deduce the 0 data in the following form:

$$\frac{\partial u}{\partial \xi} = \varphi(\xi) \in C^1, \quad \frac{\partial u}{\partial \eta} = \psi(\eta) \in C^1 \quad \text{along the } x\text{-axis.}$$

Then, the function

$$\psi(\xi, \eta) = u(\xi_0, \eta_0) + \int_{\xi_0}^{\xi} \varphi(\xi) d\xi + \int_{\eta_0}^{\eta} \psi(\eta) d\eta$$

satisfies the same Cauchy data on the x-axis and $\dfrac{\partial^2 \psi}{\partial \xi \partial \eta} = 0$.
Recalling this fact, we set

$$v(\xi, \eta) = u(\xi, \eta) - \psi(\xi, \eta). \tag{3.2}$$

Then, (3.1) becomes

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = a(u) \frac{\partial v}{\partial \xi} + b(u) \frac{\partial v}{\partial \xi} + c(u) + f \tag{3.3}$$

where $f = a(u)\psi_\xi + b(u)\psi_\eta$.
Now we put in $a(u)$, $b(u)$, $c(u)$, f the solution $u(\xi, \eta)$ then these functions of (ξ, η) remain *bounded* according to Theorem 2.1.

Namely

$$|a(u(\xi, \eta))|, \ |b(u(\xi, \eta))|, \ |c(u(\xi, \eta))| \leq K,$$
$$|f| \leq M.$$

Now, let us consider the majorant equation of (3.3):

$$\frac{\partial^2 U}{\partial \xi \partial \eta} = K\left(U + \frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta}\right) + M. \tag{3.4}$$

In fact, the Picard's approximation process shows that the solution U of (3.4), with 0 Cauchy data on the line $\xi + \eta = 0$, majorize u with its first derivatives:

$$|v(\xi, \eta)| \leq U(\xi, \eta), \quad \left|\frac{\partial v}{\partial \xi}\right| \leq \frac{\partial U}{\partial \xi}, \quad \left|\frac{\partial v}{\partial \eta}\right| \leq \frac{\partial U}{\partial \eta}.$$

Evidently, the solution U exists. This show that, the first derivatives of u also remain bounded as far as $u \in C^2$ is continued.

B) Now we pass to the domain (II). By virtue of (A), $u(\xi, \eta)$ and its first derivatives have an a priori estimate on the x-axis. *Hereafter we write* (x, y) *in the place of* (ξ, η).

Problem. Let us consider the solution $u(x, y)$ of

(3.1) $\dfrac{\partial^2 u}{\partial x \partial y} = a(u)\dfrac{\partial u}{\partial x} + b(u)\dfrac{\partial u}{\partial y} + c(u)$, with the boundary data $u(x, 0) = \varphi_1(x)$, $u(y, y) = \varphi_2(y)$.

Obtain an a priori estimate of first derivatives of u, by means of that of u itself and of $\varphi_1(x)$ and $\varphi_2(y)$.

We reduce this Goursat-type problem to zero boundary value. Namely, by defining,

$$\psi(x, y) = \varphi_1(x) + \varphi_2(y) - \varphi_1(y)\,,$$

we set

$v = u - \psi$, then (3.1) becomes

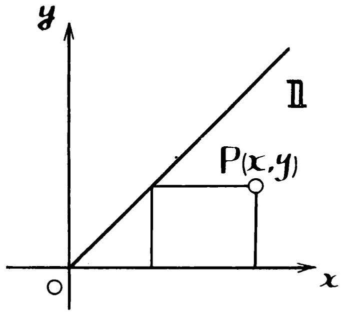

Fig. 2

$$\text{(3.3)} \qquad \frac{\partial^2 v}{\partial x \partial y} = a(u)\frac{\partial v}{\partial x} + b(u)\frac{\partial v}{\partial y} + c(u) + f\,.$$

where $f = a(u)\psi_x + b(u)\psi_y$.

where $v(x, 0) = v(y, y) = 0$.

It suffices to prove the following lemma.

Let us consider the solution $u(x, y)$ of

$$\text{(3.5)} \qquad \frac{\partial^2 u}{\partial x \partial y} = a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} + c(x, y)u + f(x, y)$$

defined in (II), with zero Goursat data: $u(x, 0) = u(y, y) = 0$

We assume

$$\text{(3.6)} \qquad |a(x, y)|,\ |b(x, y)|,\ |c(x, y)| \leqq K,\ |f(x, y)| \leqq M.$$

Now we consider the following auxiliary equation:

$$\text{(3.7)} \qquad \frac{\partial^2 U}{\partial x \partial y} = K\left(U + 2\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y}\right) + M,$$

whose $U(x, y)$ is considered in $\{x\geq 0,\ y\geq 0\}$ with zero boundary data:

$$U(x, 0) = U(0, y) = 0\,.$$

Lemma 3.1. *$u(x, y)$ is majorized by $U(x, y)$ in the following form:*

$$|u(x, y)|\leq U(x, y)\,, \quad \left|\frac{\partial u}{\partial x}\right|\leq\frac{\partial U}{\partial x}\,, \quad \left|\frac{\partial u}{\partial y}\right|\leq\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}\,.$$

Proof. Picard's successive approximation itself shows the above inequality.

Set

$$u_0(x, y) = \iint_R f(\xi, \eta)d\xi d\eta\,, \quad u_1(x, y) = \iint_R \left(a\frac{\partial u_0}{\partial x}+b\frac{\partial u_0}{\partial y}+cu_0\right)d\xi d\eta\,, \cdots$$

where R denote the rectangle showed in Fig. 2.

Set

$$U_0(x, y) = \int_0^x\int_0^y M d\xi d\eta\,, \quad U_1(x, y) = K\int_0^x\int_0^y\left(U_0+2\frac{\partial U_0}{\partial x}+\frac{\partial U_0}{\partial y}\right)d\xi d\eta\,, \cdots.$$

It is evident that

$$|u_0|\leq U_0\,, \quad \left|\frac{\partial u_0}{\partial x}\right|\leq\frac{\partial U_0}{\partial x}\,, \quad \left|\frac{\partial u_0}{\partial y}\right|\leq\frac{\partial U_0}{\partial y}\,.$$

In general we have

$$(3.8)\qquad |u_n|\leq K^n U_n\,, \quad \left|\frac{\partial u_n}{\partial x}\right|\leq K^n\frac{\partial U_n}{\partial x}\,, \quad \left|\frac{\partial u_n}{\partial y}\right|\leq K^n\left(\frac{\partial U_n}{\partial x}+\frac{\partial U_n}{\partial y}\right).$$

This relation can be shown by induction on n. In fact, recalling the relation

$$(3.9)\qquad u(x, y) = \iint_R f(\xi, \eta)d\xi d\eta\,, \quad \frac{\partial u}{\partial x} = \int_0^y f(x, \eta)d\eta\,,$$

$$\frac{\partial u}{\partial y} = \int_y^x f(\xi, y)d\xi-\int_0^y f(y, \eta)d\eta\,,$$

and supposing (3.8) valid for $n=n$, we have

$$|u_{n+1}|\leq\iint_R\left|a\frac{\partial u_n}{\partial x}+b\frac{\partial u_n}{\partial y}+cu_n\right|d\xi d\eta\leq K^{n+1}\int_0^x\int_0^y\left(2\frac{\partial U_n}{\partial x}+\frac{\partial U_n}{\partial y}+U_n\right)$$
$$\times d\xi d\eta = K^{n+1}U_{n+1}$$

$$\left|\frac{\partial u_{n+1}}{\partial x}\right| \leq \int_0^y \left| a\frac{\partial u_n}{\partial x} + b\frac{\partial u_n}{\partial y} + cu_n \right| d\eta \leq K^{n+1}\int_0^y 2\frac{\partial U_n}{\partial x} + \frac{\partial U_n}{\partial y} + U_n dy$$

$$= K^{n+1}\frac{\partial U_{n+1}}{\partial x}$$

$$\left|\frac{\partial u_{n+1}}{\partial y}\right| \leq \int_x^y |\cdots| d\xi + \int_0^y |\cdots| d\eta \leq K^{n+1}\left(\frac{\partial U_{n+1}}{\partial x} + \frac{\partial U_{n+1}}{\partial y}\right).$$

(3.8) shows that

$$\frac{\partial u}{\partial x} = \sum_{n=0}^{\infty}\frac{\partial u_n}{\partial x}, \quad \frac{\partial u}{\partial y} = \sum_{n=0}^{\infty}\frac{\partial u_n}{\partial y} \qquad \text{are majorized by}$$

$$\frac{\partial U}{\partial x}, \quad \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} \qquad \text{respectively, where}$$

$$U = U_0 + KU_1 + K^2U_2 + \cdots + K^nU_n + \cdots$$

where existence theorem is already classical. (cf. E. Picard [2], p. 106–113). Summing up the above results, we have

Theorem 3.1. *Under the same assumption as in Theorem* 2.1, *the first derivatives of* $u(x, t)$ *have also an a priori estimate. Namely, there exists a constant* C *such that*

$$\left|\frac{\partial u}{\partial x}(x, t)\right|, \quad \left|\frac{\partial u}{\partial t}(x, t)\right| < C < +\infty$$

as far as the solution u *of* (1.1) *is continued.*

4. Local existence theorem.

In this section, we shall consider mainly Goursat's problem. Let us consider

$$\text{(4.1)} \qquad \frac{\partial^2 u}{\partial x \partial y} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \qquad \text{or} \quad = f(x, y, u, p, q).$$

We assume that $f \in C^2$, for all arguments. Our problem is the following: Find a solution $u(x, y)$ of (4.1) in the domain (II): $0 \leq y \leq x \leq a$, which vanishes on the boundary: $u(x, 0) = u(x, x) = 0$.

We use Picard's method. (cf. Picard. [2])

Define

$$u_0(x, y) = \iint_R f(\xi, \eta, 0, 0, 0)d\xi d\eta ,$$

$$u_1(x, y) = \iint_R f\left(\xi, \eta, u_0(\xi, \eta) \frac{\partial u_0}{\partial \xi}(\xi, \eta) , \frac{\partial u_0}{\partial \eta}(\xi, \eta)\right)d\xi d\eta ,$$

....

$$u_n(x, y) = \iint_R f(\xi, \eta, u_{n+1}, p_{n-1}, q_{n-1})d\xi d\eta .$$

At first we restrict the domain as follows: Define

(4.2) $\quad (D) = \{x, y, u, p, q ; 0\leq y\leq x\leq a , |u|, |p|, |q|\leq b\} .$

Denote

$$M = \max_D |f(x, y, u, p, q)| .$$

We take ρ in such a way that

(4.3) $$M\rho \leq b , \quad M\rho^2 \leq b , \quad \rho \leq a .$$

We see that for $x, y\leq \rho$, u_n, p_n, q_n $(n=1, 2, \cdots)$ are remaining in (D). Now we want to prove

Proposition 4.1. *Goursat's problem to* (4.1) *of above type has a unique solution $u(x, y)$, 2 times continuously differentiable including the boundary*[2]*, in the domain $0\leq y\leq x\leq \rho$, ρ satisfying* (4.3).

Proof. At first we remark that

$$\frac{\partial u_n}{\partial x} = \int_0^y f(x, \eta, u_{n-1}(x, \eta), p_{n-1}(x, \eta), q_{n-1}(x, \eta))dy$$

$$\frac{\partial u_n}{\partial y} = \int_y^x f(\xi, y, u_{n-1}, p_{n-1}, q_{n-1})d\xi - \int_0^y f(y, \eta, u_{n-1}, \cdots)d\eta .$$

There exists a positive constant K such that

$$|f(x, y, u, p, q)-f(x, y, u', p', q')|\leq K(|u-u'| + |p-p'| + |q-q'|)$$

for any pair in (D).

Denoting $v_n(x, y) = |u_n-u_{n-1}| + |p_n-p_{n-1}| + |q_n-q_{n-1}|$,

we have

2) This means the following: $u(x, y)$ has continuous derivatives up to order 2 in $0<y<x$, and each of these derivatives has a (unique) continuous extension up to the boundary.

$$|u_{n+1}-u_n|\leq K\iint_R v_n(\xi,\eta)d\xi d\eta$$

$$|p_{n+1}-p_n|\leq K\int_0^y v_n(x,\eta)d\eta$$

$$|q_{n+1}-q_n|\leq K\int_y^x v_n(\xi,y)d\xi+K\int_0^y v_n(y,\eta)d\eta\,.$$

As we have shown in the proof of lemma 2.1, we consider a majorant equation:

$$\frac{\partial^2 U}{\partial x\partial y}=K\left(U+2\frac{\partial U}{\partial x}+\frac{\partial U}{\partial y}\right)+M, \qquad \text{for}\quad x,y\geq 0, \tag{4.4}$$

with $U(x,0)=U(0,y)=0$.

Then, the sequence of successive approximation (cf. section 2), $U_0, U_1, \cdots$ majorize $u_0, u_1, \cdots$ as follows:

$$|u_n-u_{n-1}|\leq K^n U_n$$

$$|p_n-p_{n-1}|\leq K^n\frac{\partial U_n}{\partial x}$$

$$|q_n-q_{n-1}|\leq K^n\left(\frac{\partial U_n}{\partial x}+\frac{\partial U_n}{\partial y}\right),$$

where $U=U_0+KU_1+K^2U_2+\cdots$,

Now, we know that, the solution $u(x,y,k)$ of

$$\frac{\partial^2 u}{\partial x\partial y}=k\left(u+2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)+M, \quad \text{with 0 data on the } x \text{ and } y \text{ axes,}$$

is holomorphic in k for all values of k (complex), and continuous in x, y, k, and this is also true for $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ (cf. Picard [2], 112–113).

Now, take a positive α, which is less than 1. Then there exists a constant $c(\alpha,\rho,K)$ such that

$$U_n,\ \frac{\partial U_n}{\partial x},\ \frac{\partial U_n}{\partial x}+\frac{\partial U_n}{\partial y}\leq\left(\frac{\alpha}{K}\right)^n c(\alpha,\rho,K) \qquad \text{for}\quad 0\leq y\leq x\leq\rho\,. \tag{4.5}$$

This implies that

$$v_n(x,y)=|u_n-u_{n-1}|+|p_n-p_{n-1}|+|q_n-q_{n-1}|<c\alpha^n, \quad n=1,2,\cdots \tag{4.6}$$
for $0\leq y\leq x\leq\rho$, $0<\alpha<1$, and c is a positive constant.

(4. 6) shows that $u_0+(u_1-u_0)+(u_2-u_1)+\cdots$ is a normally convergent sequence with its first derivatives. Namely $\{u_n(x, y)\}$ converges uniformly with its first derivatives. This limit function $u(x, y)\in C^1$ is a solution of (4. 1).

Now, we want to show that $u(x, y)$ is 2 times continuously differentiable. At first, let us remark that each $u_n(x, y)$ is of class C^2 including the boundary. In fact, recalling the formulae:

$$u(x, y) = \iint_R f(\xi, \eta)d\xi d\eta\,, \quad \frac{\partial^2 u}{\partial x^2}(x, y) = \int_0^y f_x(x, \eta)d\eta\,, \quad \frac{\partial^2 u}{\partial x\partial y} = f(x, y)\,,$$

$$\frac{\partial^2 u}{\partial y^2}(x, y) = -2f(y, y)+\int_y^x f_y(\xi, y)d\xi-\int_0^y f_x(y, \eta)d\eta\,,$$

we have

$$\begin{aligned}(4.7)\qquad \frac{\partial^2 u_n}{\partial x^2} &= \int_0^y f_x(x, \eta, u_{n-1}, p_{n-1}, q_{n-1}) \\ &\quad +f_u\frac{\partial u_{n-1}}{\partial x}+f_p\frac{\partial^2 u_{n-1}}{\partial x^2}+f_q\frac{\partial^2 u_{n-1}}{\partial x\partial y}d\eta\,, \\ \frac{\partial^2 u_n}{\partial x\partial y} &= f(x, y, u_{n-1}, p_{n-1}, q_{n-1})\,, \\ \frac{\partial^2 u_n}{\partial y^2} &= -2f(y, y, u_{n-1}(y, y), p_{n-1}(y, y), q_{n-1}(y, y)) \\ &\quad +\int_y^x f_y+f_u\frac{\partial u_{n-1}}{\partial y}+f_p\frac{\partial^2 u_{n-1}}{\partial x\partial y}+f_q\frac{\partial^2 u_{n-1}}{\partial y^2}d\xi \\ &\quad -\int_0^y f_x+f_u\frac{\partial u_{n-2}}{\partial x}+f_p\frac{\partial^2 u_{n-1}}{\partial x^2}+f_q\frac{\partial^2 u_{n-1}}{\partial x\partial y}d\eta\,.\end{aligned}$$

At first, we shall show that $\frac{\partial^2 u_n}{\partial x^2}$, $\frac{\partial^2 u_n}{\partial y^2}$ are uniformly bounded. Realling that $\frac{\partial^2 u_n}{\partial x\partial y}$ are uniformly bounded, let us consider a sequence of functions $A_n(y)$ by $A_n(y)=c\int_0^y\{1+A_{n-1}(\eta)\}d\eta$, for $y\geq 0$, with sufficiently large c. If we take

$A_0(x)=N\geq\max\left|\frac{\partial^2 u_0}{\partial x^2}\right|$, take we shall have in general $A_n(y)\geq\left|\frac{\partial^2 u_n}{\partial x^2}(x, y)\right|$.

It is easy to see that $A_n(y)$ are uniformly bounded for $0\leq y\leq\rho$.

In the same way, we see that, $\frac{\partial^2 u_n}{\partial y^2}$ are uniformly bounded.

Now, we want to show that $\sum_n \left| \frac{\partial^2(u_n - u_{n-1})}{\partial x^2} \right|$, $\sum \left| \frac{\partial^2(u_n - u_{n-1})}{\partial y^2} \right|$ are uniformly convergent. Denote by K_1 a common Lipschitz constant, with respect to u, p, q, of the functions $f, f_x, f_y, f_u, f_p, f_q$, when (x, y, u, p, q) in (D). Denote by M_1 a positive constant which majorizes 1° all these functions when (x, y, u, p, q) in (D), 2° all derivatives of $u_n(x, y)$ $(n=0, 1, 2, \cdots)$ up to order 2.

Then, taking into account of (4.6), (4.7) implies

$$\left| \frac{\partial^2(u_{n+1} - u_n)}{\partial x^2}(x, y) \right| \leq \int_0^y K_1 c\alpha^n + M_1 c\alpha^n + 4M_1 K_1 c\alpha^n + M_1 \left| \frac{\partial^2(u_n - u_{n-1})}{\partial x^2} \right| d\eta$$

Namely, if we denote $\varphi_n(y) = \max_{y \leq x \leq \rho} \left| \frac{\partial^2(u_n - u_{n-1})}{\partial x^2}(x, y) \right|$, we have

$$\text{(4.8)} \quad \varphi_{n+1}(y) \leq \int_0^y c_1 \alpha^n + M_1 \varphi_n(\eta) d\eta \text{ , where } c_1 \text{ is a positive constant.}$$

In the same way, we shall have

$$\left| \frac{\partial^2(u_{n+1} - u_n)}{\partial y^2}(x, y) \right| \leq K_1 c\alpha^n + \int_y^x c_2 \alpha^n + M_1 \left| \frac{\partial^2(u_n - u_{n-1})}{\partial y^2} \right| d\xi + \int_0^y c_3 \alpha^n + M_1 \left| \frac{\partial^2(u_n - u_{n-1})}{\partial x^2}(y, \eta) \right| d\eta$$

Denote by $\psi_n(x) = \max_{0 \leq y \leq x} \left| \frac{\partial^2(u_n - u_{n-1})}{\partial y^2}(x, y) \right|$, then we have

$$\leq K_1 c\alpha^n + \int_y^x c_2 \alpha^n + M_1 \psi_n(\xi) d\xi + \int_0^y c_3 \alpha^n + M_1 \varphi_n(\eta) d\eta .$$

Taking into account of $y \leq x$, we shall have

$$\text{(4.9)} \qquad \psi_{n+1}(x) \leq K_1 c\alpha^n + \int_0^x (c_2 + c_3)\alpha^n + M_1(\psi_n(\xi) + \varphi_n(\xi)) d\xi .$$

Now, let us consider a series of functions $\Phi_n(x)$ defined by

$$\Phi_n(x) = c_0 \alpha^{n-1} + \int_0^x c_0 \alpha^{n-1} + M_1 \Phi_{n-1}(\xi) d\xi ,$$

where $c_0 = \max[c_1, K_1 c, c_2 + c_3]$

Then evidently, both $\varphi_n(x)$ and $\psi_n(x)$ are majorized by $\Phi_n(x)$ if we define

$$\Phi_0(x) = N > \max|\varphi_0(x)| + \max|\psi_0(y)| .$$

We see easily that $\sum \Phi_n(x) < +\infty$. This shows a fortiori that $\sum \left|\frac{\partial^2(u_n - u_{n-1})}{\partial x^2}\right|$ and $\sum \left|\frac{\partial^2(u_n - u_{n-1})}{\partial y^2}\right|$ are uniformly convergent. This shows that the solution, limit of $u_n(x, y)$, is 2 times continuously differentiable.

The uniqueness of solution is evident. In fact, let u_1 and u_2 be two solutions. Denoting $v = u_1 - u_2$, v satisfies the following linear equation:

$$\frac{\partial^2 v}{\partial x \partial y} = \bar{f}_u v + \bar{f}_p \frac{\partial v}{\partial x} + \bar{f}_q \frac{\partial v}{\partial y}$$

$\bar{f}_u$, $\bar{f}_p$, $\bar{f}_q$ denote the mean values. If v has boundary value 0, then $v \equiv 0$. Our proof is thus complete.

Now, we return to non zero Goursat data. Namely, given two functions $\varphi_1(x)$ and $\varphi_2(x)$ which are 2 times continuously differentiable for $x \geq 0$, find a solution $u(x, y)$ of (4.1) in the domain (II): $0 \leq y \leq x \leq a$, such that $u(x, 0) = \varphi_1(x)$, $u(x, x) = \varphi_2(x)$.
Then we have

Theorem 4.1. *The above problem has a unique solution $u(x, y)$, 2 times continuously differentiable including the boundary, for $0 \leq y \leq x \leq \rho'$. Here, a positive constant ρ' may depend on the maximum norm of $\varphi_1(x)$ and $\varphi_2(x)$ up to their first derivatives, but it does not depend on the maximum norm of their second derivatives.*

Proof. This is an immediate consequence of the above proposition. In fact, defining

$$\psi(x, y) = \varphi_1(x) + \varphi_2(y) - \varphi_1(y) ,$$

set $v(x, y) = u(x, y) - \psi(x, y)$, then (4.1) becomes

$$\frac{\partial^2 v}{\partial x \partial y} = f(x, y, v + \psi,\ v_x + \psi_x,\ v_y + \psi_y) ,$$

where v vanishes on the boundary. c.q.e.d.
As regards to the Cauchy problem, the same proof can be carried

out literally, even simpler[3]. We restict ourselves to state the following.

Theorem 4.2. *Let us consider the equation*

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=f\left(x,\, t,\, u,\, \frac{\partial u}{\partial x},\, \frac{\partial u}{\partial y}\right), \tag{4.10}$$

where $f\in C^2$ (for all arguments). Given the Cauchy data

$$u(x,\,0)=\varphi_0(x)\in C^2\,, \quad \text{and} \quad u_t(x,\,0)=\varphi_1(x)\in C^1\,,$$

then, there exists a unique solution $u(x,\,t)\in C^2$, in the domain defined by $|x-t|\leq\rho$, $|x+t|\leq\rho$. ρ may depend on the maximum norm of φ_0, φ_0', and φ_1, but it does not depend on that of φ_0'' and φ_1'.

5. Final remarks.

a) Smooth continuation along a characteristic.

Let us consider the following situation: u_1 and u_2 are solutions of

$$\frac{\partial^2 u}{\partial x\partial y}=f\left(x,\, y,\, u,\, \frac{\partial u}{\partial x},\, \frac{\partial u}{\partial y}\right) \tag{5.1}$$

in the domain (I) and (II) respectively. (See Fig. 1) (We write here x, y in place of ξ, η). We assume u_1 and u_2 are two times continuously differentiable including the boundary (cf. foot note 3)). Let us assume that, at the origin, the two functions have the same derivatives up to order 2, moreover $u_1=u_2$ along the x-axis. Then, u_1 and u_2 are continuously connected up to second derivatives along the x-axis. In other words, the function $u(x,\,y)$ defined in (I)+(II) by joining u_1 and u_2 is of class C^2. This is a well-known fact. However, since this is a important fact, we give here the proof. Denote by γ the trace operator on the x-axis. We have evidently, $\gamma\frac{\partial u_i}{\partial x}=\frac{d}{dx}\gamma u_i$, $\gamma\frac{\partial^2 u}{\partial x\partial y}=\frac{d}{dx}\left(\frac{\partial u_i}{\partial y}\right)_{y=0}$. Hence, taking the trace of the functions of (5.1), we have

3) Here we use the transformation (3.2), and reduce the problem to the zero Cauchy data.

4) Recently, R. Courant showed this fact for general semi-linear hyperbolic systems ([1])

$$\frac{d}{dx}\left(\frac{\partial u_i}{\partial y}\right) = f\left(x, 0, \gamma u_i, \frac{d}{dx}\gamma u_i, \left(\frac{\partial u_i}{\partial y}\right)_{y=0}\right)$$

Taking the difference, denoting

$$\left(\frac{\partial u_1}{\partial y}\right)_{y=0} - \left(\frac{\partial u_2}{\partial y}\right)_{y=0} = \psi(x)\,,$$

we have

$$\frac{d}{dx}\psi(x) = \bar{f}_q \cdot \psi(x)\,.$$

Since, at $x=0$, $\psi(x)=0$, we have $\psi(x)\equiv 0$.
Starting from the derived equation

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y^2}\right) = f_y + f_u\frac{\partial u}{\partial y} + f_p\frac{\partial u}{\partial x\partial y} + f_q\frac{\partial^2 u}{\partial y^2}$$

and using the above result, we have moreover $\left(\frac{\partial^2 u_1}{\partial y^2}\right)_{y=0} = \left(\frac{\partial^2 u_2}{\partial y^2}\right)_{y=0}$.

b) Since our equation is semi-linear, to prove the Main Theorem in the Introduction, it suffices to prove it supposing Cauchy data has compact support. Then, under this assumption, an a priori estimate of u and its first derivatives (Theorem 2.1, and 3.1), connected with the local existence Theorems (Theorem 4.1, and 4.2), enables us the step-by-step construction of the solution in any finite interval of t.

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