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On a ring of bounded continuous functions on an open Riemann surface (supplements and corrections to my former paper)

By

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I. Introduction

("nt"("nb") means the n-th line from top (bottom respectively)) In my paper "On a compactification of an open Riemann surface and its application", this Journal vol. 1, No. 1, (1961) [*],

(1) 17t, p. 22 "We denote by $\mathfrak{F}$ the family of real-valued bounded, continuous functions on R each of which has the radial limits in K for almost all $e^{i\theta}$"$\cdots$to this sentence the following should be added: moreover the radial limit function $f\circ T(e^{i\theta})$ $(0\leqq\theta<2\pi)$ is invariant under the group of the cover transformations except for a null-set.

(2) 8t, p. 24 Delete the sentence "From this, we know that$\cdots$ belongs to $\mathfrak{F}$. Hence$\cdots$belongs to $\mathfrak{F}$"

(3) 9t, p. 27 "Then $\bar{D}-\overline{\partial D}$ meets $\Delta_{\mathfrak{F}}$" to be corrected as follows: "The $\bar{D}$ meets $\Delta_{\mathfrak{F}}$." (The proof will be given in supplements from the more general point of view.)

(4) 14b, p. 27 lemma 2.2 will be verified in supplements from the more general point of view. (The (2), (3), (4) will be studied from the other standpoint in chapter **III.**)

II. Supplements

Here we shall give some notes supplementary to the paper [*].

1. We shall use the same notations as in [*]. Let R be an open Riemann surface of hyperbolic type and let u be a subharmonic function bounded, upper semicontinuous on R. Then we have the following

PROPOSITION 2.1. *Let D be a subset in R such as $D=\{p\in R;\ u(p)>c\}$. Then the closure $\bar{D}$ of D with respect to $R^*_{\mathfrak{F}}$ meets $\Delta_{\mathfrak{F}}$, where $\inf_R u<c<\sup_R u$.*

Proof. We suppose that $\bar{D}\cap\Delta_{\mathfrak{F}}=\phi$. Then $\bar{D}\cap\Gamma_{\mathfrak{F}}=\gamma$ is compact and $\gamma\cap\Delta_{\mathfrak{F}}=\phi$, consequentry there exists a non-negative function $\varphi(\in\mathfrak{F}_0)$ such as $\varphi=0$ on $\Delta_{\mathfrak{F}}$, $=1$ or γ. Let $E=\left\{p\in R;\ \varphi(p)>\frac{1}{2}\right\}$, then $\bar{E}\cap\Delta_{\mathfrak{F}}=\phi$. Let $K;\ |z|<1$ be the conformal image of the universal covering surface of R and let $T(z)$ be the conformal mapping from K onto R. Then the radial limits of $\varphi\circ T(re^{i\theta})$ are zero for almost all θ $(0\leqq\theta<2\pi)$. Now we define a function $\tilde{u}(p)$ on R such as

$$\begin{aligned}\tilde{u}(p) &= u(p) \quad \text{on } D\\ &= c \qquad\ \ \text{on } p\in R-D\,.\end{aligned}$$

Then $\tilde{u}$ is a bounded subhamonic function on R, consequently $\tilde{u}\circ T(re^{i\theta})$ has the radial limits for each θ except for a null-set (Littlewood [6, 7]). Let $\lim\limits_{\gamma\to 1}\tilde{u}\circ T(re^{i\theta})>c$ for some θ, then the image $\sigma_\delta=\{p\in R;\ p=T(re^{i\theta}),\ 0<\delta<r<1\}$ is contained in D for a suitable number δ, that is, $\sigma_\delta\subset D$. Therefore $\bar{\sigma}_\delta\cap\Gamma_{\mathfrak{F}}\subset\gamma$ and from this we know that the set $\{e^{i\theta}\}$ such as $\lim\limits_{r\to 1}\tilde{u}\circ T(re^{i\theta})>c$ is of linear measure zero since φ belongs to $\mathfrak{F}_0$. Thus we conclude that $u(p)\leqq c$ on R. This is absurd, that is, $\bar{D}\cap\Delta_{\mathfrak{F}}\neq\phi$. (q. e. d.)

Let $\bar{u}^*(q^*)$ be the superior limit of u at q^* $(\in\Delta_{\mathfrak{F}})$. Then $\bar{u}^*(q^*)$ is the upper semicontinuous bounded function on $\Delta_{\mathfrak{F}}$, therefore is the measurable function on $\Delta_{\mathfrak{F}}$. We have the following

THEOREM 2.1. *Let u be bounned, upper (lower) semicontinuous subharmonic (superharmonic) function. Then*

$$L.H.M.\ u=\int_{\Delta_{\mathfrak{F}}}\bar{u}^*(q^*)\,d\mu(q^*;\ p) \qquad (p\in R)$$

$$(G.H.M.\ u=\int_{\Delta_{\mathfrak{F}}}\underline{\lim}\ u(q^*)\,d\mu(q^*;p))$$

Proof. In the following, we shall deal with a bounded upper semicontinuous function u. Let $\lambda(p)=L.H.M.\ u(p)$ and $\bar{u}(p) =\int_{\Delta_{\mathfrak{F}}}\bar{u}^*(q^*)\,d\mu(q^*\,;\ p)$ $(p\in R)$. We shall verify that $\lambda(p)=\bar{u}(p)$. At first, we note that $\lambda(p)\geqq\bar{u}(p)$, since $\lambda(q^*)\geqq\bar{u}^*(q^*)$ and $\bar{u}(q^*) =\bar{u}^*(q^*)$ except for a null-set. In the following, we shall see that $\bar{u}(p)\geqq u(p)$ $(p\in R)$. We suppose that at some point p_0 in R $u(p_0)-\bar{u}(p_0)=\varepsilon>0$.

Then

$$D = \left\{p\in R\,;\ u(p)-\bar{u}(p)>\frac{\varepsilon}{2}\right\}$$

is a non-compact subset in R and the closure $\bar{D}$ (in $R^*_{\mathfrak{F}}$) meets $\Delta_{\mathfrak{F}}$ by proposition 2.1. Let $q^*(\in\Delta_{\mathfrak{F}})$ be such a point that $\bar{u}(q^*) =\bar{u}^*(q^*)$, then there exists an open set σ_1 (in $R^*_{\mathfrak{F}}$) such that $q^*\in\sigma_1$ and

$$\bar{u}^*(q^*)-\frac{\varepsilon}{4}<\bar{u}(p)<\bar{u}^*(q^*)+\frac{\varepsilon}{4}$$

for every point $p\in\sigma_1$ because $\bar{u}$ is continuous on $R^*_{\mathfrak{F}}$. On the other hand, there exists an open set σ_2 such that $q^*\in\sigma_2$ and

$$u(p)<\bar{u}^*(q^*)+\frac{\varepsilon}{4}$$

for every point $p\in\sigma_2\frown R$ From these inequalities, we have

$$u(p)-\bar{u}(p)<\frac{\varepsilon}{2}$$

for every point $p\in(\sigma_1\frown\sigma_2)\frown R$. From this, we conclude that $q^*\notin\bar{D}\frown\Delta_{\mathfrak{F}}$. Considering that $\bar{u}(q^*)=\bar{u}^*(q^*)$ except for a null-set, we know that $\bar{D}\frown\Delta_{\mathfrak{F}}$ is a null-set.

Now we define the subharmonic function $v(p)$ on R such as

$$\begin{aligned} v(p) &= u(p)-\bar{u}(p)-\frac{\varepsilon}{2} && \text{on } D \\ &= 0 && \text{on } R-D-\partial D\,, \end{aligned}$$

Then we know that the least harmonic majorant of $v(p)$ is identically zero on R. Suppose that $L.H.M.\ v(p)=\mu(p)>0$. Then $\Delta_\mu=\{q^*\in\Delta_{\mathfrak{F}}\,;\ \mu(q^*)>0\}$ is of positive measure, consequently there exists a simultaneously open and closed subset γ ($\subset \Delta_\mu$) such as $\gamma \cap (\bar{D} \cap \Delta_{\mathfrak{F}}) = \phi$. Then $\mu(q^*)$ is continuous on γ, consequently $\min_\gamma \mu(q^*) = \mu_0 > 0$. Then the function

$$\tilde{\mu}(p) = \mu(p) - \mu_0 \omega_\gamma(p)$$

is positive harmonic on R, and moreover $\tilde{\mu}(p) \geqq v(p)$ on R, where $\omega_\gamma(p)$ is the harmonic measure of γ. Indeed, it is evident that $\tilde{\mu}(p)$ is a positive harmonic function. We shall show that $\tilde{\mu}(p) \geqq v(p)$ on R. Suppose that at some point p ($\in R$)

$$v(p) - \tilde{\mu}(p) = \varepsilon > 0\,,$$

then

$$E = \left\{ p \in R\,;\ v(p) - \tilde{\mu}(p) > \frac{\varepsilon}{2} \right\}$$

is a non-compact subset in R and the closure $\bar{E}$ meets $\Delta_{\mathfrak{F}}$. Let q^* ($\in \Delta_{\mathfrak{F}}$) be a point such as $q^* \notin \gamma$, then there exists an open set σ such that $q^* \in \sigma$ and

$$\mu(p) > v(p)$$
$$0 < \mu(p) - \tilde{\mu}(p) < \frac{\varepsilon}{2}$$

for every point $p \in \sigma$ because of $\omega_\gamma(q^*) = 0$. From this, we know that

$$v(p) - \tilde{\mu}(p) < \frac{\varepsilon}{2} \quad (p \in \sigma \cap R)\,,$$

that is, $\bar{E} \cap \Delta_{\mathfrak{F}}$ is the subset of γ. Since $\gamma \subset \Delta_{\mathfrak{F}} - \bar{D}$, any point $q^* \in \bar{E} \cap \Delta_{\mathfrak{F}}$ does not belong to $\bar{D} \cap \Delta_{\mathfrak{F}}$. Consequently there exists an open set $\tilde{\sigma}$ such that $q^* \in \tilde{\sigma}$ and $\tilde{\sigma} \cap \bar{D} = \phi$. From this, we know that $\overline{\lim}_{p \to q^*} v(p) = 0$. On the contrary, $\overline{\lim}_{p \to q^*} v(p) \geqq \frac{\varepsilon}{2}$ because of $q^* \in \bar{E} \cap \Delta_{\mathfrak{F}}$. This is absurd. Thus we know that $\tilde{\mu}(p) \geqq v(p)$ on R, and that considering that $\mu(p) = L.\,H.\,M.\ v(p)$ we conclude that $v(p) \equiv 0$ on R. Thus we have verified that $u(p) \leqq \bar{u}(p)$. From this, we know that

$$\bar{u}^*(q^*) = \overline{\lim_{p \to q^*}}\, u(p) \leqq \lambda(q^*) \leqq \bar{u}(q^*)$$

for almost all points $q^* \in \Delta_{\mathfrak{F}}$, where $\lambda(p) = L.\,H.\,M.\ u(p)$. Thus we have $\lambda(p) \equiv \bar{u}(p)$ $(p \in R)$. (q. e. d.)

COROLLARY. *$\bar{u}^*(q^*)$ $(= \overline{\lim}_{p \to q^*} u(p))$ $(q^* \in \Delta_{\mathfrak{F}})$ is a continuous function on $\Delta_{\mathfrak{F}}$, where $u(p)$ is a bounded semicontinuous subharmonic function.*

Proof. $\bar{u}^*(q^*) = \lambda(q^*)$ (a. e.) and $\bar{u}^*(q^*)$ is upper semicontinuous on $\Delta_{\mathfrak{F}}$, consequently $\bar{u}^*(q^*) = \lambda(q^*)$ for all points of $\Delta_{\mathfrak{F}}$.

Thus **I.** (3) has been verified and **I.** (4) also verified under the wide interpretation (cf. chapter **III**). From theorem 2.1 we have the following

THEOREM. *Let u be a positive superharmonic function on R. Then $u \equiv +\infty$, provided that $e_\infty = \{q^* \in \Delta_{\mathfrak{F}}\,;\ \underline{u}^*(q^*) = +\infty\}$ is of positive measure, where $\underline{u}^*(q^*) = \underline{\lim}_{p \to q^*} u(p)$.*

Proof. Let $u_n(p) = \min\,[u(p),\ n]$, where the n are positive integers. Then, by theorem 2.1

$$\tilde{u}_n(p) = G.\,H.\,M.\ u_n(p) = \int_{\Delta_{\mathfrak{F}}} \underline{u}_n^*(q^*)\, d\mu(q^*\,;\ p) \quad (\underline{u}_n^*(q^*) = \underline{\lim}_{p \to q^*} u_n(p))$$

and $\{\tilde{u}_n(p)\}$ is the non-decreasing sequence. Therefore

$$\lim_{n \to \infty} \tilde{u}_n(p) = \int_{\Delta_{\mathfrak{F}}} \lim \underline{u}_n^*(q^*)\, d\mu(q^*\,;\ p) \leqq u(p)$$

and from this we know that $u(p) \equiv +\infty$ provided that e_∞ is of positive measure.

III. Ring of bounded continuous functions on *R*.

1. In this chapter we shall give another compactification R_F^* of an open Riemann surface R $(\notin O_G)$ such as bounded continuous subharmonic (superharmonic) functions will be extended continuously onto R_F^*. To define it we use the Royden's decomposition [9] without using the universal covering surface as in my former paper. Now let $R \notin O_G$ be an open Riemann surface and let f be a bounded continuous function on R. Let $\{R_n\}$ be an exhaustion of R such as ∂R_n consists of a finite number of mutually disjoint Jordan closed curves. We denote by H^{f}_{Rn} the harmonic function in R_n which is the solution of the Dirichlet problen with respect to the boundaɹy value function f. Then the f is decomposed as follows;

$$f = u_n + (f - u_n)$$

where $u_n = H^{f}_{Rn}$ on R_n, $=f$ on $R - R_n$. Let $\{u_{n_k}\}$ be a subsequence of $\{u_n\}$ such that u_{n_k} converges uniformly on any compact subdomain in R. Then

$$f = u_{n_k} + (f - u_{n_k}) = u + \varphi$$

where $u = \lim_{k\to\infty} u_{n_k}$ and $\varphi = f - u$. This decomposition is not always unique. Now let $\boldsymbol{F}$ be the family of bounded continuous functions on R for each of which the above decomposition is unique for any exhaustion $\{R_n\}$ of R. We denote by $\mathfrak{D}_H[f]$ the harmonic part of the decomposition of f. It is evident that $\boldsymbol{F}$ contains the superharmonic and subharmonic functions. We denote by $\bar{K}$ the subfamily of $\boldsymbol{F}$ consisting of φ such as $\mathfrak{D}_H[\varphi] = 0$. Now we shall prove that $\boldsymbol{F}$ is a ring with respect to the usual multiplication and sum.

Property 1. $f \in \boldsymbol{F} \Rightarrow \lambda f \in F$ $\quad$ (λ ; *real*)
Property 2. $f, g \in \boldsymbol{F} \Rightarrow f + g \in \boldsymbol{F}$
Property 3. $f, g \in \boldsymbol{F} \Rightarrow fg \in \boldsymbol{F}$

Properties 1 and 2 are trivial. We shall verify the property 3. Let $f \in \boldsymbol{F}$, $\varphi \in \bar{K}$ and $\varphi \geqq 0$ on R. At first, we show that $f \cdot \varphi \in \bar{K}$. Let $M = \sup_R |f|$, then

$$-M\varphi \leqq f\varphi \leqq M\varphi\,,$$

consequently for an exhaustion $\{R_n\}$

$$-MH^{\varphi}_{Rn} \leqq H^{f\varphi}_{Rn} \leqq MH^{\varphi}_{Rn}\,,$$

that is, $\lim H^{f\varphi}_{Rn} = 0$ and from this we know that $f\varphi \in \bar{K}$. Next, in case that φ ($\in \bar{K}$) is general, we note that φ^+ and φ^- belong to $\bar{K}$ respectively, where $\varphi^+ = \max(\varphi, 0)$ and $\varphi^- = \max(-\varphi, 0)$. Indeed,

$$H^{\varphi+}_{Rn} = H^{\varphi}_{Rn} \vee 0\,, \quad H^{\varphi-}_{Rn} = H^{-\varphi}_{Rn} \vee 0\,,$$

and that $H_{Rn}^{\varphi+} \wedge H_{Rn}^{\varphi-}=0$, $H_{Rn}^{\varphi}=H_{Rn}^{\varphi+}-H_{Rn}^{\varphi-}$. From this, we know that φ^{+}, φ^{-} belong to $\bar{K}$ respectively. Thus we can see that $f\varphi \in \bar{K}$ for any φ $(\ni \bar{K})$. Let f, g be any elements of $\boldsymbol{F}$. Then $f=u+\varphi$ $(u=\mathfrak{D}_H[f])$ and $g=v+\psi$ $(v=\mathfrak{D}_H[g])$, consequently $fg=uv+u\psi+v\varphi+\varphi\psi$. From this we know that $fg \in \boldsymbol{F}$ because of $uv \in \boldsymbol{F}$ $(4uv=(u+v)^2-(u-v)^2)$. Next we denote by $\|f\|=\sup_R |f|$ the norm of f. Then we have the following

PROPOSITION 3. 1. $\|f\| \geqq \|u\|$, *where* $u=\mathfrak{D}_H[f]$.

PROPOSITION 3. 2. *Let* $\{f_n\}(f_n \in \boldsymbol{F})$ *be a Cauchy sequence with respect to the above norm. Then there exists a function* $f(\in \boldsymbol{F})$ *such as* $\|f_n-f\| \to 0$ *as* $n \to \infty$.

Proof. We know easily that $f_n(p)$ converges uniformly a function f on R. Let $f_n=u^{(n)}+\varphi^{(n)}$ be the decomposition of f_n, then $f_n-f_m=(u^{(n)}-u^{(m)})+(\varphi^{(n)}-\varphi^{(m)})$ and $\|f_n-f_m\| \geqq \|u^{(n)}-u^{(m)}\|$ by proposition 3. 1. Therefore $u^{(n)}$ converges uniformly to an HB-function u. Thus we have

$$\varphi=\lim_{n\to\infty} \varphi^{(n)}=\lim_{n\to\infty}(f_n-u^{(n)})=f-u ,$$

and the convergence is uniform. We must prove that $\varphi \in \bar{K}$. Now we suppose hat $\varphi=v+\psi$ $(v \in HB)$. Then noting the $\mathfrak{D}_H[\varphi]=\mathfrak{D}_H[\varphi-\varphi^{(n)}]$, we have

$$\|v\| \leqq \|\varphi-\varphi^{(n)}\| \to 0 \quad (\text{as } n \to \infty)$$

that is, $v \equiv 0$. (q. e. d.)

Thus we know that $\boldsymbol{F}$ is a normed ring with respect to the above norm. By means of Gelfand's method, we obtain the compact Hausdorff space R_F^*. R is mapped topologically in R_F^* and and its image is open and dense in R_F^*. We denote by Γ_F the R_F^*-R and Γ_F is called the ideal boundary of R. All of the maximal ideals each of which contains the ideal $\bar{K}$ construct the harmonic boundary of R and is denoted by Δ_F $(\subset \Gamma_F)$.

REMARK. The ring $\boldsymbol{F}$ contains the bounded continuous subharmonic and superharmonic functions. From this, we see the similarity with Y. Kusunoki's ring [5]. I do not know whether they are identical or not.

LEMMA 3.1. *A bounded continuous subharmonic (superharmonic) function attains the maximum (minimum) on* Δ_F.

LEMMA 3.2. *Let* e_1, e_2 *be closed subsets in* Δ_F *such as* $e_1 \cap e_2 = \phi$. *Then there exists a positive bounded harmonic function* u *such af* $u=1$ *on* e_1, $=0$ *on* e_2.

LEMMA 3.3. *Let* u, v *be HB-functions on* R, *then* $u \vee v = \max(u, v)$ *and* $u \wedge v = \min(u, v)$ *on* Δ_F *respectively.*

We note that proposition 2.1 and lemma 2.2 in my former paper [*] are established this time. Now let γ be any subset in Δ_F, then we can give the outer harmonic measure μ_γ with respect to γ by the same method as in [*]. We shall see that all of the results in [*] are established, because the above three lemmas hold in R_F^*. The following lemma will be used in the succeeding sections.

LEMMA 3.4. *Let* γ *be a subset of* Δ_F *such as* $\mu_\gamma=0$. *Then* $\gamma \subset \overline{(\Delta_F-\gamma)}$. (cf. proposition 4.3 [*])

2. Martin boundary and harmonic boundary Δ_F. Let $R \notin 0_G$ and let Δ_1 be Martin boundary consisting of the minimal points. Let G be an open set in R. After Constantinescu-Cornea, we call that s $(\in \Delta_1)$ belongs to $\Delta_1(G)$, provided that $I_G K_s > 0$ and denote it by $s \in \Delta_1(G)$. Let $\dot{s} = \bigcap_{s \in \Delta_1(G)} \bar{G}$, where $\bar{G}$ is the closure of G in R_F^* and s is a minimal point in Δ_1. We call $\dot{s}$ the image of s and denote by $\dot{s} = \Psi(s)$. Let γ be a subset of Δ_1, then $\Psi(\gamma') = \{\dot{s}\,;\ \dot{s} = \Psi(s), s \in \gamma\}$ is called the image of γ in R_F^*. The $\dot{s}$ is not empty as is verified easily under the considerations of Folgesatz 2 [1] and the compactness of R_F^*.

PROPOSITION 3.3. *The image* $\dot{s}$ *is connected in* R_F^*.

Proof. We note that $I_G K_s > 0$ implies that there exists the only one component $\tilde{G}$ of G such as $I_{\tilde{G}} K_s > 0$ and $I_{G'} K_s = 0$ for any other component G' of G different from $\tilde{G}$ [1]. Consequently, in the following, we assume that G is connected. Now suppose that $\mathring{s}$ is disonnected. Then $\mathring{s}$ is decomposed to $\sigma_1 \cup \sigma_2$, where σ_1, σ_2 are both compact and $\sigma_1 \frown \sigma_2 = \phi$. Let f $(\in \boldsymbol{F})$ be a function on R such as $f > 0$ on σ_1, $=0$ on σ_2 and $\inf_{\sigma_1} f = c > 0$. Then $D = \left\{ p \in R\,;\ f(p) > \frac{c}{2} \right\}$ is open and $\bar{D} \supset \sigma_1$, $\bar{D} \frown \sigma_2 = \phi$. Now we show that for any domain G $(s \in \Delta_1(G))$,

$$\bar{G} \frown \overline{\partial D} \neq \phi\,. \quad (*)$$

Suppose that $\bar{G} \frown \overline{\partial D} = \phi$, then $H_1 = \bar{G} \frown \bar{D}$ and $H_2 = \bar{G} \frown \overline{(R - \bar{D})}$ are not empty respectively and $H_1 \frown H_2 = \phi$, consequently $\bar{G} = H_1 \cup H_2$. This is absurd, because G is connected. Thus we know that $\bar{G} \frown \overline{\partial D} \neq \phi$. Now let $\tilde{G} = \bar{G} \frown \overline{\partial D}$, then we can prove that $\bigwedge \tilde{G} \neq \phi$, where G varies in $\mathfrak{G} = \{G\,;\ s \in \Delta_1(G),\ G \text{ connected}\}$. Suppose that $\bigwedge \tilde{G} = \phi$, then $\bigvee \tilde{G}^c = \overline{\partial D}$, where $\tilde{G}^c$ is the complementary set of $\tilde{G}$ with respect to $\overline{\partial D}$. From the compactness of $\overline{\partial D}$

$$\overline{\partial D} = \tilde{G}_1^c \cup \tilde{G}_2^c \cup \tilde{G}_3^c \cup \cdots \cup \tilde{G}_n^c\,,$$

that is,

$$\bigwedge_{i=1}^{n} \tilde{G}_i = \phi\,, \quad (**)$$

where $\tilde{G}_i = \overline{\partial D} - \tilde{G}_i^c$. On the other hand, for any G_1, $G_2 (\in \mathfrak{G})$

$$I_{G_1 \frown G_2} K_s > 0$$

because of $I_{G_1} K_s > 0$ and $I_{G_2} K_s > 0$ [1], consequently there cxists a component G_0 of $G_1 \frown G_2$ such as $I_{G_0} K_s > 0$, that is, $\bar{G}_0 \frown \overline{\partial D} \neq \phi$ (cf. $(*)$). From this, we know that

$$\tilde{G}_i \frown \tilde{G}_j = (\bar{G}_i \frown \overline{\partial D}) \frown (\bar{G}_j \frown \overline{\partial D}) = (\bar{G}_i \frown \bar{G}_j) \frown \overline{\partial D} \supset \overline{(G_i \frown G_j)} \frown \overline{\partial D} \neq \phi$$

for any i and j, that is, $\bigwedge_{i=1}^{n} \tilde{G}_i \neq \phi$. This contrudicts with $(**)$. Therefore $\mathring{s} \frown \overline{\partial D} \neq \phi$. This is absurd, because of $\mathring{s} \frown \overline{\partial D} = \phi$.

LEMMA 3.5. *Let D be a non-compact subregion in R and let the relative boundary ∂D consists of the regular points with respect to the Dirichlet problem. Then $I_D K_s > 0$ if and only if $\mathring{s} \subset \bar{D} - \overline{\partial D}$.*

Proof. Let $I_D K_s > 0$, then $w(p)$ $(= I_D K_s)$ is minimal in D and it vanishes continuously on ∂D. Let E be a subregion of D such as $E=\{p\in D\,;\ w(p)>c>0\}$. Since $I_E K_s>0$ and $\bar{E}\frown\overline{\partial D}=\phi$, we know that $\dot{s}\subset\bar{D}-\overline{\partial D}$, where $\bar{E}$, $\bar{D}$, $\overline{\partial D}$ are the closure in R_F^*. Conversely, let $\dot{s}\subset\bar{D}-\overline{\partial D}$, then

$$(\bigcap_{s\in\Delta_1(G)}\bar{G})\frown\overline{\partial D}=\phi\,,$$

therefore

$$(\bigcup\bar{G}^c)\smile\overline{\partial D}^c=R_F^*,$$

that is, $(\bigcap_{i=1}^{n}\bar{G}_i)\frown\overline{\partial D}=\phi$, where $\bar{G}^c=R_F^*-\bar{G}$ and $\overline{\partial D}^c=R_F^*-\overline{\partial D}$. From this, we know that

$$\overline{(\bigcap_{i=1}^{n}G_i)}\frown\overline{\partial D}=\phi\,.$$

We note that $I_{G_i}K_s>0$ for each G_i, consequently $I_{\tilde{G}}K_s>0$ where $\tilde{G}=\bigcap_{i=1}^{n}G_i$. Let G_0 be a component of $\tilde{G}$ such as $I_{G_0}K_s>0$, then $G_0\subset D$ because of $\bar{G}_0\frown\overline{\partial D}=\phi$ and $\dot{s}\subset\bar{D}-\overline{\partial D}$. From this, we know that $I_D K_s>0$. (q. e. d.)

PROPOSITION 3. 4. *Let γ be a X-measurable subset of Δ_1 with positive measure and let $\omega_\gamma(p)$ be the harmonic measure* [1]. *Let $D_n=\left\{p\in R\,;\ \omega_\gamma(p)>1-\dfrac{1}{n}\right\}$ $(n=2, 3, \cdots)$. Then $\bigcap^{\infty}\bar{D}_n$ contains the image of γ except for a null-set of X-measure zero.*

Proof. By means of Constantinescu-Cornea [1],

$$\omega_\gamma(p)=\int_{\Delta_1}K_s\theta(s)\,dX(s)\qquad \begin{aligned}\theta(s)&=1\quad s\in\gamma\\ &=0\quad s\in\Delta_1-\gamma\,,\end{aligned}$$

and

$$A_\alpha\subset\Delta_1(G_\alpha)\quad [X]$$

where $A_\alpha=\{s\in\Delta_1\,;\ \theta(s)>\alpha\}$ and $G_\alpha=\{p\in R\,;\ \omega_\gamma(p)>\alpha\}$. From this and lemma 3. 5, we know that $\bar{G}_\alpha\supset\Psi(\gamma)$ $[X]$, that is, $\bar{G}_\alpha$ contains the image of γ except for a null-set in γ. Let $\alpha=1-\dfrac{1}{n}$, then $\bigcap^{\infty}\bar{D}_n\supset\Psi(\gamma)$ $[X]$. (q. e. d.)

PROPOSITION 3. 5. *The harmonic boundary Δ_F is contained in the closure of $\Psi(\Delta_1)$.*

Proof. Suppose that the harmonic measure of $\overline{\Psi(\Delta_1)}\cap\Delta_F$ is constant 1. Then that $\Delta_F\subset\overline{\Psi(\Delta_1)}$ is valid (cf. lemma 3. 4). Suppose that the harmonic measure of $\overline{\Psi(\Delta_1)}\cap\Delta_F$ is not constant. Then $\Delta_F-\overline{\Psi(\Delta_1)}$ contains a subset σ simultaneously open and closed. Let $\omega_\sigma(p)$ be the harmonic measure of σ. Since

$$\omega_\sigma(p) = \int_A K_s d\chi(s)$$

for a suitable set $A(\subset\Delta_1)$, $\Psi(A)\subset\bigcap^{\infty}\bar{D}_n$ $[\chi]$ (cf. prop. 3. 4). Let

$$\tilde{\sigma} = \{\Psi(s)\,;\ \Psi(s)\subset\bigcap^{\infty}\bar{D}_n,\ s\in A\}\ ,$$

then $\omega_\sigma=1$ at every point of $\tilde{\sigma}$. On account of the assumption, $\overline{\Psi(\Delta_1)}\cap\sigma=\phi$, $\bar{\tilde{\sigma}}\cap\sigma=\phi$ (where σ is open and closed). Next, noting that ω_σ vanishes on $\Delta_F-\sigma$, while ω_σ is 1 on $\tilde{\sigma}$, we know that $\bar{\tilde{\sigma}}\cap(\Delta_F-\sigma)=\phi$. Thus $\bar{\tilde{\sigma}}\cap\Delta_F=\phi$, from which we can conclude that A is of χ-measure zero. Indeed, on account of $\bar{\tilde{\sigma}}\cap\Delta_F=\phi$, there exists a non-negative continuous function f on R_F^* such as $f\geqq c$ (>0) on $\tilde{\sigma}$, $=0$ on Δ_F. Let D be an open set in R such as $D=\left\{p\in R\,;\ f(p)>\dfrac{c}{2}\right\}$. Then there exists an open set $\tilde{D}$ such that $\tilde{D}\supset D\cup\partial D$, $\bar{\tilde{D}}\cap\Delta_F=\phi$ and that $\partial\tilde{D}$ consists of regular points. It is clear that $\tilde{D}$ belongs to the class SO_{HB}. Consequently the set $\gamma=\{s\in\Delta_1\,;\ I_{\tilde{D}}K_s>0\}$ is of χ-measure zero by [1]. On the contrary, the set A is different from γ by χ-measure zero by means of lemma 3. 4. This is absurd, because A is of positive χ-measure. Thus we conclude that $\Delta_F\subset\overline{\Psi(\Delta_1)}$. (q. e. d.)

PROPOSITION 3. 6. *Let γ be a subset of the harmonic boundary Δ_F of R with positive measure and let $\omega_\gamma(p)$ $(p\in R)$ be the harmonic measure of γ. Let $\tilde{\gamma}$ be the subset of $\Psi(\Delta_1)$ such that ω_γ attains 1 on $\tilde{\gamma}$, that is,*

$$\tilde{\gamma} = \{\Psi(s)\,;\omega_\gamma = 1 \ \mathit{on}\ \Psi(s),\ s\in\Delta_1\}\ .$$

Then the γ is contained in the closure of $\tilde{\gamma}$ except for a set of the harmonic measure zero.

Proof. Let $\bar{\tilde{\gamma}}$ be the closure of $\tilde{\gamma}$. Suppose that $\gamma-\bar{\tilde{\gamma}}$ is of positive measure. Then there exists a simultaneously open and closed subset of $\gamma-\tilde{\gamma}$, We denote it by σ. Then there exists a subset A of Δ_1 such as $\omega_A(p)=\omega_\sigma(p)$. According to Constantinescu-Cornea [1] and Proposition 3.4, $\Psi(A)\subset\bigcap^{\infty}\bar{E}_n$ [χ], where $E_n=\left\{p\in R;\ \omega_\sigma(p)>1-\frac{1}{n}\right\}$. Now the set

$$\eta = \{\Psi(s);\ \omega_\sigma=1 \ on\ \Psi(s),\ s\in A\}$$

is a subset of $\bar{\tilde{\gamma}}$ and $\tilde{\gamma}\cap\sigma=\phi$ by the assumption. Consequently $\bar{\eta}\cap\sigma=\phi$ and further $\bar{\eta}\cap\Delta_F=\phi$, because $\omega_\sigma=1$ on σ, $=0$ on $\Delta-\sigma$. Thus there exists a function φ continuous on R_F^* such as $\varphi=1$ on η, $=0$ on Δ_F. Let D_1, D_2 be open sets snch as

$$D_1 = \left\{p\in R;\ \varphi(p)>\frac{1}{2}\right\}$$
$$D_2 = \left\{p\in R;\ \varphi(p)>\frac{1}{3}\right\}.$$

Since $\varphi(p)$ is continuous on R_F^*, $D_2\supset D_1\cup\partial D_1$. There exists an open set D such as $D_1\subset D\subset D_2$ and that ∂D consists of regular points of the Dirichlet problem with respect to D. It is clear that $\bar{D}-\overline{\partial D}\supset\eta$ and $\bar{D}\cap\Delta_F=\phi$. The former implies that $D\notin SO_{HB}$ by lemma 3.4 and Constantinescu-Cornea [1], and the latter implies that $D\in SO_{HB}$. This is absurd. (q. e. d.)

NOTE. I conjecture that the following proposition will be hold; Let A be a χ-measurable subset of Δ_1, and let $\Psi(A)$ be the image in R_F^*. Then

$$\omega_A = \int_A K_s d\chi(s)$$

coincides with the harmonic measure of $\overline{\Psi(A)}\cap\Delta_F$.

LEMMA 3.6. *Let $S(p)$ be a lower semi-continuous, bounded superhamonic function. Then $\underline{S}(q^*)$ $(\varliminf_{p\to q^*} S(p))$ is continuous on Δ_F and*

$$G.\ H.\ M.\ \underline{S}(p) = \int_{\Delta_F}\underline{S}(q^*)\,d\mu(q^*\,;\ p)\,.$$

Proof. Let $u(p)=G.\ H.\ M.\ S(p)$, then

$$U(p) = \int_{\Delta_F} \underline{S}(p^*)\, d\mu(p^*\,;\, p) \geqq u(p)\,,$$

and $U(p^*) = \underline{S}(p^*)$ except for a null-set, that is, the harmonic measure of

$$\mathfrak{N} = \{p^* \in \Delta_F;\ U(p^*) \neq S(p^*)\}$$

is zero. Now we suppose that $U(p) \neq u(p)$. Let $\tilde{\sigma} = \{p^* \in \Delta_F;\ U(p^*) > u(p^*)\}$, then $\tilde{\sigma}$ is open in Δ_F. Consequently $\tilde{\sigma}$ contains a simultaneously open and closed subset σ with respect to Δ_F. Since $\mathfrak{N}$ is a null-set, there exists a simultaneously open and closed subset γ $(\subset \Delta_F)$ such that $\sigma - \gamma$ is of positive harmonic measure and at each point of $\sigma - \gamma$

$$U(p^*) = \underline{S}(p^*) > u(p^*)\,.$$

Therefore

$$\inf_{\sigma-\gamma} (\underline{S}(p^*) - u(p^*)) = \tilde{c} > 0\,.$$

Let $\omega_{\sigma-\gamma}(p)$ be the harmonic measure of $\sigma-\gamma$, then we can prove that

$$S(p) \geqq u(p) + c\omega_{\sigma-\gamma}(p) \quad (0 < c < \tilde{c})\,.$$

To prove this, suppose that $u(p) + c\omega_{\sigma-\gamma}(p) - S(p) > 0$ at some point in R. Then the set $D = \{p \in R:\ u(p) + c\omega_{\sigma-\gamma}(p) - S(p) > \varepsilon\}$ is not empty for a suitable number $\varepsilon > 0$. It is clear that $\bar{D} \cap \Delta_F = \phi$. Indeed, for any $p^* \in \Delta_F \subset \bar{D}$ we have

$$\begin{aligned}\varliminf_{\substack{p\to p^* \\ p\in D}} (S(p) - u(p) - c\omega_{\sigma-\gamma}(p)) &\geqq \varliminf_{\substack{p\to p^* \\ (p\in R)}} (S(p) - (u(p) + c\omega_{\sigma-\gamma}(p))) \\ &= \underline{S}(p^*) - u(p^*) - c\omega_{\sigma-\gamma}(p^*) \geqq 0\,.\end{aligned}$$

This is absurd, that is, $\bar{D} \cap \Delta_F = \phi$. Hence there exists a non-negative bounded continuous function f on R_F^* such as

$$\begin{aligned} f &> 0 \quad \text{on } \bar{D} \\ &= 0 \quad \text{on } \Delta_F\,. \end{aligned}$$

The function f belongs to $\bar{K}$ since $f = 0$ on Δ_F. Then

$$\lambda f(p) \geqq \max\,[u(p) + c\omega_{\sigma-\gamma}(p) - S(p) - \varepsilon,\ 0] \quad (p \in R)$$

for a suitable positive number λ. Consequently $(u+c\omega_{\sigma-\gamma}-S-\mathcal{E})\vee 0=0$, because $\mathfrak{D}_H[\lambda f]=0$. This is absurd. Thus we know that $S(p)\geqq u(p)+c\omega_{\sigma-\gamma}(p)$ at every point in R. This is absurd, because of $u=G.H.M.\ S(p)$. Consequently

$$U(p) = \int_{\Delta_F} \underline{S}(p^*)\, d\mu(p^* ;\ p) = u(p)\,.$$

From this, we know that $\underline{S}(p^*)$ is continuous on Δ_F since $\underline{S}(p^*) = u(p^*)$ (a. e.) and that $\underline{S}(p^*)$ is lower semi-continuous on Δ_F. (q. e. d.)

Corollary 1. *Let $S(p)$ be a lower semi-continuous superharmonic function. Then the quasi-bounded component of G. H. M. $S(p)$ is equal to*

$$\int_{\Delta_F} \underline{S}(p^*)\, d\mu(p^* ;\ p)$$

provided that $S(p)$ is bounded below.

Corollary 2. *Let $S(p)$ be a lower semi-continuous positive superharmonic function. Then $S(p)\equiv +\infty$, provided that the set $e_\infty=\{p^*\in \Delta_F ;\ \underline{S}(p^*)=+\infty\}$ is of positive harmonic measure.*

3. On the Lindelöfian mapping. Let R. R' be open Riemann surfaces and let R be of hyperbolic type, while R' be unrestricted. Let f be an analytic mapping of R into R'. Let p^* be a point of the harmonic boundary of R and $\mathfrak{U}=\{U(p^*)\}$ be the family of open sets in R_F^* each of which contains the p^*. Now we define the image of the p^* by the mapping f. At first, we assume that R' is of hyperbolic type. We define the set in $R_F'^*$ such as

$$M_f(p^*) = \bigcap_{U(p^*)\in \mathfrak{U}} \overline{f(U(p^*))}\,,$$

where $f(U(p^*))$ is the image of $U\cap R$ by f and $\overline{f(U(p^*))}$ is the closure of $f(U(p^*))$ in $R_F'^*$. $M_f(p^*)$ is not empty, because $(U_1\cap R)\cap(U_2\cap R)\neq\phi$ for any $U_1,\ U_2\in \mathfrak{U}$. We call $M_f(p^*)$ the image of p^*. From the fact that the set $\bigcap_{U\in\mathfrak{U}} U(p^*)$ consists of the single point p^*, we know that $u'\in HB(R')$ converges as $p\to p^*$

along the filter $\mathfrak{U}$. Consequently every bounded harmonic function on R' is constant on $M_f(p^*)$.

PROPOSITION 3.6. *$M_f(p^*)$ contains the only one harmonic boundary point of R', provided that $M_f(p^*) \cap \Delta'_F \neq \phi$.*

PROPOSITION 3.7. *$M_f(p^*)$ consists of a single point, provided that $M_f(p^*) \cap R' \neq \phi$.*

Proof. At first, we show that $M_f(p^*) \cap R'$ is connected. Suppose that $M_f(p^*) \cap R'$ is disconnected. Then there exists disjoint closed sets A and B such as $A \cup B = M_f(p^*) \cap R'$. Let G be a Jordan domain in R' whose relative boundary is smooth and $A \cap G \neq \phi$, while $B \cap G = \phi$. Let $S'(p')$ be the superharmonic function in R' such that $S'(p') = 1$ on G and $S'(p') = H^1_{R'-G}$ on $R' - \bar{G}$. Then $S' \circ f$ is a continuous superharmonic function in R, consequently $S' \circ f$ converges along the filter $\mathfrak{U}$. This is absurd, that is, $M_f(p^*) \cap R'$ is connected. By the same manner, we know that $M_f(p^*)$ consists of a single point in R'. (q. e. d.)

In the following, we shall study the mapping of type-Bl under the condition $R' \notin 0_G$. Now let $\mathfrak{G}_R(p\,;\,q)$, $\mathfrak{G}_{R'}(p'\,;\,q')$ be the Green functions of R, R' respectively. According to M. Heins, $\mathfrak{G}_{R'}(f(p)\,;\,q') = \sum_{f(r)=q'} n(r)\mathfrak{G}_R(p\,;\,r) + u_{q'}(p)$ and $u_{q'}(p)$ is the greatest harmonic minorant of $\mathfrak{G}_{R'}(f(p)\,;\,q)$. The $u_{q'}(p)$ has the Parreau's decomposition: $u_{q'}(p) = v_{q'}(p) + w_{q'}(p)$, where $v_{q'}$ is the quasi-bounded component and $w_{q'}$ is the singular component. Then we have the following

PROPOSITION 3.8. $v_{q'}(p) = \int_{\Delta_F} \varliminf_{\tilde{p} \to p^*} \mathfrak{G}_{R'}(f(\tilde{p})\,;\,q')\,d\mu(p^*\,;\,p)$

$$(p \in R).$$

THEOREM 3.1. *The analytic mapping f of R into R' is of type-Bl if and only if $M_f(p^*) \subset (\mathfrak{G}_{R'})_0$ for every p^* $(\in \Delta_F)$, where $(\mathfrak{G}_{R'})_0$ is the subset of the ideal boundary Γ'_F of R' on which Green function of R' vanishes.*

Proof. We note that $(\mathfrak{G}_{R'})_0$ is independent of the singular point q' of $\mathfrak{G}_{R'}(p'\,;\,q')$. From this and proposition 3.8, it is evident that f is of type-Bl provided that $M_f(p^*) \subset (\mathfrak{G}_{R'})_0$. Conversely let f be of type-Bl at some points $q'(\in R')$. Then $v_{q'}=0$, consequently $\overline{\lim_{p\to p^*}} \mathfrak{G}_{R'}(f(p)\,;\, q')$ must vanish at every points of Δ_F because it is continuous on Δ_F by lemma 3.5. On the other hand, we know easily that $\mathfrak{G}_{R'}(f(p)\,;\, q')$ $(p \in R)$ is extended continuously (admitting $+\infty$) onto R_F^*. From this we know that $M_f(p^*) \subset (\mathfrak{G}_{R'})_0$. (q. e. d.)

THEOREM 3.2. *The closure of* $\bigcup_{p^* \in \Delta_F} M_f(p^*)$ *contains the harmonic boundary of* R', *that is,* $\Delta'_F \subset \overline{\bigcup M_f(p^*)}$.

Proof. We denote by γ the set $\bigcup M_f(p^*)$. Suppose that $\Delta'_F - \bar{\gamma}$ is of positive harmonic measure. Then it contains a simultaneously open and closed subset σ. It is evident that $\sigma=\bar{\sigma}$, $\bar{\sigma}\cap\bar{\gamma}=\phi$. Let $\omega_{R'}(p'\,;\,\sigma)$ $(p'\in R')$ be the harmonic measure of σ. Then $\omega_{R'}(f(p)\,;\,\sigma)$ is an HB-function on R. Let $\tilde{\sigma}$ be the subset of Δ_F such as $\tilde{\sigma}=\{p^*\in\Delta_F\,;\,(\omega_{R'}\circ f)(p^*)=1\}$. The $\tilde{\sigma}$ is of the harmonic measure positive, more exactly, $\omega_{R'}\circ f$ is the harmonic measure of $\tilde{\sigma}$. Indeed, $(1-\omega_{R'}\circ f))\wedge\omega_R\circ f=0$. Let $M_{\tilde{\sigma}}$ be the image of $\tilde{\sigma}$, that is ,$M_{\tilde{\sigma}}=\{M_f(p^*)\,;\,p^*\in\tilde{\sigma}\}$. Then the closure $\bar{M}_{\tilde{\sigma}}$ of $M_{\tilde{\sigma}}$ does not meet σ by the assumption. Furthermore $\bar{M}_{\tilde{\sigma}}$ does not meet Δ'_F, because $\omega_{R'}(p'\,;\,\sigma)$ attains 1 at every points of $M_{\tilde{\sigma}}$. This is absurd by means of the following lemma.

LEMMA 3.7. *Let f be of type-Bl. Let Δ_F, Δ'_F be the harmonic boundaries of R and R' respectively. Let γ be a subset of Δ_F whose harmonic measure is positive, and let M_γ be the set such as $M_\gamma=\{M_f(p^*)\,;\,p^*\in\gamma\}$. Then the closure of M_γ (with respect to $R_F'^*$) meets Δ'_F.*

Proof. Suppose that $\bar{M}_\gamma$ does not meet Δ'_F. Then there exists a non-negative function φ' continuous on $R_F'^*$ such as $\varphi'\geqq c$ (>0) on $\bar{M}_\gamma$, $=0$ on Δ'_F, Let D' be an open set in R' such as $D'=\left\{p'\in R'\,;\,\varphi'(p')>\frac{c}{2}\right\}$. Then $\bar{D}'\cap\Delta'_F=\phi$ and $\bar{D}'\supset\bar{M}_\gamma$. We can construct the open set D'' such that $\partial D''$ consists of the Jordan curves and $D''\supset D'\cup\partial D'$, furthermore $\bar{D}''\cap\Delta'_F=\phi$. Then there exists a positive bounded superharmonic function $S'(p')$ continuous on R' such as

$$S'(p') = 0 \quad \text{on } \Delta'_F$$
$$= 1 \quad \text{on } D'' \cup \partial D''.$$

Now $S(p) = S' \circ f(p)$ is superharmonic and continuous on R. It is clear that $S(p)$ is positive on γ. Consequently $u(p) = G.H.M.\ S(p)$ is positive on R by lemma 3.6. Let $\bar{u}(p') = \sup_{f(p)=p'} u(p)$, then $\bar{u}(p') \leqq S'(p')$ on R'. From this, we know that

$$0 < E_f u \leqq S'(p'),$$

where $E_f u$ is harmonic on R ([1], Satz 5). This is absurd, that is, $\bar{M}_\gamma \cap \Delta' \neq \phi$. (q. e. d.)

From lemma 3.7 and theorem 3.2, we know the following

THEOREM (M. Heins [4]) *If f is a conformal map of type-Bl from R into R' and w is a singular positive harmonic function on R', then $w \circ f$ is a singular positive harmonic function.*

Noting that the essential part in the proof of lemma 3.7 is that $M_f(p^*) \cap R' = \phi$ for every $p^* \in \Delta_F$, we have the following

THEOREM (M. Heins [4]) *Let f denote a conformal map of R into R' and let u denote a positive harmonic function on R'. If $u \circ f$ is singular on R, then u is singular on R' and f is of type-Bl.*

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