J. Math. Kyoto Univ.
2-2 (1963) 209-226.

On deformations of cross-sections of a differentiable fibre bundle

By

Toshimasa YAGYU

(Received Jan. 19, 1963)

Introduction

It is well-known that geometric structures on a topological space can be defined mostly through the notion of (B, Γ)-structure, where Γ is a pseudogroup of local homeomorphisms of a topological space B. Particularly for a differentiable manifold, when we take the euclidean space R^n as B and some pseudogroup Γ_d of local differentiable transformations of R^n as Γ, (R^n, Γ_d)-structures are objects of differential geometry. On the other hand, there are also structures defined by cross-sections of differentiable bundles over a differentiable manifold such as Riemannian metric structures. But they are not considered generally as (R^n, Γ_d)-structures. However if we take the space of germs of cross-sections of the product bundle over R^n as B and a suitable pseudogroup on it as Γ, we can regard the structures by cross-sections of the differentiable fibre bundle as (B, Γ)-structures. (§ 5.)

D. C. Spencer ([10]) has pointed out without proof that the set of germs of m-parameter deformations of a (B, Γ)-structure may be identified with a 1-cohomology set with coefficients on some sheaf, from the theory of A. Haefliger ([8]). Hence, we can apply this theory to deformations of a cross-section and we have a theorem on deformations of a Riemannian manifold as an example.

We give a direct formulation and proof of Spencer's proposition without such a objectionable condition for our application, that B is paracompact. Though our result (Theorem 3) can be proved more directly, we treat it from a view point of a general theory of deformations of (B, Γ)-structures, (§§ 1–4) and its application. (§§ 5–7)

§ 1. Differentiable (B, Γ)-structures

Let B be a topological space with a differentiable structure, i.e. there exists a neighborhood U of each point of B and a homeomorphism φ_U from U to an open set of n-dimensional euclidean space R^n such that $\varphi_U \cdot \varphi_V^{-1}$ is a bidifferentiable transformation on $\varphi_V(U \cap V)$ for $U \cap V \neq \Phi$. (B is not necessarily separable or paracompact.)

Let Γ be some pseudogroup of local bidifferentiable transformations of B and let M be a differentiable manifold. For each open set U of M, we set

$B(U) = \{\varphi$; a diffeomorphism, in the sense of differentiable structures of M and B, from U onto the domain of an element of $\Gamma\}$.

We define that $\varphi, \psi \in B(U)$ are equivalent if and only if $\varphi \cdot \psi^{-1} \in \Gamma$ and we denote the set of the equivalence classes of $B(U)$ by $B/\Gamma(U)$. For $U \supset U'$, the restriction induces a correspondence $r^U_{U'}: B(U) \to B(U')$ such that $r^{U'}_{U''} \cdot r^U_{U'} = r^U_{U''}$ for $U \supset U' \supset U''$ and $(r^U_{U'}\varphi)(r^U_{U'}\psi)^{-1} \in \Gamma$ if $\psi \cdot \varphi^{-1} \in \Gamma$. Therefore, there exists a correspondence $r'^U_{U'}: B/\Gamma(U) \to B/\Gamma(U')$ such that $r'^{U'}_{U''} \cdot r'^U_{U'} = r'^U_{U''}$ for $U \supset U' \supset U''$. and then $\{B/\Gamma(U)\}$ is a presheaf over M and induces a sheaf $[B/\Gamma]_M$ over M.

Definition. *A differentiable (B, Γ)-structure on M is an element s of $H^0(M, [B/\Gamma]_M)$, which is a section of $[B/\Gamma]_M$ over M.*

For a differentiable (B, Γ)-structure s, there exist a suitable open neighborhood U of each point x of M and $s_U \in B/\Gamma(U)$ such that the germ of s_U at x is $s(x)$, and we have $\varphi_U \in B(U)$ such that $p_U(\varphi_U) = s_U$ where p_U is the projection $B(U) \to B/\Gamma(U)$. U and φ_U are called a *coordinate neighborhood* of s and *coordinate map* of s, respectively. For an open covering $\{U_j, j \in J\}$ of M by coordinate neighborhoods of s and coordinate maps $\varphi_j \in B(U_j)$, $\{U_j, \varphi_j, j \in J\}$ is called a *coordinate system* of s. This definition ensures that each element of $H^0(M, [B/\Gamma]_M)$ has necessarily a coordinate system. If $\{U'_k, k\in K\}$ is a refinement of $\{U_i, j\in J\}$ (with the index injection of the refinement $\kappa; K\to J$), then $\{U'_k, \varphi_{\kappa(k)}|U'_k\}$ is also a coordinate system of s. If $\{U_j, \varphi_j, j\in J\}$ and $\{U'_k, \varphi'_k, k\in K\}$ are coordinate systems of the same element of $H^0(M, [B/\Gamma]_M)$, there exists a refinement $\{U''_l, l\in L\}$ of $\{U_j\}$ and $\{U'_k\}$ (with the index injections of the refinement $\iota: L\to J$, $\kappa: L\to K$) such that $\varphi_{\iota(l)}|U''_l$ and $\varphi'_{\kappa(l)}|U''_l$ are equivalent in $B(U''_l)$.

Lemma 1. *Let $B'(U)$ be a subset of $B(U)$ for each open set U of M such that $r^U_{U'}(B'(U))\subset B'(U')$ if $U\supset U'$, and let Γ' be a sub-pseudogroup of Γ such that $\varphi\cdot\psi^{-1}\in\Gamma'$ if $\varphi, \psi\in B'(U)$ and $\varphi\cdot\psi^{-1}\in\Gamma$. Then $[B'/\Gamma']_M$ is a sub-sheaf of $[B/\Gamma]_M$ and so $H^0(M, [B'/\Gamma']_M)$ can be identified with a subset of $H^0(M, [B/\Gamma])_M)$.*

Proof. If $\varphi, \psi\in B'(U)$ are equivalent in $B(U)$, they are equivalent in $B'(U)$, and then $B'/\Gamma'(U)\subset B/\Gamma(U)$. Since $r^U_{U'}(B'(U))\subset B'(U')$, $r'^U_{U'}: B/\Gamma(U)\to B/\Gamma(U')$ maps $B'/\Gamma'(U)$ into $B'/\Gamma'(U')$. Therefore, $\{B'/\Gamma'(U)\}$ is a sub-presheaf of $\{B/\Gamma(U)\}$ and so $[B'/\Gamma']_M$ is a sub-sheaf of $[B/\Gamma]_M$.

When W is an open set of M, we define similarly a coordinate system of a section $s|W$ of $[B/\Gamma]_M$ over W.

Lemma 2. *Let η be a diffeomorphism of W onto an open set of M. Then η induces a map $\bar{\eta}$ of sections over $\eta(W)$ into sections over W.*

Proof. If $\{U_j, \varphi_j\}$ is a coordinate system of a section $s|\eta(W)$ over $\eta(W)$, $\varphi_j\cdot\eta: \eta^{-1}(U_j)\to B$ is an element of $B(\eta^{-1}(U_j))$ and $(\varphi_i\eta)\cdot(\varphi_j\eta)^{-1}=\varphi_i\cdot\varphi_j^{-1}\in\Gamma$ for $U_i\cap U_j(\neq\Phi)$. Therefore $\{\varphi_j\cdot\eta, \eta^{-1}(U_j)\}$ is a coordinate system of a section over W which is denoted by $\bar{\eta}(s|W)$.

Remark. If η is a diffeomorphism of W into M such that $\varphi\cdot\eta\in B'(U)\subset B(U)$ for any $\varphi\in B'(\eta(U))\subset B(\eta(U))$ and any open set U included in W, then Lemma 2 ensures that η induces a map $\bar{\eta}$ of sections of $[B'/\Gamma']_M$ over $\eta(W)$ into sections of $[B'/\Gamma']_M$ over W.

§ 2. Differentiable deformations of (B, Γ)-structures

Let I be the open interval $(-1, 1)$ of real numbers. The product space $B\times I$ is naturally a topological space with a differentiable structure. Let $\Gamma\times I$ denote the pseudogroup of local bidifferentiable transformations γ of $B\times I$ such that

1°. $t=\gamma_t(x, t)$,

2°. For every fixed t, the local bidifferentiable transformation $\gamma_x(x, t)$ of B is an element of the given pseudogroup Γ of B.

where $\gamma(x, t)=(\gamma_x(x, t), \gamma_t(x, t))$, $x\in B$, $t\in(-1, 1)$.

For each open set U of $M\times I$, we set

$B\times I(U)=\{\varphi$; diffeomorphisms of U onto domains of elements of $\Gamma\times I$ such that $\varphi_t(x, t)$ are independent of x where $\varphi(x, t)=(\varphi_x(x, t), \varphi_t(x, t))$ and $(x, t)\in U\}$.

φ, $\psi\in B\times I(U)$ are said to be equivalent if and only if $\varphi\cdot\psi^{-1}\in\Gamma\times I$. We set $B\times I/\Gamma\times I(U)=\{$equivalence classes of $B\times I(U)\}$. Similarly as in § 1, $\{B\times I/\Gamma\times I(U)\}$ is a presheaf over $M\times I$, and induces a sheaf $[B\times I/\Gamma\times I]_{M\times I}$ over $M\times I$.

Let $\{U_j, \varphi_j, j\in J\}$ be a coordinate system of $s\in H^0(M\times I, [B\times I/\Gamma\times I]_{M\times I})$. By the properties of $B\times I(U)$ and $\Gamma\times I$, t' of $(y', t')=\varphi_j(x, 0)$ is a constant for any $j\in J$ and moreover depends only on s. We call t' the *parameter* of s. We set

$$D=\{s\in H^0(M\times I, [B\times I/\Gamma\times I]_{M\times I}) \text{ whose parameter is zero}\}$$

Let $\{U_j, \varphi_j, j\in J\}$ denote a coordinate system of an element s of D. Setting $V_j=U_j\cap(M\times 0)$ and identifying $M\times 0$ with M, we have $\varphi_j|V_j\in B(V_j)$ and $\varphi_i\cdot\varphi_j^{-1}|\varphi_j(V_i\cap V_j(\neq\Phi))\in\Gamma$ since $\varphi_j(V_j)\subset B\times 0$. Therefore $\{V_j, \varphi_j, j\in J\}$ is a coordinate system of an element s_0 of $H^0(M, [B/\Gamma]_M)$ i.e. a differentiable (B, Γ)-structure. Obviously s_0 depends only on the element s of D and so we have a map $i: D\to H^0(M, [B/\Gamma]_M)$.

Lemma 3. *The map i maps D onto $H^0(M, [B/\Gamma]_M)$.*

Proof. Let $\{V_\alpha, \psi_\alpha, \alpha\in A\}$ be a coordinate system of an element s_0 of $H^0(M, [B/\Gamma]_M)$ and φ_α denote a map of $V_\alpha \times I$ into $B \times I$ defined by

$$\varphi_\alpha(x, t) = (\psi_\alpha(x), t), \qquad (x \in V_\alpha, t \in I).$$

Hence,

$$(\varphi_\alpha | (V_\alpha \times I) \cap (V_\beta \times I))(\varphi_\beta | (V_\alpha \times I) \cap (V_\beta \times I))^{-1} \in \Gamma \times I$$

where $V_\alpha \cap V_\beta \neq \Phi$, and thus $\{\varphi_\alpha, V_\alpha \times I, \alpha \in A\}$ is a coordinate system of an element of $H^0(M \times I, [B \times I/\Gamma \times I]_{M \times I})$ and determines d of D. Since $\varphi_\alpha | V_\alpha = \psi_\alpha$, then $i(d) = s$.

Definition. *Differentiable deformations of a given differentiable (B, Γ)-structure s_0 are elements d of D such as $i(d) = s_0$. We denote their set by $D(s_0)$, i.e. $D(s_0) = i^{-1}(s_0)$.*

Let d_ε be a section of $[B \times I/\Gamma \times I]_{M \times I}$ over $M \times (-\varepsilon, \varepsilon)$ where ε is an arbitrary positive number (< 1). d_ε also determines an element of $H^0(M, [B/\Gamma]_M)$.

Lemma 4. *When d_ε determines an element s_0, d_ε can be extended to a section of $[B \times I/\Gamma \times I]_{M \times I}$ over $M \times I$ which is an element of $D(s_0)$.*

Proof. It is well-known that there exists a diffeomorphism η of $M \times I$ on $M \times (-\varepsilon, \varepsilon)$ such that $\eta | M \times (-\varepsilon, \varepsilon) = \text{identity}$, $\eta_x(x, t)$ is independent of t and $\eta_t(x, t)$ is independent of x, where $\eta(x, t) = (\eta_x(x, t), \eta_t(x, t))$. If we apply Lemma 2 and Remark of §1, to $M \times I$ and $\Gamma \times I$, then η induces a map $\bar{\eta}$ of sections over $M \times (-\varepsilon, \varepsilon)$ into sections over $M \times I$, since $\varphi \cdot \eta \in B \times I(U)$ for each open set U of $M \times I$ and for $\varphi \in B \times I(\eta(U))$. Then $\bar{\eta}(d_\varepsilon) \in H^0(M \times I, [B \times I/\Gamma \times I]_{M \times I})$ and moreover $\bar{\eta}(d_\varepsilon) \in D(s_0)$ since $\bar{\eta}(d_\varepsilon) | M \times (-\varepsilon, \varepsilon) = d_\varepsilon$.

Henceforth, *we suppose that M is compact.*

A diffeomorphism φ from an open set V of M to an open set of B is said a *regular map* on V for a differentiable (B, Γ)-structure s_0 if $(\varphi_j | V_j \cap V)(\varphi | V_j \cap V)^{-1} \in \Gamma$ for a coordinate system $\{V_j, \varphi_j\}$ of s_0 and for any j such as $V_j \cap V \neq \Phi$. This definition is independent of a coordinate system of s_0.

For each open set V of M (identified with $M \times 0$), we set $\Pi(V) = \{(\psi, \bar{\gamma})\}$ where ψ is a regular map on V for the given s_0

and $\bar{\gamma}$ is the germ of $\gamma\in\Gamma\times I$ on $\psi(V)$ where the domain of γ includes $\psi(V)$. For $(\psi^1, \bar{\gamma}^1)$, $(\psi^2, \bar{\gamma}^2)\in\Pi(V)$, let the product $(\psi^2, \bar{\gamma}^2)\cdot(\psi^1, \bar{\gamma}^1)$ be defined if and only if the regular map $(\gamma'|\psi'(V))\cdot\psi'$ on V is equal to ψ^2, in this case $\gamma^2\cdot\gamma^1$ can be combined in the sense of the pseudogroup $\Gamma\times I$ by a suitable restricution of domain, and we set

$$(\psi^2, \bar{\gamma}^2)\cdot(\psi^1, \bar{\gamma}^1) = (\psi^1, \text{germ of } \gamma^2\cdot\gamma^1 \text{ on } \psi'(V))\in\Pi(V),$$

where germs of γ^i on $\psi^i(V)$ is $\bar{\gamma}^i(i=1, 2)$. By this product $\pi(V)$ is a groupoid. For $V\supset V'$, the restriction of ψ, $\bar{\gamma}$ defines a map $\Pi(V)\to\Pi(V')$ and $\{\Pi(V)\}$ is a presheaf over M and it induces a sheaf $[\Pi]$ of groupoid over M.

For an open covering $\mathfrak{V}=\{V_\alpha, \alpha\in A\}$ of M, let $C^1(\mathfrak{V}, \Pi)$ denote the set of systems $\{\bar{\psi}_{\alpha\beta}\in\Pi(V_\alpha\cap V_\beta), (V_\alpha\cap V_\beta\neq\Phi)\}$ such that

$$\bar{\psi}_{\alpha\beta}\cdot\bar{\psi}_{\beta\gamma} = \bar{\psi}_{\alpha\gamma} \quad \text{for} \quad V_\alpha\cap V_\beta\cap V_\gamma\neq\Phi.$$

$\{\bar{\psi}_{\alpha\beta}\}$, $\{\bar{\psi}'_{\alpha\beta}\}\in C^1(\mathfrak{V}, \Pi)$ are said to be *cohomologous* if there exists $\bar{\psi}_\alpha\in\Pi(V_\alpha)$ for each α such as $\bar{\psi}_\alpha\cdot\bar{\psi}_{\alpha\beta}=\bar{\psi}'_{\alpha\beta}\bar{\psi}_\beta$ for $V_\alpha\cap V_\beta(\neq\Phi)$ and we denote by $\mathfrak{H}^1(\mathfrak{V}, \Pi)$ the set of cohomologous classes of $C^1(\mathfrak{V}, \Pi)$. For a refinement $\mathfrak{V}'=\{V'_{\alpha'}, \alpha'\in A'\}$ of $\mathfrak{V}$, (with the index injection of the refinement $\mathfrak{a}: A'\to A$),

$$\{\bar{\psi}_{\mathfrak{a}(\alpha')\mathfrak{a}(\beta')}|V'_{\alpha'}\cap V'_{\beta'}(\neq\Phi)\}\in C^1(\mathfrak{V}', \Pi)$$

and if $\{\bar{\psi}_{\alpha,\beta}\}$, $\{\bar{\psi}'_{\alpha,\beta}\}$ are cohomologous, then $\{\bar{\psi}_{\mathfrak{a}(\alpha')\mathfrak{a}(\beta')}|V'_{\alpha'}\cap V'_{\beta'}\}$, $\{\bar{\psi}'_{\mathfrak{a}(\alpha')\mathfrak{a}(\beta')}|V'_{\alpha'}\cap V'_{\beta'}\}$ are cohomologous in $C^1(\mathfrak{V}', \Pi)$. Therefore we have a correspondence

$$\bar{\gamma}^{\mathfrak{V}}_{\mathfrak{V}'}: \mathfrak{H}^1(\mathfrak{V}, \Pi)\to\mathfrak{H}^1(\mathfrak{V}', \Pi)$$

such that $r^{\mathfrak{V}'}_{\mathfrak{V}''}r^{\mathfrak{V}}_{\mathfrak{V}'}=r^{\mathfrak{V}}_{\mathfrak{V}''}$ for $\mathfrak{V}>\mathfrak{V}'>\mathfrak{V}''$ ($>$; refinement of coverings) and the system $\{\mathfrak{H}^1(\mathfrak{V}, \Pi), \mathfrak{V}\in$ the systems of open coverings of $M\}$ forms a direct system. We denote its inductive limit by $H^1(M, [\Pi])$. (The Čech cohomology set of 1-dim with coefficients in the sheaf $[\Pi]$). An element $\{\bar{\psi}_{\alpha\beta}\}$ of $C^1(\mathfrak{V}, \Pi)$ is called a *cocycle* of $\bar{\psi}$ for $\mathfrak{V}$ if $\bar{\psi}$ is the element of $H^1(M, [\Pi])$ determined by the inductive limit of cohomologous class of $\{\bar{\psi}_{\alpha\beta}\}$.

Lemma 5. *There exists a map δ from $D(s_0)$ onto $H^1(M, [\Pi])$.*

Proof. If $\{U_j, \varphi_j, j\in J\}$ is a coordinate system of $d\in D(s_0)$, then $\varphi_j | V_j : V_j \to B\times 0$ (identified with B) are regular maps for s_0 where V_j denote $U_j \cap (M\times 0)$ $(\neq \Phi)$, and $(\varphi_i \cdot \varphi_j^{-1}) \in \Gamma \times I(U_i \cap U_j \neq \Phi)$. Therefore if we denote by $\bar{\varphi}_{ij}$

$$(\varphi_j | V_j, \text{ germ of } \varphi_i \varphi_j^{-1} \text{ on } \varphi_j(V_i \cap V_j))$$

then $\quad \bar{\varphi}_{ij} \in \Pi(V_i \cap V_j)$ and $\bar{\varphi}_{ij} \cdot \bar{\varphi}_{jk} = \bar{\varphi}_{ik}$ on $V_i \cap V_j \cap V_k \neq \Phi$. Since $\{V_j = U_j \cap (M\times 0)\}$ is a covering of M, $\{\bar{\varphi}_{ij}\}$ is a cocycle of an element $\bar{\varphi}$ of $H^1(M, [\Pi])$. If we take other coordinate system $\{U'_k, \varphi'_k, k\in K\}$ of d, there exists a refinement covering $\{U''_l, l\in L\}$ of the coverings $\{U_j, j\in J\}$ and $\{U'_k, k\in K\}$ (with index injections of refinement $\iota : L\to J$, $\kappa : L \to K$), then

$$(\varphi_{\iota(l)} | U''_l)(\varphi'_{\kappa(l)} | U''_l)^{-1} \in \Gamma \times I$$

and

$$(\varphi_{\iota(l)} | U''_l \cap U''_m)\cdot(\varphi'_{\kappa(l)} | U''_l \cap U''_m)^{-1}\cdot(\varphi'_{\kappa(m)} | U''_l \cap U''_m)\cdot(\varphi'_{\kappa(m)} | U''_l \cap U''_m)^{-1}$$
$$= (\varphi_{\iota(m)} | U''_l \cap U''_m)\cdot(\varphi_{\iota(m)} | U''_l \cap U''_m)^{-1}\cdot(\varphi_{\iota(m)} | U''_l \cap U''_m)\cdot(\varphi'_{\kappa(m)} | U''_l \cap U''_m)^{-1}$$

on $U''_l \cap U''_m \neq \Phi$. If we set

$$\bar{\psi}_l = (\varphi'_{\kappa(l)} | V''_l, \text{ germ of } \varphi_{\iota(l)}\cdot(\varphi'_{\kappa(l)})^{-1} \text{ on } \varphi'_{\kappa(l)}(V''_l))$$

where $V''_l = U''_l \cap (M\times 0)$ $(\neq \Phi)$, then $\bar{\psi}_l \in \Pi(V''_l)$ and

$$(\bar{\psi}_l | V''_l \cap V''_m)\cdot(\bar{\varphi}'_{\kappa(l)\kappa(m)} | V''_l \cap V''_m) = (\bar{\varphi}_{\iota(l)\iota(m)} | V''_l \cap V''_m)\cdot(\bar{\psi}_m | V''_l \cap V''_m).$$

Since $\{V''_l\}$ is a refinement of the coverings $\{V_j \equiv U_j \cap (M\times 0)\}$ and $\{V'_k \equiv U'_k \cap (M\times 0)\}$ of $M\times 0 (\equiv M)$, then $\{\bar{\varphi}'_{kn}\}$ is a cocycle of the same element $\bar{\varphi}$ and the correspondence $d\to \bar{\varphi}$ defines a map $\delta : D(s_0) \to H^1(M, [\Pi])$.

Next, if $\{\bar{\psi}_{\alpha\beta}\} = \{(\psi_{\alpha\beta}, \bar{\gamma}_{\alpha\beta})\}$ is a cocycle of an element $\bar{\psi} \in H^1(M, [\Pi])$ for an open finite covering $\{V_\alpha, \alpha \in A\}$ of M, there exists a finite covering $\{U_j, j\in J\}$ of $M\times 0$ by open sets of $M\times I$ satisfying following conditions:

1) $\{U_j \cap (M\times 0)\}$ considered as a covering of M, is a refine- of $\{V_\alpha\}$ (with the index injection of the refinement $\alpha : J \to A$)

2) there exist $\varphi_{ij} \in B\times I(U_i \cap U_j (\neq \Phi))$ and $\gamma_{\alpha\beta} \in \Gamma \times I$ such that (domain of $\gamma_{\alpha(i)\alpha(j)}) \supset \varphi_{ij}(U_i \cap U_j)$, $\varphi_{ij}(V'_j) \subset B\times 0$ and

$$\psi_{\alpha(i)\alpha(j)} | V'_i \cap V'_j = \varphi_{ij} | V'_i \cap V'_j,$$

where $V'_j = U_j \cap (M\times 0)$ and

$$\bar{\gamma}_{\alpha(i)\alpha(j)} | \varphi_{ij}(V'_i \cap V'_j) = (\text{germ of } \gamma_{ij} \text{ on } \varphi_{ij}(V'_i \cap V'_j)) .$$

Since $\psi_{\alpha\beta}\psi_{\beta\gamma} = \psi_{\alpha\gamma}$ on $V_\alpha \cap V_\beta \cap V_\gamma (\neq \Phi)$ and by the definition of the product in $\Pi(V)$, we can choose these objects such that $\varphi_{ii}\varphi_{jj}^{-1} = \gamma_{ij} \in \Gamma \times I$ for $U_i \cap U_j \neq \Phi$ and $\gamma_{ij}\gamma_{jk} = \gamma_{jk}$ for $U_i \cap U_j \cap U_k (\neq \Phi)$. Since $\{U_j\}$ is a finite covering, we can take a positive number $\varepsilon (<1)$ such that $M \times (-\varepsilon, \varepsilon) \subset \bigcup_{j \in J} U_j$. If we set $\varphi_j = \varphi_{jj} | U_j \cap (M \times (-\varepsilon, \varepsilon))$, $\{\varphi_j, U_j \cap (M \times (-\varepsilon, \varepsilon), j \in J\}$ is a coordinate system of a section d_ε of $[M \times I / \Gamma \times I]_{M \times I}$ over $M \times (-\varepsilon, \varepsilon)$. By Lemma 4 there is a $d \in D(s_0)$ such that $d | M \times (-\varepsilon, \varepsilon) = d_\varepsilon$. Since $\gamma_{ij} = \varphi_i \varphi_j^{-1}$ for $(U_i \cap M \times (-\varepsilon, \varepsilon)) \cap (U_j \cap (M \times (-\varepsilon, \varepsilon)) \neq \Phi$ and so

$$\varphi\{(\varphi_{ij} | V'_i \cap V'_j, \text{ germ of } \gamma_{ij} \text{ on } \varphi_{ij}(V'_i \cap V'_j))\} = \psi_{\alpha(i)\alpha(j)} | V'_i \cap V'_j ,$$

then we have $\delta(d) = \psi$.

§ 3. Classes of locally equivalent deformations

Elements d of $D(s_0)$ being sections of the sheaf $[B \times I / \Gamma \times I]_{M \times I}$, let $d | W$ denote their restrictions on an open set W of $M \times I$ and set $D(s_0) | W = \{d | W ; d \in D(s_0)\}$. If η is a diffeomorphism from an open set W of $M \times I$ into $M \times I$ such that

[1] $\eta_t(x, t)$ *is independent of* x

where $\eta(x, t) = (\eta_x(x, t), \eta_t(x, t))$, $((x, t) \in W,\ x \in M,\ t \in I)$, then η induce a map $\bar{\eta}$ from $D(s_0) | \eta(W)$ into $D(s_0) | W$.

Definition. *Two differentiabl edeformation* d^1 *and* d^2 *of* s_0 *are locally equivalent if there exist a positive number* $\varepsilon > 1$ *and a diffeomophism* η *from* $M \times (-\varepsilon, \varepsilon)$ *into* $M \times I$ *such that* η *satisfies* [1] *and also the following two conditions,*

[2] $\eta(x, 0)$ *is identity,*

[3] $\bar{\eta}(d^2 | \eta(M \times (-\varepsilon, \varepsilon)) = d^1 | M \times (-\varepsilon, \varepsilon)$.

The local equivalence of deformations satisfies the equivalence relation and their equivalence classes are called *classes of locally equivalent deformations* and the set of these classes is denoted by $\bar{D}(s_0)$.

Proposition 1. *The map* $\delta : D(s_0) \to H^1(M, [\Pi])$ *induces a bijection* $\bar{\delta} : \bar{D}(s_0) \to H^1(M, [\Pi])$.

Proof. Let $\{U_j, \varphi_j^1, j \in J\}$, $\{U_j, \varphi_j^2, j \in J\}$ denote coordinate systems of deformations d^1, d^2, respectively, for a suitable common covering $\{U_j, j \in J\}$ of $M \times I$. If d^1, d^2 are locally equivalent, there exists a covering $\{W_l, l \in L\}$ of $M \times 0$ (identified with M) by open sets of $M \times I$, such that

(1) $\{W_l\}$ is a refinement of $\{U_j ; U_j \cap (M \times 0) \neq \Phi\}$ as a covering of $M \times 0$ by open sets of $M \times I$ (with the index injection of the refinement $\mu : L \to J' \subset J$),

(2) $W_l \subset M \times (-\varepsilon, \varepsilon)$ for each $l \in L$,

(3) $\eta(W_l) \subset U_{\mu(l)}$

where η is the diffeomorphism from $M \times (-\varepsilon, \varepsilon)$ into $M \times I$ which gives the local equivalence of d^1, d^2. Since $\{\eta(W_l), \varphi_{(l)}^2 | \eta(W_l)\}$ is a coordinate system of $d^2 | \eta(\bigcup_{l \in L} W_l) \subset d^2 | \eta(M \times (-\varepsilon, \varepsilon))$, we see that $\{W_l, \varphi_{\mu(l)}^2 \cdot \eta | W_l\}$ is a coordinate system of $\bar{\eta}(d^2 | \eta(\bigcup_{l \in L} W_l))$. On the other hand, $\{W_l, \varphi_{\mu(l)}^1 | W_l\}$ is a coordinate system of $d^1 | \bigcup_{l \in L} W_l$ and $\bar{\eta} \cdot d^2 | \bigcup_{l \in L} W_l = d^1 | \bigcup_{l \in L} W_l$. Therefore, for each $l \in L$, the local diffeomorphism $\varphi_{\mu(l)}^2 \cdot \eta \cdot (\varphi_{\mu(l)}^1)^{-1} | \varphi_{\mu(l)}^1(W_l)$ of $B \times I$ is an element of $\Gamma \times I$, and is denoted by γ_l. The image $\varphi_{\mu(l)}^1(W_l)$ is the domain of γ_l and $\varphi_{\mu(l)}^1(W_l) \supset \varphi_{\mu(l)}^1(V_l)$ where $V_l = W_l \cap (M \times 0)$. Then $(\varphi_{\mu(l)}^1 | V_l$, germ of γ_l on $\varphi_{\mu(l)}^1(V_l)) \in \Pi(V_l)$. Since

$$\varphi_{\mu(l)}^1 (\varphi_{\mu(m)}^1)^{-1} | \varphi_{\mu(m)}^1 (W_l \cap W_m) \in \Gamma \times I\,,$$

$$\eta(W_l \cap W_m) = (\varphi_{\mu(l)}^2)^{-1} \cdot \gamma_l \cdot \varphi_{\mu(l)}^1 | W_l \cap W_m = (\varphi_{\mu(m)}^2)^{-1} \cdot \gamma_m \cdot \varphi_{\mu(m)}^1 | W_l \cap W_m$$

and the range of γ_l is $\varphi_{\mu(l)}^2$, then

$$\gamma_l \cdot \varphi_{\mu(l)}^1 \cdot (\varphi_{\mu(m)}^1)^{-1} = \varphi_{\mu(l)}^2 \cdot (\varphi_{\mu(m)}^2)^{-1} \cdot \gamma_m \quad \text{on} \quad \varphi_{\mu(m)}^1 (W_l \cap V_m)$$

and so $\psi_l \cdot \bar{\varphi}_{lm}^1$, $\bar{\varphi}_{lm}^2 \cdot \psi_m$ are defined on $V_l \cap V_m \neq \Phi$ and are equal, where

$$\begin{aligned}
\psi_l &= (\varphi_{\mu(l)}^1 | V_l, \ \text{germ of} \ \gamma_l \ \text{on} \ \varphi_{\mu(l)}^1(V_l)), \\
\bar{\varphi}_{lm}^1 &= (\varphi_{\mu(m)}^1 | V_m, \ \text{germ of} \ \varphi_{\mu(l)}^1 \cdot (\varphi_{\mu(m)}^1)^{-1} \ \text{on} \ \varphi_{\mu(m)}^1(V_m)), \\
\varphi_{lm}^2 &= (\varphi_{\mu(m)}^2 \cdot \eta | V_m, \ \text{germ of} \ \varphi_{\mu(l)}^2 (\varphi_{\mu(m)}^2)^{-1} \ \text{on} \ \varphi_{\mu(m)}^2(V_m)).
\end{aligned}$$

Therefore, $\{\bar{\varphi}_{lm}^1\}$ and $\{\bar{\varphi}_{lm}^2\}$ are cohomologous in $\mathcal{C}^1(\{V_i\}, \Pi)$, where the former determines $\delta(d^1)$ and the latter determines $\delta(d^2)$ because η is identity on $M \times 0$, that is $\delta(d^1) = \delta(d^2)$.

Conversely, we suppose $\delta(d^1)=\delta(d^2)$. Since M is compact, there exists a finite covering $\{V_k,\ k\in K\}=\mathfrak{V}$ of M by open sets of M which is a refinement of $\{U_j\cap M\times 0(\neq\Phi)\}$ as an open covering of M (with the index injection of the refinement $\lambda: K\to J$), such that

$$\{\varphi^1_{kl}\}=\{(\varphi^1_{\lambda(l)}\mid V_k\cap V_l(\neq\Phi),\ \text{germ of } \varphi^1_{\lambda(k)}\cdot(\varphi^1_{\lambda(l)})^{-1} \text{ on } \varphi^1_{\lambda(l)}(V_k\cap V_l))\}$$

and

$$\{\bar{\varphi}^2_{kl}\}=\{(\varphi^2_{\lambda(l)}\mid V_k\cap V_l,\ \text{germ of } \varphi^2_{\lambda(k)}\cdot(\varphi^2_{\lambda(l)})^{-1} \text{ on } \varphi^2_{\lambda(l)}(V_k\cap V_l))\}$$

are cohomologous in $C^1(\{V_k\},\pi)$. Then we have a element $\bar{\gamma}_k$ of $\Pi(V_k)$ for each $k\in K$ such as

$$\bar{\gamma}_k\cdot\bar{\varphi}^1_{kl}=\bar{\varphi}^2_{kl}\cdot\bar{\gamma}_l \qquad \text{for} \quad V_k\cap V_l\neq\Phi\,.$$

From the definition of $\pi(V_k)$ and the product in it,

$$\bar{\gamma}_k=(\varphi^1_{\lambda(k)}\mid V_k,\ \text{germ of } \gamma_k \text{ on } \varphi^1_{\lambda(k)}(V_k))$$

where $\gamma_k\in\Gamma\times I$, (the domain of $\gamma_k)\cap(B\times 0)=\varphi^1_{\lambda(k)}(V_k)$, (the range of $\gamma_k)\cap(B\times 0)=\varphi^2_{\lambda(k)}(V_k)$, and $\gamma_k\varphi^1_{\lambda(k)}\mid V_k=\varphi^2_{\lambda(k)}\mid V_k$. If we set

$$W_k=(\varphi^1_{\lambda(k)})^{-1}\cdot(\text{the domain of } \gamma_k)\cap(\varphi^2_{\lambda(k)})^{-1}\cdot(\text{the range of } \gamma_k) \subset M\times I\,,$$

then $W_k\cap(M\times 0)=V_k$, $\{W_k,\ k\in K\}$ is a finite covering of $M\times 0$ by open sets of $M\times I$, and $(\varphi^2_{\lambda(k)})^{-1}\cdot\gamma_k\cdot\varphi^1_{\lambda(k)}$ can be defined on W_k. Since

$$\bar{\gamma}_k\cdot\bar{\varphi}^1_{\lambda(k)\lambda(l)}=\bar{\varphi}^2_{\lambda(k)\lambda(l)}\cdot\bar{\gamma}_l$$

then

$$(\varphi^2_{\lambda(k)})^{-1}\cdot\gamma_k\cdot\varphi^1_{\lambda(k)}=(\varphi^2_{\lambda(k)})^{-1}\cdot\gamma_l\cdot\varphi^1_{\lambda(l)} \quad \text{on} \quad W_k\cap W_l\,(\neq\Phi)\,.$$

Therefore, there exist a positive number ε and a homeomorphism η from $M\times(-\varepsilon,\varepsilon)$ into $M\times I$ such that $M\times(-\varepsilon,\varepsilon)\subset\bigcup_{k\in K}W_k$,

$$\eta\mid M\times(-\varepsilon,\varepsilon)\cap W_k=(\varphi^2_{\lambda(k)})^{-1}\cdot\gamma_k\cdot\varphi^1_{\lambda(k)}\mid(M\times(-\varepsilon,\varepsilon))\cap W_k\,,$$
$$\gamma_k\cdot\varphi^1_{\lambda(k)}\mid M\times(-\varepsilon,\varepsilon)\cap W_k=\varphi^2_{\lambda(k)}\eta\mid M\times(-\varepsilon,\varepsilon)\cap W_k$$

and $\eta_t(x,t)$ is independent of x where $\eta(x,t)=(\eta_x(x,t),\eta_t(x,t))$. Here, $\{\gamma_k\cdot\varphi_{(k)},\ (M\times(-\varepsilon,\varepsilon))\cap W_k)\}$ and $\{\varphi^2_{\lambda(k)}\cdot\eta,\ (M\times(-\varepsilon,\varepsilon)\cap W_k)\}$ are coordinate systems of $d^1\mid M\times(-\varepsilon,\varepsilon)$ and $\bar{\eta}d^2\mid M\times(-\varepsilon,\varepsilon)$, respectively, i.e. $d^1\mid M\times(-\varepsilon,\varepsilon)=\bar{\eta}d^2$. Since $\varphi^2_{\lambda(k)}\cdot\gamma_k\cdot\varphi^1_{\lambda(k)}=\text{identity}$

on V_k, then $\eta | M\times 0=$identity. Therefore, η gives the local equivalence of d^1 and d^2.

§4. Germs of local automorphisms depending differentiably on 1-parameter for the differentiable (B, Γ)-structure

A diffeomorphism ξ of an open set V of M to an open set of M is called a *local automorphism for the differentiable* (B, Γ)-*structure* s_0 if $\xi\cdot s_0=s_0$ on V, i.e. for a regular map φ of s_0 on a neighborhood of each point $x\in\xi(V)$, $\varphi\cdot\xi$ is a regular map on a neighborhood of $\xi^{-1}(x)$.

A diffeomorphism ζ of $V\times(-\varepsilon, \varepsilon)$ into $M\times(-\varepsilon, \varepsilon)$ is said a *local automorphism of V depening differentiably on* 1-*parameter for* s_0, if

$$\zeta(x, 0) = \text{identity } (x\in V), \ \zeta_t(x, t) = t$$

and if $\zeta_x(x, t)$ is local automorphism of M for each fixed t where

$$\zeta(x, t) = (\zeta_x(x, t), \zeta_t(x, t)), \qquad x\in V, t\in(-\varepsilon, \varepsilon).$$

For each open set V of M, we set

$$A(V) = \{\text{germ of } \zeta \text{ on } V\times 0\}.$$

which is a group. By the restriction $A(V)\rightarrow A(V')$ for $V\supset V'$, $\{A(V)\}$ is a presheaf of group over M and induces a sheaf $[A]$ over M.

Definition. *The sheaf* $[A]$ *is the sheaf of germs of local automorphisms depending differentiably on* 1-*parameter for* (B, Γ)-*structure* s_0.

Lemma 6. *For each open set V of M where V has a regular map* ψ *of* s_0, *there exists an onto-map* $\pi: \Pi(V)\rightarrow A(V)$.

Proof. For $\tilde{\psi}=(\psi$, germ of γ on $\psi(V))\in\Pi(V)$, $(\gamma\in\Gamma\times I)$, if we set $\tilde{\psi}(x, t)=(\psi(x), t)$, $(x\in V, t\in I)$ and $\tilde{\gamma}(y, t)=(\gamma(y, 0), t)$, $(y\in\psi(V), t\ni I)$, then $\tilde{\psi}\in B\times I(V\times I)$, $\tilde{\gamma}\in\Gamma\times I$ and $(\tilde{\gamma})^{-1}\cdot\gamma\in\Gamma\times I$. Hence, there exists an open set W of $M\times I$ such that $W\cap(M\times 0)=V$, $\tilde{\psi}(W)\subset$ (domain of γ) and $\tilde{\psi}^{-1}\cdot\tilde{\gamma}^{-1}\cdot\gamma\cdot\tilde{\psi}$ can be defined on W. Since $\tilde{\gamma}^{-1}\cdot\gamma|\psi(V)=$identity and since $\tilde{\psi}^{-1}\cdot\tilde{\gamma}^{-1}\cdot\gamma\cdot\tilde{\psi}$ is a local automorphism of V depending differentiably 1-parameter for s_0, we see that (germ of $\tilde{\psi}^{-1}\cdot\tilde{\gamma}^{-1}\cdot\gamma\cdot\tilde{\psi}$ on V) is an element $\pi\psi$ of $A(V)$ and the correspondence $\psi\to\pi\cdot\psi$ gives a map $\pi:\Pi(V)\to A(V)$.

Conversely, let (germ of ζ on V) be an element $\bar{\zeta}$ of $A(V)$ where ζ is a local diffeomorphism of an open set of $M\times I$ including V such that ζ gives a local automorphism of V depending 1-parameter. For a regular map φ of s_0 on V,

$$(\varphi,\ \text{germ of } \tilde{\varphi}\zeta\tilde{\varphi}^{-1} \text{ on } \varphi(V)) \quad \text{where} \quad \tilde{\varphi}(x,t)=(\varphi(x),t)$$

is an element ψ of $A(V)$ such as $\pi\psi=\zeta$, that is, π is onto.

We define $H^1(M,[A])$ from the presheaf $\{A(V)\}$ in the same manner as we did for $H^1(M,[\Pi])$, and we have

Proposition 2. *The map π induces a bijection $\pi^*: H^1(M,[\Pi])\to H^1(M,[A])$.*

Proof. For an element $\{\psi_{\alpha\beta}\}=\{\psi_{\alpha\beta}$, germ of $\gamma_{\alpha\beta}$ on $\psi_{\alpha\beta}(V_\alpha\cap V_\beta))\}\in C^1(\mathfrak{V},\Pi)$ where $\mathfrak{V}=\{V_\alpha\}$ and $\gamma_{\alpha\beta}\in\Gamma\times I$, we have

$$\tilde{\gamma}_{\alpha\beta}(y,t)=\tilde{\psi}_\alpha\tilde{\psi}_\beta^{-1}(y,t) \quad \text{where} \quad y\in\psi_\beta(V_\beta),\ t\in I$$

and

$$\tilde{\psi}_{\alpha\beta}=\tilde{\psi}_\beta|V_\alpha\cap V_\beta)\times I \quad \text{where} \quad \tilde{\psi}_\alpha=(\psi_{\alpha\alpha}(x),t)\ (x\in V_\alpha)$$

because $\psi_{\alpha\alpha}\cdot\psi_{\alpha\beta}=\psi_{\alpha\beta}\cdot\psi_{\beta\beta}$. Since

$$\begin{aligned}(\tilde{\psi}_{\alpha\beta}^{-1}\cdot(\tilde{\gamma}_{\alpha\beta})^{-1}\cdot\gamma_{\alpha\beta}\cdot\tilde{\psi}_{\alpha\beta})\cdot(\tilde{\psi}_{\beta\gamma}^{-1}\cdot(\tilde{\gamma}_{\beta\gamma})^{-1}\cdot\gamma_{\beta\gamma}\cdot\tilde{\psi}_{\beta\gamma}) &= \tilde{\psi}_\alpha^{-1}\cdot\gamma_{\alpha\beta}\cdot\gamma_{\beta\gamma}\cdot\tilde{\psi}_\gamma\\ =\tilde{\psi}_\gamma^{-1}\cdot\tilde{\psi}_\gamma\cdot\tilde{\psi}_\alpha^{-1}\cdot\gamma_{\alpha\gamma}\cdot\tilde{\psi}_\gamma = \tilde{\psi}_{\alpha\gamma}^{-1}\cdot(\tilde{\gamma}_{\alpha\beta})^{-1}\cdot\gamma_{\alpha\gamma}\cdot\tilde{\psi}_{\alpha\gamma}\end{aligned}$$

then $\{\pi\psi_{\alpha\beta}\}$ is an element of $C^1(\mathfrak{V},A)$ and moreover this correspondence $\{\psi_{\alpha\beta}\}\to\{\pi\psi_{\alpha\beta}\}$ gives a map from $C^1(\mathfrak{V},\Pi)$ onto $C^1(\mathfrak{V},A)$ by Lemma 5. If two elements $\{\psi_{\alpha\beta}^1\}$ and $\{\psi_{\alpha\beta}^2\}$ of $C^1(\mathfrak{V},\Pi)$ are cohomologous, then there exists an element $\psi_\alpha=(\psi_{\alpha\alpha}^1$, germ of γ_α on $\psi_{\alpha\alpha}^1(V_\alpha))$ of $\Pi(V_\alpha)$ for each V_α, such that $\psi_\alpha\cdot\psi_{\alpha\beta}^1=\psi_{\alpha\beta}^2\cdot\psi_\beta$ and $\gamma_\alpha\cdot\gamma_{\alpha\beta}^1=\gamma_{\alpha\beta}^2\cdot\gamma_\beta$ on a suitable domain including $\psi_{\alpha\beta}^1(V_\alpha\cap V_\beta)$. Then

$$\begin{aligned}((\tilde{\psi}_\alpha^2)^{-1}\gamma_\alpha\tilde{\psi}_\alpha^1)\cdot((\tilde{\psi}_\beta^1)^{-1}\cdot(\tilde{\gamma}_{\alpha\beta}^1)^{-1}\cdot\gamma_{\alpha\beta}^1\cdot\tilde{\psi}_\beta^1) &= (\tilde{\psi}_\alpha^2)^{-1}\cdot\gamma_\alpha\cdot\gamma_{\alpha\beta}^1\tilde{\psi}_\beta^1\\ =(\tilde{\psi}_\alpha^2)^{-1}\cdot\gamma_{\alpha\beta}^2\cdot\gamma_\beta\cdot\tilde{\psi}_\beta^1 = ((\tilde{\psi}_\beta^2)^{-1}\cdot(\tilde{\gamma}_{\alpha\beta}^2)^{-1}\cdot\gamma_{\alpha\beta}^2\cdot\tilde{\psi}_\beta^2)\cdot((\tilde{\psi}_\beta^2)^{-1}\cdot\gamma_\beta\cdot\tilde{\psi}_\beta^1)\end{aligned}$$

on a suitable open set of $M\times I$ including $V_\alpha\cap V_\beta(\neq\Phi)$. Therefore, if we set

$$\bar{\zeta}_\alpha=(\text{germ of } (\tilde{\psi}_\alpha^2)^{-1}\cdot\gamma_\alpha\cdot\tilde{\psi}_\alpha^1 \text{ on } V_\alpha)\in A(V_\alpha),$$

we have $\bar{\zeta}_\alpha(\pi\psi_{\alpha\beta}^1)=(\pi\psi_{\alpha\beta}^1)\bar{\zeta}_\beta$ on $V_\alpha\cap V_\beta$, i.e. $\{\pi\psi_{\alpha\beta}^1\}$, $\{\pi\psi_{\alpha\beta}^2\}$ are cohomologous.

Conversely, if $\{\pi\bar{\psi}^1_{\alpha\beta}\}$ and $\{\pi\bar{\psi}^2_{\alpha\beta}\}$ are cohomologous in $\mathcal{C}^1(\mathfrak{V}, A)$, then there exists, for each α, a local diffeomorphism ζ_α on an open set W_α of $M\times I$ including V_α such that $\bar{\zeta}_\alpha(\pi\bar{\psi}^1_{\alpha\beta})=(\pi\bar{\psi}^2_{\alpha\beta})\bar{\zeta}_\beta$ where $\bar{\zeta}_\alpha$ is the germ of ζ_α on V_α, and such that $\zeta_\alpha((\tilde{\psi}^1)^{-1}\cdot(\tilde{\gamma}^1)^{-1}\cdot\gamma^1_{\alpha\beta}\cdot\tilde{\psi}^1_\beta)$ and $((\tilde{\psi}^2_\beta)^{-1}\cdot(\tilde{\gamma}^2_{\alpha\beta})^{-1}\cdot\gamma^2_{\alpha\beta}\cdot\tilde{\psi}^2)\zeta_\beta$ can be defined and are equal on $W_\alpha\cap W_\beta(\neq\Phi)$. If we set $\gamma_\alpha=\tilde{\psi}^2_\alpha\zeta_\alpha(\tilde{\psi}^1_\alpha)^{-1}$ on W_α, then

$$\gamma_\alpha\gamma^1_{\alpha\beta} = \gamma_\alpha\tilde{\psi}^1_\alpha(\tilde{\psi}^1_\beta)^{-1}(\tilde{\gamma}^1_{\alpha\beta})\gamma^1_{\alpha\beta} = \tilde{\psi}^2_\alpha(\psi^2_\beta)^{-1}(\tilde{\gamma}^2_{\alpha\beta})^{-1}\gamma^2_{\alpha\beta}\gamma_\beta = \gamma^2_{\alpha\beta}\gamma_\beta\,.$$

Therefore, $(\psi_\alpha$, germ of γ_α on $\psi_\alpha(V_\alpha))\cdot\bar{\psi}^1_{\alpha\beta}=\bar{\psi}^2_\alpha\cdot(\psi_\beta$, germ of γ_β on $\psi_\beta(V_\beta(V_\beta))$, that is, $\{\bar{\psi}^1_{\alpha\beta}\}$ and $\{\bar{\psi}^2_{\alpha\beta}\}$ are cohomologous in $\mathcal{C}^1(\mathfrak{V}, \Pi)$.

From Proposition 1 and Proposition 2, we have

Theorem 1. *There exists a bijection* $\bar{D}(s_0)\to H^1(M, [A])$.

§ 5. Cross-sections of a differentiable bundle

Let F be a differentiable manifold and G be an effective differentiable transformation group on F and let Γ_0 be the pseudogroup of all local diffeomorphisms of R^n. For each element γ_0 of Γ_0 whose domain is U, we define a diffeomorphism $\tau(\gamma_0): F\times U\to F\times\gamma_0(U)$ such that $\tau(\gamma_0)(x, f)=(\gamma_0(x), \tau_F(x, f))$ and for each fixed x, τ_F is a transformation of F by G. Differentiable cross-sections of $F\times R^n$ over U can be transformed to differentiable cross-sections over $\gamma_0(U)$ by $\tau(\gamma_0)$. If we denote by $\tilde{B}$ the space of germs of differentiable cross-sections of $F\times R^n$ over R^n, then $\tilde{B}$ is a topological space with a differentiable structure and $\tau(\gamma_0)$ induces a local diffeomorphism of $\tilde{B}$. Then Γ_0 defines a pseudogroup $\tilde{\Gamma}$ of local diffeomorphisms of $\tilde{B}$ associated to τ. Hence we can consider differentiable $(\tilde{B}, \tilde{\Gamma})$-structures.

On the other hand, let $\{U_i, \varphi_i\}$ be a coordinate system of the differentiable structure of M, then $\{U_i, \varphi_i\}$, F, G, and τ define a differentiable fibre bundle $\mathcal{B}$ with the fibre F, the structure group G, the base space M, the bundle space X and the projection p. We say $\mathcal{B}$ an *F-bundle τ-associated* to the differentiable structure of M (or a *differentiable F-bundle*) and $\{U_i, \varphi_i\}$ a *coordinate system* of $\mathcal{B}$. The diffeomorphism $\varphi_i: U_i\to R^n$ induces a fibre-preserving diffeomorphism $\varphi_i^*: p^{-1}(U_i)\to\varphi_i(U_i)\times F$ and

$$\varphi_j^*(\varphi_i^*)^{-1}|\varphi_i(U_i\cap U_j)\times F=\tau(\varphi_j\varphi_i^{-1}|\varphi_i(U_i\cap U_j))\quad\text{for}\quad U_i\cap U_j\neq\Phi.$$

If c is a differentiable cross-section of $\mathfrak{B}$ over M, the map $\varphi_i^*\cdot c|U_i$ can be regarded as a diffeomorphism c_i of U_i into $\tilde{B}$ and $c_j\cdot c_i^{-1}|c_i(U_i\cap U_j)\in\tilde{\Gamma}$, then $\{U_i, c_i\}$ is a coordinate system of a differentiable $(\tilde{B}, \tilde{\Gamma})$-structure s and s is independent of the coordinate system $\{\varphi_i, U_i\}$ of $\mathfrak{B}$. Therefor we have a map $C:\{c\}\to H^0(M, [\tilde{B}/\tilde{\Gamma}]_M)$ where $\{c\}$ is the set of all differentiable cross-sections of $\mathfrak{B}$ over M.

Lemma 7. *The map C is a bijection.*

Proof. We can take a coordinate system $\{U_i, \bar{\varphi}_i\}$ for $s\in H^0(M, [\tilde{B}/\tilde{\Gamma}]_M)$ such that $\{U_i, \varphi_i\}$ is a coordinate system of $\mathfrak{B}$ where $\varphi_i=p_0\cdot\bar{\varphi}_i$, p_0 is the projection of sheaf $\tilde{B}\to R^n$ and $\varphi_i^*: p^{-1}(U_i)\to\varphi_i(U_i)\times F$ is a coordinate function induced from φ_i. Then $(\varphi_i^*)^{-1}\bar{\varphi}_i(U)$ is a cross-section s_i over U_i for $\mathfrak{B}$ and

$$(\varphi_j^*)(\varphi_i^*)^{-1}|\bar{\varphi}_i(U_i\cap U_j)=\tau(\varphi_j\cdot\varphi_i^{-1})|\bar{\varphi}_i(U_i\cap U_j)=\bar{\varphi}_j\bar{\varphi}_i^{-1}|\bar{\varphi}_i(U_i\cap U_j)$$

for $U_i\cap U_j\neq\Phi$ and so

$$\begin{aligned}s_i|U_i\cap U_j)&=(\varphi_i^*)^{-1}\bar{\varphi}_i|(U_i\cap U_j)=(\varphi_j^*)^{-1}\varphi_j^*(\varphi_i^*)^{-1}\bar{\varphi}_i\bar{\varphi}_j^{-1}\bar{\varphi}_j|U_i\cap U_j\\&=(\varphi_j^*)^{-1}\bar{\varphi}_j|U_i\cap U_j=s_j|U_i\cap U_j,\end{aligned}$$

hence $\{s_i\}$ is a cross-section c over M. The correspondence $s\to c$ defines a correspondence $S: H^0(M, [\tilde{B}/\tilde{\Gamma}]_M)\to\{c\}$ and $S\cdot C=$identity, $C\cdot S=$identity.

Then we have

Theorem 2. *Differentiable cross-sections of the differentiable F-bundle are differentiable $(\tilde{B}, \tilde{\Gamma})$-structures.*

Remark. The proof of Lemma 7 ensures that C gives a bijection of the set of differentiable cross-sections over an open set U of M onto the set of sections of $[\tilde{B}/\tilde{\Gamma}]_M$ over U.

§ 6. Deformations of differentiable cross-sections of the differentiable bundle

From the differentiable F-bundle $\mathfrak{B}(X, M, F, G)$, a differentiable F-bundle $\mathfrak{B}\times I(X\times I, M\times I, F, G)$ is naturally defined. As for the coordinate system $\{U_i, \varphi_i\}$ of $\mathfrak{B}\times I$, φ_i can be taken to be diffeomorphisms of U_i into $R^n \times I$ such as $\varphi_{i,t}(x, t) = t$ where $\varphi_i = (\varphi_{i,x}(x, t), \varphi_{i,t}(x, t))$, $(x, t) \in U$. Differentiable cross-sections $\tilde{d}$ of $\mathfrak{B} \times I$ define cross-sections c of $\mathfrak{B}$ by the restriction on $M \times 0$, and $\tilde{d}$ is called a (*differentiable*) *deformation of c*.

Definition. *A deformation $\tilde{d}$ of a given cross-section c_0 of $\mathfrak{B}$ is locally trivial if there exist an open neighborhood U relative to $M \times I$ for each point of M and a diffeomorphism ξ from U into $M \times I$ such as*

$$\xi_t(x, t) = t\,, \quad \xi(x, 0) = \textit{identity and } \tilde{d} = \xi^* c_0\,,$$

where ξ^* is a local bundle-automorphism induced by ξ, $\tilde{c}_0(x, t) = (c_0(x), t)$ and $\xi(x, t) = (\xi_x(x, t), \xi_t(x, t))$, $((x, t) \in U)$.

Now, we take $\tilde{B}$, $\tilde{\Gamma}$ as B, Γ in §§2–3, then $\tilde{B} \times I$, $\tilde{\Gamma} \times I$, $[\tilde{B} \times I / \tilde{\Gamma} \times I]_{M \times I}$, $D(\tilde{s}_0)$ ($\tilde{s}_0 \in H^0(M, [\tilde{B}/\tilde{\Gamma}]_M)$), $[\tilde{\Pi}]$, $\tilde{D}(s_0)$ and $[\tilde{A}]$ take the place of $B \times I$, $\Gamma \times I$, $[B \times I / \Gamma \times I]_{M \times I}$, $D(s_0)$, $[\Pi]$, $\bar{D}(s_0)$ and $[A]$, respectively. If we apply Theorem 1 to this case, we have

Proposition 3. *We have a bijection $\tilde{D}(s_0) \rightarrow H^1(M, [A])$.*

Let $\widetilde{B \times I}$ be the space of germs of differentiable cross-sections of the product bundle $F \times (R^n \times I)$ over $R^n \times I$ and let $\widetilde{\Gamma \times I}$ be the pseudogroup of local diffeomorphisms of $\widetilde{B \times I}$ induced by local diffeomorphisms of $R^n \times I$ as in §5. Then

Lemma 8. $H^0(M \times I, [\tilde{B} \times I / \tilde{\Gamma} \times I]_{M \times I})$ is a sub-set of $H^0(M \times I, [\widetilde{B \times I} / \widetilde{\Gamma \times I}]_{M \times I})$.

Proof. $\tilde{B} \times I$ is a sub-space of $\widetilde{B \times I}$ and $\tilde{\Gamma} \times I$ is a sub-pseudo-proup of $\widetilde{\Gamma \times I}$. The set $\tilde{B} \times I(U)$ is a sub-set of $\widetilde{B \times I}(U)$ for each open set U of $M \times I$. If

$$\varphi,\ \psi \in \tilde{B} \times I(U) \quad \text{and} \quad \varphi \cdot \psi^{-1} = \gamma \in \widetilde{\Gamma \times I}\,,$$

then $\gamma \in \tilde{\Gamma} \times I$ and therefore $\tilde{B} \times I / \tilde{\Gamma} \times I(U) \subset \widetilde{B \times I} / \widetilde{\Gamma \times I}(U)$ by Lemma 1. Therefore, $[\tilde{B} \times I / \tilde{\Gamma} \times I]_{M \times I} \subset [\widetilde{B \times I} / \widetilde{\Gamma \times I}]_{M \times I}$ since $r^U_{U'}(\tilde{B} \times I(U)) \subset \tilde{B} \times I(U')$ for $U \supset U'$.

If we apply Lemma 7 to the set $\{\tilde{c}\}$ of differentiable cross-sections of $\mathfrak{B} \times I$ and $H^0(M \times I, [\widetilde{B \times I} / \widetilde{\Gamma \times I}]_{M \times I})$, we have a bijection

$$\{\tilde{c}\} \underset{\tilde{S}}{\overset{\tilde{C}}{\rightleftarrows}} H^0(M\times I,\ [\widetilde{B\times I}/\widetilde{\Gamma\times I}]_{M\times I})\,.$$

Definition. *Locally trivial deformations $\tilde{d}^1$ and $\tilde{d}^2$ of c_0 are locally equivalent if there exist a positive number number $\varepsilon<1$ and a diffeomorphism ξ from $M\times(-\varepsilon, \varepsilon)$ into $M\times I$ such that*

1. *$\xi_t(x, t)$ is independent of x for $(x, t)\in M\times(-\varepsilon, \varepsilon)$,*
2. *$\xi(x, 0) = identity$,*
3. *$\tilde{d}^1|\xi(x, t) = \xi^*(\tilde{d}^2(x, t))$,*

where $\xi(x, t)=(\xi_x(x,t),\ \xi_t(x, t))$ and ξ^* is a bundle map induced by ξ.

If we set $\tilde{s}_0=C(c_0)$ where c_0 is a given cross-section of $\mathfrak{B}$, then $\tilde{S}$ maps bijectively $D(\tilde{s}_0)$ onto a sub-set $E(c_0)$ of the set of locally trivial deformations of c_0.

Lemma 9. *For each locally trivial deformation $\tilde{d}$ of c_0, there exists an element $\tilde{d}'$ of $E(c_0)$ such that $\tilde{d}$ and $\tilde{d}'$ are locally equivalent.*

Proof. Let $\{U_j, \varphi_j, j\in J\}$ be a coordinate system of $\mathfrak{B}$. Since $\tilde{d}$ is a locally trivial deformation of c_0 and since M is compact, there are a finite covering $\{U'_k, k\in K\}$ of $M\times 0$ by open sets of $M\times I$ and diffeomorphisms ξ_k of U_k into $M\times I$ for each $k\in K$, such that the covering $\{U'_k\}$ is a refinement of the covering $\{U_j\,;\ U_j\cap(M\times 0)\neq\Phi,\ j\in J'\}$ of $M\times 0$ (with the index injection of the refinement $\kappa: K\rightarrow J'$), $\xi^*_k\tilde{c}_0=\tilde{d}$ on U'_k and $\xi_k(U'_k)\subset U_k$. Then

$$\begin{aligned}\varphi^*_{\kappa(k)}\tilde{d}(x, t) &= \varphi^*_{\kappa(k)}(\xi_k\tilde{c}_0)(x, t) = \varphi^*_{\kappa(k)}\tilde{c}_0(\xi_k(x, t)) = \varphi^*_{\kappa(k)}\tilde{c}_0(\xi_{k,x}(x, t), t)\\ &= \varphi^*_{\kappa(k)}\tilde{c}_0(\xi_{k,x}(x, t)), t))\subset(\tilde{s}_{\kappa(k)}(\xi_{k,x}(x, t)))\times I\subset\tilde{B}\times I\end{aligned}$$

where $(x, t)\in U'$ and $\xi_k(x, t)=(\xi_{k,x}(x, t), t)$, hence $\tilde{C}(\tilde{d}|U'_k)$ is a section of $[\tilde{B}\times I/\tilde{\Gamma}\times I]_{M\times I}$ over U'_k by Lemark in §5. If we take a positive number ε such as $M\times(-\varepsilon, \varepsilon)\subset\bigcup_{k\in K} U'_k$, then $\tilde{C}(\tilde{d}|M\times(-\varepsilon, \varepsilon))$ is a section of $[\tilde{B}\times I/\tilde{\Gamma}\times I]_{M\times I}$ over $M\times(-\varepsilon, \varepsilon)$. By Lemma 4, this section can be extended over $M\times I$ which is an element d of $D(s_0)$. Then $\tilde{S}\cdot d\in E(c_0)$ and

$$\tilde{S}\cdot d|M\times(-\varepsilon, \varepsilon) = \tilde{S}\cdot\tilde{C}(\tilde{d}|M\times(-\varepsilon, \varepsilon)) = \tilde{d}|M\times(-\varepsilon, \varepsilon)$$

i.e. $\tilde{S}\cdot d$ and $\tilde{d}$ are equal on $M\times(-\varepsilon, \varepsilon)$.

By definitions, the local equivalence of locally trivial deformations of c_0 applied to $E(c_0)$ and local equivalence of $D(\tilde{s}_0)$ are compatible with the bijection $E(c_0)\to D(\tilde{s}_0)$. Then, by Lemma 9 we have

Proposition 4. *The set of local equivalence clases of all locally trivial deformations of c_0 can be identified with the set $\tilde{D}(\tilde{s}_0)$ of local equivalence classes of $D(\tilde{s}_0)$.*

A local diffeomorphism ξ_V of an open set V of M into M is said to be a *local automorphism* of V for the cross-section c_0, if $c_0|\xi_V(x)=\xi_V^*\cdot c|x$ where ξ_V^* is a local bundle map induced by ξ_V.

Definition. *A local diffeomorphism ζ of an open set $V\times(-\varepsilon, \varepsilon)$ of $M\times I$ into $M\times I$ is a local automorphism on V depending differentiably on 1-parameter for the cross-section c_0 if $\zeta_t(x, t)=t$, and for each fixed t, $\zeta_x(x, t)$ is a local automorphism of V for c_0, where $\zeta(x, t)=(\zeta_x(x, t), \zeta_t(x, t))$.*

From the definition of the map C(§5.), local automorphisms on V depending differentiably on 1-parameter for c_0 are local automorohisms on V depending differentiably on 1-parameter for the $(\tilde{B}, \tilde{\Gamma})$-structure $\tilde{s}_0=C(c_0)$. Then, the sheaf $[\mathfrak{M}]$ of germs of local automorphisms depending differentiably for the given cross-section c_0 of $\mathfrak{B}$ is isomorphic to the sheaf $[A]$ for $C(c_0)$.

Therefore, from Proposition 3 and Proposition 4, we have

Theorem 3. *There is a one-to-one correspondence between the set of local equivalence classes of locally trivial deformations of the cross-section c_0 of $\mathfrak{B}$ and the cohomology set $H^1(M, [\mathfrak{M}])$.*

§ 7. Remarks

1. The fibre bundle of positive definite symmetric tensors of the differentiable manifold M is a fibre bundle associated to the differentiable structure of M and its cross-sections are Riemannian metrices on M. In this case, our sheaf $[\mathfrak{M}]$ is the sheaf of germs of motions depending differentiably on 1-parameter for the given Riemannian metric g_0.

2. Though we have discussed "1-parameter" to simplify the

exposition, our theory is valid for "m-parameter" by taking I^m as the parameter space.

REFERENCES

[1] Chern, S. S. Pseudo-groupes continus infinis, Colloques de géométrie différentielle, Strasbourg, C.N.R.S. (1953) p.p. 119-136.

[2] Dedecker, P. Introduction aux structures locales, Colloques de géometrie différentielle globale, Bruxelles, C.B.R.M. (1958) p.p. 103-135.

[3] Ehresmann, C. Structures locales et structures infinitésimales, Comptes Rendus, 234 (1952) p.p. 587-589.

[4] ———, Introduction à la théorie des structures infinitésimales et des pseudo-groupes de Lie, Colloque de géométrie différentielle, Strasbourg, C.N.R.S. (1953), p.p. 97-110.

[5] ———, Catégories topologique et catégories différentiables, Colloque de géométrie différentielle globale, Bruxelles, C.B.R.M. (1958) p.p. 137-150.

[6] Gray, J. W. Abstract theory of preudo-grougs. Reports seminor in topology, part II. (X), Univ. of Chicago. (1957).

[7] ———, Some global properties of contact structures. Ann. of Math., 69 (1959), p.p. 421-329.

[8] Haefliger, A. Structures feuilletées et cohomologie à valeur dans un faiceau de groupoïdes. Comment. Math. Helv., 32 (1957-1958), p.p. 248-329.

[9] Spencer, D. C. Some remarks on perturbation of structure. "Analytic function" Princeton Univ. press, (1960).

[10] ———, Homological analysis and structures. Differential Geometry, Proceedings of Symposia in pure mathematics, III (1961) p.p. 56-86.