On homotopy groups of $S^3$-bundles over spheres

By
Hiroshi Toda

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§ 1. Statement of results

We shall consider the $p$-primary components of the homotopy groups of a cell complex

$$B(p) = S^3 \cup e^{2p+1} \cup e^{2p+1}$$

having the cohomology ring $(\mathcal{O}^1 = Sq^2$ if $p = 2) \mod p$

$$H^*(B(p), \mathbb{Z}_p) = \Lambda(u, \mathcal{O}^1u), \quad u \in H^3(B(p), \mathbb{Z}_p).$$

The existence of such a complex $B(p)$ is provided by an $S^3$-bundle over a $(2p + 1)$-sphere $S^{2p+1}$ with a characteristic class $\alpha \in \pi_{2p}(S^3)$ of a non-trivial mod $p$ Hopf invariant [12].

Denote by $X_p$ the 3-connective fibre space over $B(p)$ Then

$$\pi_i(X_p) \approx \pi_i(B(p)) \quad \text{for} \quad i > 3$$

and we have

**Theorem 1.** $H^*(X_p, \mathbb{Z}_p) = \Lambda(a, \mathcal{O}^p a) \otimes \mathbb{Z}_p[b]$, where $a \in H^{2p+1}(X_p, \mathbb{Z}_p)$ and the relation $\Delta b = \mathcal{O}^p a$ holds $(\Delta = Sq^1$ and $\mathcal{O}^p = Sq^i$ if $p = 2)$.

Denote by $\mathcal{C}$ the class of the finite abelian groups without $p$-torsion, then by use of Serre’s $\mathcal{C}$-theory [9], it follows from the theorem the following

**Corollary.** There is a mapping $g : S^{2p+1} \to B(p)$ which induces $\mathcal{C}$-isomorphisms $g_* : \pi_i(S^{2p+1}) \to \pi_i(B(p))$ for $3 \leq i \leq 2p^2 - 1$. 
As a space of paths in the mapping-cylinder of \( g \), we have a space \( Y_p \) which is a fibre of a fibering equivalent to \( g \) and also which is the total space of a fibering \( \pi : Y_p \rightarrow S^{2p+1} \) of a fibre \( \Omega(B(p)) \). Then we have an exact sequence

\[
\cdots \rightarrow \pi_{i-1}(B(p)) \rightarrow \pi_i(Y_p) \xrightarrow{\pi_*} \pi_i(S^{2p+1}) \xrightarrow{g_*} \pi_i(B(p)) \rightarrow \cdots
\]

Let \( f : S^n \rightarrow S^n \), \( n=2p^2-1 \), be a mapping of degree \( p \) and let \( Z_f = S^n \cup S^n \times (0,1] \) be the mapping-cylinder of \( f \). By shrinking \( S^n = S^n \times (1) \) to a point, we have a mapping-cone \( C_f = Z_f / S^n \) of \( f \). Let \( p : Z_f \rightarrow C_f \) be the shrinking map.

**Theorem 2.** There exists a mapping \( h \) of \( C_f \) into \( Y_p \) satisfying the following conditions. The composition \( h \circ p \) induces \( \mathbb{C} \)-isomorphisms \( (h \circ p)_* : \pi_i(Z_f, S^{2p+1}) \rightarrow \pi_i(Y_p) \) for \( 3 \leq i \leq 2p^2-2 \). A mapping-cone of \( \pi \circ h \) is a cell complex \( S^{2p+1} \cup e^{2p^2} \cup e^{p^2+1} \) with non-trivial \( \Delta \) and \( \Theta^p \), and the restriction \( \pi \circ h|S^{2p+1} \) represents an element of order \( p \) in \( \pi_2(S^{2p+1}) \approx Z_p \).

Denote by \( p\pi_i(B(p)) \) the \( p \)-primary component of \( \pi_i(B(p)) \), then the explicit value of it is given as follows.

**Theorem 3.** \( p\pi_{2p^2-1}(B(p)) \approx Z_p \) for \( 1 \leq i < 2p \) and \( i \neq p \), \( p\pi_{2p^2-2p+1}(B(p)) \approx Z_p^2 \), \( p\pi_{2p^2-2p+2j-1}(B(p)) \approx Z_p \) for \( 2 \leq j < p \), \( p\pi_k(B(p)) = 0 \) otherwise for \( k < 2p + 4p(p-1) - 3 \).

These results can be applied to compute the homotopy groups of Lie groups by use of the following \( \mathbb{C} \)-isomorphisms:

\( (1.4) \quad \pi_i(SU(p+1)) \approx \pi_i(S) \oplus \pi_i(S) \oplus \cdots \oplus \pi_i(S^{p-1}) \oplus \pi_i(B(p)) \),

\( (1.5) \quad \pi_i(Sp\left(\frac{p+1}{2}\right)) \approx \pi_i(SO(p+2)) \approx \pi_i(S) \oplus \pi_i(S) \oplus \cdots \oplus \pi_i(S^{p-3}) \oplus \pi_i(B(p)) \) for odd \( p \),

\( (1.6) \quad \pi_i(G_2) \approx \pi_i(B(5)) \) for \( p = 5 \).
§ 2. Proof of Theorem 1

We have two fiberings:

\[ p : X_p \to B(p) \]

with fibre \( K(Z, 2) \)

\[ p' : B'(p) \to K(Z, 3) \]

with fibre \( X_p \),

where \( K(Z, n) \) denotes Eilenberg-MacLane space of type \( (Z, n) \) and \( B'(p) \) has the same homotopy type as \( B(p) \).

Let \( (E^*_p) \) be the cohomological spectral sequence with the coefficient \( Z_p[7] \) associated with the first fibering, then

\[ E^*_p \cong H^*(B(p), Z_p) \otimes H^*(Z, 2 ; Z_p) \cong \Lambda(u, \mathcal{P}'u) \otimes Z_p[v], \]

\[ v \in H^2(Z, 2 ; Z_p). \]

By concerning the dimensions of the elements of \( \Lambda(u, \mathcal{P}'u) \), we have that the coboundary \( d_r \) is trivial except for \( r = 3, 2p+1, 2p+4. \) Thus \( E^*_p = E^*_3, E^*_4 = E^*_2p+1, E^*_2p+2 = E^*_2p+4 \) and \( E^*_2p+5 = E^*_5. \)

Since \( X_p \) is a \( 3 \)-connective fibering, the generator \( v \) can be chosen such that \( d_3(1 \otimes v) = u \otimes 1. \) Then \( d_3(x \otimes v^n) = n(xu \otimes v^{n-1}) \) for \( x \in \Lambda(u, \mathcal{P}'u). \) Hence we have the following isomorphism, by means of the cup-product,

\[ \Lambda(\mathcal{P}'u \otimes 1, u \otimes v^{p-1}) \otimes Z_p[1 \otimes v^p] \cong H(E^*_p) = E^*_p = E^*_{2p+1}. \]

Since the transgression commutes with the operation \( \mathcal{P}' \) and since \( \mathcal{P}'v = v^p \), we have \( d_{2p+1}(1 \otimes v^p) = \mathcal{P}'u \otimes 1 \) and \( d_{2p+1}(u \otimes v^{p-1}) \in E^*_{2p+1} \otimes 1 = \mathcal{P}'u \otimes 1 \). Thus \( d_{2p+1}(1 \otimes v^p) = m(\mathcal{P}'u \otimes v^{(m-1)p}) \) and \( d_{2p+1}(u \otimes v^{p-1}) = (m-1)(u \otimes \mathcal{P}'u \otimes v^{(m-1)p}). \) It follows that

\[ \Lambda(u \otimes v^{p-1}, \mathcal{P}'u \otimes v^{p-1} \otimes Z_p[1 \otimes v^p]) \cong H(E^*_p) = E^*_p = E^*_{2p+4}. \]

Finally, the triviality of \( d_{2p+4} \) is easily seen, and \( E^*_p = E^*_{2p+4} \) is a graded ring associated with \( H^*(X_p, Z_p). \) Thus we have obtained

\[
\begin{equation}
H^*(X_p, Z_p) = \Lambda(a, c) \otimes Z_p[b],
\end{equation}
\]

where \( a, c \) and \( b \) correspond to \( u \otimes v^{p-1}, \mathcal{P}'u \otimes v^{(p-1)p} \) and \( 1 \otimes v^p \), respectively.

Next consider the spectral sequence \( (E^*_p) \) associated with the second fibering \( p' : B'(p) \to K(Z, 3). \) \( E^*_p \cong H^*(Z, 3 ; Z_p) \otimes H^*(X_p, Z_p). \)
By Cartan’s results [3], \( H^*(Z, 3; \mathbb{Z}_p) = \Lambda(u, \varphi^1 u, \varphi^p \varphi^1 u, \ldots) \otimes Z_p[\Delta \varphi^1 u, \Delta \varphi^p \varphi^1 u, \ldots] \) for odd \( p \) and \( H^*(Z, 3; \mathbb{Z}_2) = \mathbb{Z}_2[\Lambda u, Sq^1 u, Sq^q \varphi^1 u, \ldots] \), where \( u \) is the fundamental class.

It is easy to see that \( d_*(1 \otimes a) = 0 \) for \( r < 2p + 2 \). Then \( E^{3,2p+1}_{2p+2} = 0 \). Since \( H^{2p+2}(B(p), \mathbb{Z}_p) = 0 \), \( E^{2p+2,0}_{2p+3} = E^{2p+2,0}_{2p+3} = 0 \). The element \( \Delta \varphi^1 u \otimes 1 \) is not a \( d_* \)-image for \( r < 2p + 2 \). Thus it has to be a \( d_{2p+2} \)-image. By changing the coefficient of \( a \), if it is necessary, we have that

\[
d_{2p+2}(1 \otimes a) = \Delta \varphi^1 u \otimes 1 \quad (= Sq^1 u \otimes 1 = u^2 \otimes 1 \quad \text{for} \quad p = 2).
\]

By Adem’s relation [1], [4], \( \varphi^p(\Delta \varphi^1 u) = \Delta \varphi^p \varphi^1 u \) for odd \( p \) and \( Sq^1 Sq^q u = Sq^q Sq^q u = (Sq^q u)^2 \). Then \( \varphi^p a \) is transgressive and

\[
d_{2p+2}(1 \otimes \varphi^p a) = \Delta \varphi^p \varphi^1 u \otimes 1 \quad (d_0(1 \otimes Sq^1 a) = (Sq^q u)^2 \otimes 1).
\]

The element \( \Delta \varphi^p \varphi^1 u \otimes 1 \) is not a \( d_* \)-image for \( r < 2p^2 + 2 \). This shows that \( \varphi^p a = 0 \) and we can replace \( c \) by \( \varphi^p a \) in (2.1).

It is checked directly that \( d_*(1 \otimes b) = 0 \) for \( r \leq 2p + 2 \). Then it is verified that \( E^*_{2p+2} = E^*_{2p+2} \) and that

\[
E^*_{2p+3} = \Lambda(u, \varphi^1 u, \varphi^p \varphi^1 u, \ldots) \otimes Z_p[\Delta \varphi^p \varphi^1 u, \ldots] \otimes \Lambda(c) \otimes Z_p[b],
\]

\[
(p: \text{odd})
\]

\[
E^*_{2p} = \Lambda(u) \otimes Z_p[Sq^1 u, Sq^q Sq^q u, \ldots] \otimes \Lambda(c) \otimes Z_p[b] \quad (p = 2).
\]

\( \varphi^p \varphi^1 u \) is not a \( d_* \)-image for \( r < 2p^2 + 1 \), but it is a \( d_* \)-image for \( r = 2p^2 + 1 \) since \( H^*(B(p), \mathbb{Z}_p) = E^*_{2p^2} = E^*_{2p^2+1} = 0 \) for \( r = 2p^2 + 1 \).

By changing the coefficient of \( b \), if it is necessary, we have that

\[
d_{2p+2}(1 \otimes b) = \varphi^p \varphi^1 u \otimes 1 \quad (= Sq^1 Sq^q u \otimes 1 \quad \text{for} \quad p = 2).
\]

Since the Bockstein operation \( \Delta \) commutes with the transgression, we have

\[
(2.2) \quad \Delta b = c = \varphi^p a \quad (Sq^1 b = c = Sq^q a \quad \text{for} \quad p = 2),
\]

where the elements \( a, b, c \) are different only in coefficients \( \equiv 0 \) from those in (2.1).

Consequently we have proved Theorem 1.
§ 3. Proof of Theorem 2

The space $X_p$ is a homology $(2p+1)$-sphere mod $p$, by Theorem 1, for dimensions $< 2p^2$ and 3-connected. By Serre's $C$-theory, $\pi_i(S^{2p+1})$ is $C$-isomorphic to $\pi_i(X_p)$ for $i < 2p^2 - 1$, by a homomorphism $g'_* \Rightarrow$ induced by a representative $g' : S^{2p+1} \to X_p$ of an element of $\pi_{2p+1}(X_p)$ not divisible by $p$.

Then Corollary to Theorem 1 is proved by taking $g$ as the composition of $g'$ and the 3-connective fibering $: X_p \to B(p)$.

In order to prove Theorem 2, we may replace $Y_p$ by a 2-connective fibre space $Y'_p$ over $Y_p$, whence $B(p)$ in (1.3) may be replaced by $X_p$.

The space $Y'_p$ is given as follows. Let $Z_g' = X_p \cup S^{2p+1} \times (0, 1]$ be the mapping cylinder of $g'$. Then $Y'_p$ is the set of paths $(I, 0, 1) \to (Z_g', S^{2p+1}, \ast)$. The paths $(I, 0, 1) \to (Z_g', S^{2p+1}, Z_g')$ form a fibre space over $Z_g'$ with a fibre $Y'_p$. Consider a spectral sequence $(E^*_{*})$ associated with this fibering, then $E^2_{*} \cong H^*(X_p, Z_p) \otimes H^*(Y'_p, Z_p)$ and $E^\infty_{*} \cong H^*(S^{2p+1}, Z_p)$. We shall prove the following lemma

(3.1). There exists an element $w$ of $H^{2p^2-1}(Y'_p, Z_p)$ such that $H^*(Y'_p, Z_p)$ is isomorphic to $\Lambda(w) \otimes Z_p[\Delta w]$ for dimensions less than $2p^2$.

By a simple computation of the spectral sequence, we have that $b$ and $\Delta b = \Theta^p a$ are transgression images of $w$ and $\Delta w$, i.e., $d_n(1 \otimes w) = b \otimes 1$ and $d_{n+1}(1 \otimes \Delta w) = \Theta^p a \otimes 1$, $n = 2p^2$, for suitable choice of $w$. Construct a formal spectral sequence $(E^\infty_{*})$ with the above $d_n$, $d_{n+1}$ and $E^2_{*} = H^*(X_p, Z_p) \otimes (\Lambda(w) \otimes Z_p[\Delta w])$. The spectral sequence is well-defined for dimensions less than $2p^2$ and the final term is $E^\infty_{*} = \Lambda(a \otimes 1)$. Comparing $E^\infty_{*}$ with $E^2_{*}$, it follows that (3.1) is true (cf. [16]).

By generalized Hurewicz theorem in $C$-theory, $\pi_{2p^2-1}(Y'_p)$ is $C$-isomorphic to $Z_p$ and there exists a mapping

$h' : S^{2p^2-1} \to Y'_p$

such that $h'^* : H^{2p^2-1}(Y'_p, Z_p) \cong H^{2p^2-1}(S^{2p^2-1}, Z_p)$ and the composi-
H"o's f is homotopic to zero.

Let $S$ be a space consists of pairs $(l, s)$ of paths $l : I \to Y'_p$ and points $s$ of $S^{2p^2 - 1}$ such that $l(1) = h'(s)$. $S$ is a fibre space over $Y'_p$ with the projection $\pi_0$ given by $\pi_0(l, s) = h'(s) = l(1)$. By setting $i(s) = (l_s, s)$, $l_s(I) = h'(s)$, we have an injection $i$ of $S^{2p^2 - 1}$ into $S$ which is a homotopy equivalence. Then

$h' = \pi_0 \circ i$.

Let $F = \pi^{-1}_0(*)$ be a fibre. Since $h' \circ f$ is homotopic to zero, then the injection $i$ is extended to

$k : Z_f \to S$, \quad $k | S^{2p^2 - 1} = i$,

such that $k(S^{2p^2 - 1}) \subset F$. There exists uniquely a mapping $h_0$ such that the diagram

\[
\begin{array}{ccc}
(Z_f, S^{2p^2 - 1}) & \xrightarrow{k} & (S, F) \\
\downarrow \phi & & \downarrow \pi_0 \\
(C_f, *) & \xrightarrow{h_0} & (Y'_p, *)
\end{array}
\]

is commutative. $h_0$ is an extension of $h'$.

We shall prove

(3.2). The restriction $k_o = k | S^{2p^2 - 1} : S^{2p^2 - 1} \to F$ induces isomorphisms $H^i(F, Z_p) \approx H^i(S^{2p^2 - 1}, Z_p)$ for $i < 2p^2 - 1$.

Consider a spectral sequence $(E^*_r)$ associated with the fibering $\pi_0 : S \to Y'_p$, then $E^{*}_2 \approx H^*(Y'_p, Z_p) \otimes H^*(F, Z_p)$ and $E^{*}_2 \approx H^*(S, Z_p) \approx H^*(S^{2p^2 - 1}, Z_p)$.

Let $n = 2p^2 - 1$. First we have easily that $H^i(F, Z_p) = E^{0,i}_2 = 0$ for $i < n$. Since $E^{*}_2$ is equivalent to $h^*$, we have that $E^{*}_{2,0} \approx H^*(Y'_p, Z_p) \approx Z_p$ is mapped isomorphically onto $E^{*}_{2,0} \approx H^*(S, Z_p)$. Then it follows that $H^*(F, Z_p) \approx E^{0,n}_2$ is isomorphic to $Z_p$ and generated by an element $x$ such that $d_{n+1}(x \otimes x) = (\Delta w) \otimes 1$. Thus $d_{n+1}((\Delta w)^k \otimes x) = (\Delta w)^{k+1} \otimes 1$ and $d_{n+1}(w \cdot (\Delta w)^k \otimes x) = w \cdot (\Delta w)^{k+1} \otimes 1$.

This shows that $E^{0,i}_r = E^{r,i}_2 = 0$ for $r > n + 2$, $s \leq n$ and $n < t + s < 2p^2$.

Let $y \in H^i(F, Z_p)$ be a non-zero element of minimum $i > n$. If $i < 2p^2 - 1$, then it is easily seen that $d_{r}(1 \otimes y) = 0$ for all $r \geq 2$,
On homotopy groups of $S^3$-bundles over spheres

and thus $E_n^i = 0$. But this contradicts to $H^i(S, Z_p) = 0$. We have obtained $H^i(F, Z_p) = 0$ for $n < i < 2p^2 - 1$.

Now, it is sufficient to prove that $k\#: H^i(F, Z_p) \to H^i(S^n, Z_p)$, $n = 2p^2 - 1$, is an isomorphism. $h\#: H^i(Y'_p, Z_p) \to H^i(C_f, Z_p)$ is equivalent to $h\#: H^i(Y'_p, Z_p) \to H^i(S^n, Z_p)$ and it is an isomorphism. By the naturality of $\Delta$, it follows that $h\#: H^{i+1}(Y'_p, Z_p) \cong H^{i+1}(C_f, Z_p)$. Also we have isomorphisms $p\#: H^i(C_f, Z_p) \cong H^i(Z_f, S^n; Z_p)$ and $\pi_i\#: H^i(Y'_p, Z_p) \cong H^i(S, F; Z_p)$ for $i = n, n + 1$.

Then, by the commutativity of the previous diagram, we have isomorphisms $k\#: H^i(S, F; Z_p) \cong H^i(Z_f, S^n; Z_p)$ for $i = n, n + 1$. Since $k: Z_f \to S$ is a homotopy equivalence, we have $H^*(S, Z_p) \cong H^*(Z_f, Z_p)$. By applying the five lemma, we have that $k\#: H^*(F, Z_p) \to H^*(S^n, Z_p)$ is an isomorphism onto. This completes the proof of (3.2).

By generalized J.H.C. Whitehead's theorem in $\mathfrak{C}$-theory, it follows from (3.2) that $k_{0\#}: \pi_i(S^n) \to \pi_i(F)$ is a $\mathfrak{C}$-isomorphism for $i < 2p^2 - 2$ and a $\mathfrak{C}$-onto for $i \leq 2p^2 - 2$. Since $k$ is a homotopy equivalence, $k\#: \pi_i(Z_f) \cong \pi_i(S)$ for all $i$. By the five lemma, we have

$$(3.3) \quad (h_0 \circ h)^\#: \pi_i(Z_f, S^n; Z_p) \to \pi_i(S, F) \cong \pi_i(Y'_p)$$

is a $\mathfrak{C}$-isomorphism onto for $i \leq 2p^2 - 2$.

Let $h: C_f \to Y_p$ be the composition of $h_0$ and the 2-connective fibering of $Y'_p$ onto $Y_p$. Then the first assertion of Theorem 2 is proved.

The composition $\pi \circ h$ in Theorem 2 coincides with the composition of $h: C_f \to Y'_p$ and a fibering $\pi': Y'_p \to S^{2p+1}$ given by $\pi'(l) = l(0)$, $l \in Y'_p$. Let $W = S^{2p+1} \cup e^{2p^2} \cup e^{2p^2+1}$ be a mapping cone of $\pi \circ h$. Since the image of each point of $C_f$ under $h_0$ is a path $l: (I, 0, 1) \to (Z_g', S^{2p+1}, *)$, $h_0$ defines a mapping

$$H: W \to Z_g'$$

such that $H|S^{2p+1}$ is the identity and that $H$ induces a mapping of paths $\Omega(H): \Omega(W, S^{2p+1}) \to Y'_p$ with $\Omega(H)|C_f = h_0$, where $\Omega(W, S^{2p+1}) = \{ l: (I, 0, 1) \to (W, S^{2p+1}, *) \}$ and each point $x$ of $C_f$ is identified with a path $x \times [0, 1]$ in $W$.

Then it is verified that, for dimensions less than $2p^2 + 2p - 2$,
the mappings $h$, $\Omega(H)$ and $H$ induces isomorphisms of the cohomology groups mod $p$. Since $X_p$ is a deformation retract of $Z_{g'}$, it follows from Theorem 1 that $\Delta = 0$ and $\partial_p = 0$ in $W$. This proves the second assertion of Theorem 2.

Let $\beta \in \pi_z(S^{2p+1})$ be the class of the restriction $\pi \circ h|S^{2p+1}$.

Assume that $p$ is odd and $\beta = 0$. Then $W$ is homotopy equivalent to a complex $W' = (S^{2p+1} \vee S^{2p}) \cup e^{2p+1}$. Then $\partial_p = 0$ in $W'/S^{2p} = S^{2p+1} \cup e^{2p+1}$. But this contradicts the non-existence of non-trivial mod $p$ Hopf invariant in $\pi_z(S^{2p+1})$ [12]. Thus $\beta = 0$ for odd prime $p$ and the last assertion of Theorem 2 is proved for odd $p$.

The last assertion of Theorem 2 for $p = 2$ will be proved in the next section.

§ 4. $B(2)$

In this section, we consider the case $p = 2$.

We first consider $SU(3)$ which is one of $B(2)$, since the characteristic class for the bundle $p:SU(3) \to S^5$ is the generator $\eta_3$ of $\pi_i(S^3) \approx Z_2$.

We shall compute the following result.

\[ i \begin{array}{cccccccccc} 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \pi_i(SU(3)) \approx Z & Z_6 & 0 & Z_{12} & 0 & Z_6. \end{array} \]

This follows from the exact sequence

\[ \cdots \to \pi_{i+1}(S^3) \xrightarrow{\delta} \pi_i(S^3) \xrightarrow{i_*} \pi_i(SU(3)) \xrightarrow{i_*} \pi_i(S^3) \to \cdots \]

of the bundle and the following results (cf. [15]),

\[ i \begin{array}{cccccccc} 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \pi_{i+1}(S^3) \approx Z & Z_6 & Z_6 & Z_6 & 0 & Z_6 & Z_6 & Z_6. \\
\pi_i(S^3) \approx Z_{12} & Z_6 & Z_6 & Z_6 & Z_6 & Z_6 & Z_6 & Z_6 & Z_6, \end{array} \]

where $\delta$ satisfies the relation $\delta(E\alpha) = \eta_3^p \alpha$ for $\alpha \in \pi_i(S^3)$. It is sufficient to show that $\delta: \pi_{i+1}(S^3) \to \pi_i(S^3)$ is not trivial for
On homotopy groups of $S^3$-bundles over spheres

In the notations of [15], we have non-trivial $\partial$-images: $\partial(\iota_6) = \eta_3$, $\partial(\iota_8) = \eta_3^2$, $\partial(\iota_9) = 2\nu'$, $\partial(\nu_4) = \nu_5 \circ \nu_4 = \nu' \circ \eta_5$, and $\partial(\nu_4 \circ \nu_4) = \eta_5 \circ \nu_4 \circ \eta_4 = \nu' \circ \eta_5^2$. Thus (4.1) is computed.

Next we prove

(4.2). The homotopy groups of $B(2)$ and $SU(3)$ are $\mathbb{C}$-isomorphic to each other.

Consider 5-skeleton $S^5 \cup e^5$ of $B(2)$ which has non-trivial $Sq^5$. The homotopy type of $S^5 \cup e^5$ is characterized by $Sq^5$. Thus any $B(2)$ has the same homotopy type of a complex

$$(S^5 \cup e^5) \cup _\gamma e^8,$$

in which $e^8$ is attached to a representative of a class $\gamma$ of $\pi_7(S^3 \cup e^8)$.

Since $\pi_7(SU(3)) = 0$ by (4.1), then the injection of $S^5 \cup e^5$ into $SU(3)$ can be extended over a mapping $f : B(2) \rightarrow SU(3)$ which induces isomorphisms of homology groups of dimensions less than 8. By considering the ring structure mod 2 for $B(2)$ and $SU(3)$, it follows that $f$ induces isomorphisms of the cohomology groups mod 2 and thus $\mathbb{C}$-isomorphisms of the homotopy groups.

Consider the exact sequence (1.3), in particular,

$$\pi_*(Y_5) \xrightarrow{\pi_*} \pi_*(S^5) \xrightarrow{g_*} \pi_*(B(2)).$$

$g_*$ is trivial since $\pi_7(S^5) \approx \mathbb{Z}$ and the 2-component of $\pi_7(B(2))$ vanishes by (4.1) and (4.2). Thus $\pi_*$ is onto. It follows from the first assertion of Theorem 2 that the last assertion of Theorem 2 is true for $p=2$.

§ 5. Some results in unstable homotopy groups of spheres

In this section we assume that $p$ is an odd prime. First we recall the following results from Theorem 8.3 of [13].

(5.1) Let $m$ be sufficiently large integer, then

$$p^i \pi_{2m+i-2(p-1)}(S^{2m+i}) \approx \mathbb{Z}_p$$

for $1 \leq i \leq 2p-1$ and $i \neq p$, $p^i \pi_2(S^{2m+i}) \approx \mathbb{Z}_p^2$, $p^i \pi_2(S^{2m+i}) \approx \mathbb{Z}_p$.
\[ p^{\pi_{2m+1+k}(p+1)(p-1)-2}(S^{2m+1}) \approx Z_p \]

and

\[ p^{\pi_{2m+1+k}(S^{2m+1})} = 0 \quad \text{otherwise for } k \leq 4p(p-1)-4. \]

In the exact sequence

\[(5.2) \quad \cdots \to \pi_{i+1}(\Omega^2(S^{2m+1}), S^{2m-1}) \to \pi_i(S^{2m-1}) \xrightarrow{E^2} \pi_{i+2}(S^{2m+1}) \to \pi_i(\Omega^2(S^{2m+1}), S^{2m-1}) \to \cdots,\]

we have the following \( C \)-isomorphism, by (8.7)' of [11],

\[(5.3) \quad \pi_i(\Omega^2(S^{2m+1}), S^{2m-1}) \cong \pi_{i+1}(Z_f, S^{2pm-1}) \text{ for } i < 2p^2m-3, \text{ where } Z_f \text{ is the mapping-cylinder of a mapping } f: S^{2pm-1} \to S^{2pm-1} \text{ of degree } p, \]

If \( i < 2mp-2 \), then the groups in (5.3) are finite without \( p \)-torsions. Thus \( E^2: \pi_i(S^{2m+1}) \to \pi_{i+2}(S^{2m+1}) \) are \( C \)-isomorphisms onto for \( i < 2(m+1)p-3 \), and we have

\[(5.1)' \quad (5.1) \text{ is true for } 2n+1 \geq (k+2)/(p-1). \]

For \( m = p \), we have

\[(5.4) \quad p^{\pi_{2p+1+k}(p+1)(p-1)-2}(S^{2p+1}) \approx Z_p \quad \text{for } i = 1, 2, \ldots, p-1, \]

\[ p^{\pi_{2p-1}(S^{2p+1})} \approx Z_p, \]

\[ p^{\pi_{2p-2}(S^{2p+1})} \approx Z_p \quad \text{for } i = 1, 2, \ldots, p-1, \]

\[ p^{\pi_{2p+1+k}(S^{2p+1})} = 0 \quad \text{otherwise for } k < 2p^2-4. \]

Furthermore, we shall prove

\[(5.5) \quad p^{\pi_{2p+2+k}(p+1)(p-1)-2}(S^{2p+1}) \approx Z_p \quad \text{for } i = p+1, p+2, \ldots, 2p-1, \]

\[ p^{\pi_{2p+2+k}(p+1)(p-1)-1}(S^{2p+1}) \approx Z_p \quad \text{for } i = p+1, p+2, \ldots, 2p-1, \]

\[ p^{\pi_{2p+1+k}(S^{2p+1})} = 0 \quad \text{otherwise for } 2p^2-4 \leq k < 4p(p-1)-4. \]

More generally, we shall prove the following (5.6) by decreasing induction on \( j \).

\[(5.6) \quad p^{\pi_{2p+2j+2+k}(p+1)(p-1)-2}(S^{2p+2j+1}) \approx Z_p \quad \text{for } p+1 \leq i \leq 2p-1 \text{ and } 0 \leq j < i-p, \]

\[ p^{\pi_{2p+2j+2+k}(p+1)(p-1)-1}(S^{2p+2j+1}) \approx Z_p \quad \text{for } p+1 \leq i \leq 2p-1 \text{ and } 0 \leq j < i-p \]

\[ p^{\pi_{2p+1+k}(S^{2p+2j+1})} = 0 \quad \text{otherwise for } 2p^2-4 \leq k < 4p(p-1)-4 \text{ and } j \geq 0. \]
On homotopy groups of $S^3$-bundles over spheres

(5.6) is true for sufficiently large $j$, for example $j \geq p$, by (5.1)'. By (5.3), (5.1)' and by (5.2), we have the following exact sequence.

$$
\cdots \to 0 \to p\pi_{2p+2i(j-1)+2k(p-1)}(S^{2p+2j-1}) \to E^2_{p, p+2i(j-1)+2k(p-1)-1}(S^{2p+2j+1}) \to Z_p \to p\pi_{2p+2i(j-1)+2k(p-1)-1}(S^{2p+2j-1}) \to E^2_{p, p+2i(j-1)+2k(p-1)-2}(S^{2p+2j+1}) \to Z_p \to p\pi_{2p+2i(j-1)+2k(p-1)-2}(S^{2p+2j-1}) \to E^2_{p, p+2i(j-1)+2k(p-1)-3}(S^{2p+2j+1}) \to 0 \to \cdots \to 0 \to p\pi_{2p+2i(j-1)+2k(p-1)-3}(S^{2p+2j-1}) \to E^2_{p, p+2i(j-1)+2k(p-1)-4}(S^{2p+2j+1}) \to Z_p \to p\pi_{2p+2i(j-1)+2k(p-1)-4}(S^{2p+2j-1}) \to E^2_{p, p+2i(j-1)+2k(p-1)-5}(S^{2p+2j+1}) \to Z_p \to p\pi_{2p+2i(j-1)+2k(p-1)-5}(S^{2p+2j-1}) \to E^2_{p, p+2i(j-1)+2k(p-1)-6}(S^{2p+2j+1}) \to Z_p \to p\pi_{2p+2i(j-1)+2k(p-1)-6}(S^{2p+2j-1}) \to E^2_{p, p+2i(j-1)+2k(p-1)-7}(S^{2p+2j+1}) \to 0,
$$

where $i = p$, $p+1$, $\ldots$, $2p-1$.

§ 6. Proof of Theorem 3

For the case $p = 2$, Theorem 3 is proved by (4.1) and (4.2).

In the following, we assume that $p$ is an odd prime. By Theorem 2 and (5.1)', we have that $\pi_i(Y_p)$ is finite for $3 \leq i \leq 2p^2 - 2$ and

$$
(6.1) \quad p\pi_{2p+2i(p-1)-1}(Y_p) \approx Z_p \quad \text{for} \quad i = p, p+1, \ldots, 2p-1,
$$

$$
\quad \pi_{2p+2i(p-1)-2i}(Y_p) \approx Z_p \quad \text{for} \quad i = p+1, p+2, \ldots, 2p-1,
$$

and $\pi_k(Y_p) = 0$ otherwise for $k < 2p + 4p(p-1)-3$.

Apply the results (6.1), (5.4) and (5.5) to the exact sequence (1.3), then we see that Theorem 3 is a consequence of the following lemma

$$
(6.2) \quad \text{The homomorphisms} \quad \pi_k: \pi_k(Y_p) \to \pi_k(S^{2p+1}) \quad \text{for} \quad k = 2p + 2i(p-1) - 1, i = p, p+1, \ldots, 2p-1 \quad \text{and} \quad k < 2p + 2p^2 - 4 \quad \text{are isomorphisms of the $p$-components.}
$$
i): The case $k = 2p + 2p(p-1) - 1 = 2p^2 - 1$. In this case, a generator of $p\pi_h(Y_p)$ is represented by $h|\Sigma^4$. By the last assertion of Theorem 2, we have that (6.2) is true for this case.

ii): The case $k = 2p + 2p^2 - 4$. In this case, the image of $\pi_\ast$ contains the composition $\beta \circ \alpha$ of the class $\beta \in \pi_p(S^{2p+1})$ of $\pi_\ast h|\Sigma^{2p^2 - 1}$ and a generator $\alpha$ of $p\pi_h(S^{2p^2-1}) \approx Z_p$. In the stable range, we know in [13] that the composition $E^\ast(\beta \circ \alpha) = E^\ast(\beta) \circ E^\ast(\alpha)$ is not zero. Thus $\pi_\ast$ is not trivial for $p$-components and (6.2) is true for this case.

iii): The cases $k = 2p + 2(p+j)(p-1) - 1$ and $j = 1, 2, \cdots, p-1$.

Let $K = S^{2p^2 - 1} \cup e^{2p^2 - 3}$ be the mapping-cone of a mapping of degree $p$. We may assume that $C_j$ is a three fold iterated suspension $E^3K$ of $K$. Then $\pi \circ h$ defines a mapping $\Omega^3(\pi \circ h) : K \to \Omega^3(S^{2p+1})$. Set $Q = \Omega(\Omega(S^{2p+1}), S^{2p-1})$, then the homomorphism $\pi_{i+1}(S^{2p+1}) \to \pi_i(S^{2p+1}), S^{2p-1})$ in (5.2) is equivalent to a homomorphism $i_\ast : \pi_{i-1}(\Omega(S^{2p+1})) \to \pi_{i-1}(Q)$ induced by the natural injection $i$.

Since the class of $\pi \circ h|\Sigma^{2p^2 - 1}$ is an $E^2$-image, $\Omega^3(\pi \circ h)|\Sigma^{2p^2 - 4}$ is homotopic to zero. Thus $\Omega^3(\pi \circ h)$ is factorized to $K \to S^{2p^2 - 3} \to Q$.

Next we have

(6.3). $H^\ast(Q, Z_p)$ is spanned by $1$, $w$ and $\Delta w$ for dimensions less than $4p^2 - 5$, $w \in H^{2p^2 - 1}(Q, Z_p)$.

This follows from the results on $H_\ast(\Omega^3(S^{2p+1}), Z_p)$ in [6].

Then $\pi_{2p^2 - 3}(Q)$ is $C$-isomorphic to $Z_p$. Thus $\Omega^3(\pi \circ h)$ is homotopic to the composition of a mapping $q : K \to EK$ and a mapping $g : EK \to Q$ such that $q(S^{2p^2 - 1}) = \ast$ and $q^\ast : H^n(EK, Z_p) \approx H^n(K, Z_p)$ for $n = 2p^2 - 3$. We prove

(6.4). $g$ induces isomorphism of cohomology groups mod $p$ and thus $C$-isomorphisms of homotopy groups for dimensions less than $4p^2 - 6$.

It is sufficient to prove that $g|S^{2p^2 - 3}$ is not homotopic to zero. Assume that $g|S^{2p^2 - 3}$ is homotopic to zero. Then $\Omega^3(\pi \circ h)$ is homotopic to zero in $Q$. It follows that $\Omega^3(\pi \circ h) : EK \to \Omega^3(S^{2p+1})$ is homotopic to a mapping into $S^{2p-1}$. Let $L = S^{2p-1} \cup e^{2p^2 - 2} \cup e^{2p^2 - 1}$ be the mapping-cone of the last mapping. Then the mapping-cone $S^{2p+1} \cup e^{2p^2} \cup e^{2p^2 - 1}$ of $\pi \circ h$ in Theorem 2 is homotopy equivalent to
On homotopy groups of \( S^3 \)-bundles over spheres

205

\( E^2L \). Then \( \varphi^p \mid L \) in \( E^2L \) and thus \( \varphi^p \mid L \) in \( L \). But \( \varphi^p H^{2p-1} \) (\( Z_p \)) = 0 in general. We have a contradiction, hence \( g \mid S^{2p^2-2} \) is not homotopic to zero and (6.4) is proved.

Now consider an element \( \gamma \) of \( \pi_{k-3}(K) \) such that, by shrinking \( S^{2p^2-1} \) to a point, \( \gamma \) is carried to a generator of \( \pi_{k-3}(S^{2p^2-3}) \). Then \( \gamma \) is not trivial for the case iii) and it is an isomorphism of the \( p \)-components.

Consequently, Theorem 3 has been proved.

§ 7. Remarks on homotopy groups of Lie groups

Since \( \pi_{2n}(S^{2k+1}) \) is finite and has no \( p \)-torsion if \( k < n < p \), it follows from the exact sequence for the bundle \( SU(k+1) \to S^{2k+1} = SU(k+1)/SU(k) \) that \( \pi_{2n}(SU(k+1)) \) is finite and has no \( p \)-torsion.

From the exactness of the sequence \( \pi_{2n+1}(SU(n+1)) \to \pi_{2n}(SU(n+1)) \to \pi_{2n}(SU(n+1)) \to \pi_{2n}(SU(n+1)) \), we have that if \( p < n \) then there exists a mapping \( f_n : S^{2n+1} \to SU(n+1) \) such that the mapping degree of the composition \( \pi_0 f_n : S^{2n+1} \to S^{2n+1} \) is prime to \( p \). The multiplication in \( SU(n+1) \) and the mappings \( f_1, f_2, \ldots, f_n \) define a mapping

\[ f : S^1 \times S^1 \times \cdots \times S^{2n+1} \to SU(n+1). \]

Then it is verified that \( f \) induces isomorphisms of the cohomology groups mod \( p \) and thus \( C \)-isomorphisms

\[ f^* : \pi_i(S^3) \oplus \pi_i(S^6) \oplus \cdots \oplus \pi_i(S^{2n+1}) \to \pi_i(SU(n+1)) \text{ for all } i \text{ and for } n < p. \]

We have also that \( \pi_{2p}(SU(p)) \) is finite and the injection homomorphism : \( \pi_{2p}(SU(2)) \to \pi_{2p}(SU(p)) \) is an onto map of the \( p \)-components. This injection homomorphism is equivalent to the projection homomorphism : \( \pi_{2p+1}(B_{SU(2)}) \to \pi_{2p+1}(B_{SU(p)}) \). Let \( g : S^{2p+1} \to B_{SU(p)} \) be a mapping which induces the \( SU(p) \)-bundle \( SU(p+1) \to S^{2p+1} \). Then there exists a mapping \( q : S^{2p+1} \to S^{2p+1} \) of the degree prime to \( p \) such that the composition \( g \circ q \) is homotopic to a mapping into \( B_{SU(2)} \). Let \( q : X \to SU(p+1) \) be a bundle map
induced by \( q \). Then \( X \) is equivalent to a \( SU(p) \) bundle, whose group of structure can be reduced into \( SU(2) \). Thus there exists a \( SU(2) \)-bundle \( B(p) \) over \( S^{2p+1} \) such that the diagram

\[
\begin{array}{ccc}
B(p) & \xrightarrow{g'} & SU(p+1) \\
\downarrow & \ & \downarrow \pi \\
S^{2p+1} & \xrightarrow{q} & S^{2p+1}
\end{array}
\]

is commutative, for a mapping \( g' \). By use of \( g' \) and \( f_2, \ldots, f_{p+1} \), construct a mapping

\[
f' : S^s \times \cdots \times S^{2p-1} \times B(p) \to SU(p+1)
\]

as above, then \( f' \) induces isomorphisms of the cohomology groups mod \( p \) and thus \( \mathbb{C} \)-isomorphisms

\[
(1.4) \quad f'_* : \pi_i(S^s) \oplus \cdots \oplus \pi_i(S^{2p-1}) \oplus \pi_i(B(p)) \to \pi_i(SU(p+1))
\]

for all \( i \). By [2], \( \varphi^{1+1} = 0 \) in \( SU(p+1) \). Thus \( B(p) \) satisfies (1.1).

Similarly, we have mappings

\[
f : S^s \times S^t \times \cdots \times S^{m-1} \to Sp(n)
\]

and

\[
f' : S^t \times \cdots \times S^{2p-3} \times B(p) \to Sp\left(\frac{p+1}{2}\right) \quad (p: \text{odd})
\]

which induce \( \mathbb{C} \)-isomorphisms

\[
(7.2) \quad f'_* : \pi_i(S^s) \oplus \cdots \oplus \pi_i(S^{2p-3}) \oplus \pi_i(B(p)) \to \pi_i(Sp(n))
\]

for all \( i \) and for \( p \geq 2n \) and

\[
(1.5) \quad f'_* : \pi_i(S^s) \oplus \cdots \oplus \pi_i(S^{2p-3}) \oplus \pi_i(B(p)) \to \pi_i\left(\frac{p+1}{2}\right)
\]

for all \( i \) and for odd \( p \).

By [5], we have \( \mathbb{C} \)-isomorphisms

\[
(1.5)' \quad \pi_i(Spin(n+2)) \cong \pi_i(SO(n+2)) \cong \pi_i\left(\frac{n+1}{2}\right)
\]

for odd \( n \), odd \( p \) and for all \( i \).

There is a \( G_2 \)-bundle: \( Spin(7) \to S^7 \) with a characteristic class of order 2. Then we have \( \mathbb{C} \)-isomorphisms

\[
\pi_i(G_2) \oplus \pi_i(S^7) \cong \pi_i(Spin(7)) \cong \pi_i(S^7) \oplus \pi_i(B(5))
\]
On homotopy groups of $S^3$-bundles over spheres

for all $i$ and for $p=5$. It follows

\begin{equation}
\pi_i(G_3) \approx \pi_i(B(5)) \quad (p = 5).
\end{equation}

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