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Reduction of models over a discrete valuation ring

By

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Let $\mathfrak{o}$ be a discrete valuation ring of a field k and let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$. Then the notion of a model over $\mathfrak{o}$, defined by Nagata in [4], may be considered in a sense as a complex notion of two models defined over different fields k and $\mathfrak{o}/\mathfrak{p}$ respectively. Let M be a model defined over $\mathfrak{o}$. Then M is a set of spots which dominate $\mathfrak{o}$ or k. If we denote by M_k the set of all the spots in M which dominate k, M_k is an open subset of M and is a model defined over k. On the other hand the closed subset $M-M_k$ corresponds naturally to a (not necessarily irreducible) model defined over $\mathfrak{o}/\mathfrak{p}$. Moreover each spot of $M-M_k$ is obtained as a specialization of one of M_k over $\mathfrak{o}$. Then there arises naturally a question that how the structure of M_k as a model over k is reflected in that of $M-M_k$ as a model over $\mathfrak{o}/\mathfrak{p}$ in this specialization process over $\mathfrak{o}$. This work is initiated by this question.

On the other hand, an algebraic variety defined over k is equivalent to a model defined over k (cf. Chapter 1, §9 in [4]), and a theory of the reduction of algebraic varieties of any dimension with respect to a valuation $\mathfrak{p}$ of a basic field k was developed by Shimura in [8]. In Shimura's theory, the reduction of a variety V is naturally obtained, roughly spoken, by the reduction of defining equations for V, if V is an affine variety or a projective variety with a "*fixed system of coordinates*". However when V is an abstract variety, the reduction of V depends on a choice of affine representatives of V. In other words it is impossible to define a "*canonical*" reduction of V. On the contrary our theory treats, from the first, a model defined over $\mathfrak{o}$ and the reduction process is uniquely determined on a given model. From our point of view, the construction of a $\mathfrak{p}$-variety by Shimura can be considered as follows: Let V be a variety defined over k and let M_0 be the model over k, which is equivalent to V. Then the construction of a $\mathfrak{p}$-variety having V as its underlying variety is essentially nothing but to construct a model M over $\mathfrak{o}$ such that M contains M_0 as an open subset and each spot of $M-M_0$ dominates $\mathfrak{o}$. Grothendieck indicates also this standpoint in [1] (cf. Chap. I, 3.7). For example, let V be an affine variety with a generic point $(x_1, \cdots, x_n)$ over k. Then the canonical reduction of V with respect to this system of coordinates is nothing but to construct the affine model over $\mathfrak{o}$ defined by the affine ring $\mathfrak{o}[x_1, \cdots, x_n]$.

From this point of view, our standpoint seems to be quite natural. Moreover, it is not worthless to point out that our reduction theory may be regarded in some point as an intersection theory of generalized cycles on models over $\mathfrak{o}$ (cf. §3).

Shimura's method in [8] depends on the theory of multiplicity of proper specializations over a local domain, which is a generalization of the specialization theory with respect to a field developed by Weil in [9]. On the contrary our method makes extensive use of the theory of multiplicity in local rings.

In §0, we shall recall the terminologies and the notations on models defined in [4]. In §1, we shall summarize the preliminary results, which will be necessary in the other sections. In §2, we shall investigate some calculi of generalized cycles on models over a discrete valuation ring. In §3, we shall first define an induced spot Q' of a spot Q which does not dominate the ground ring, and then we shall define a multiplicity $\mu(Q; Q')$. Using this multiplicity we shall define two operations ρ and ρ', which correspond to the operation ρ defined in [8]. Moreover we shall see that if the treated model is $\mathfrak{p}$-simple, $\mu(Q; Q')$ may be considered as a multiplicity of Q' as a component of intersection of Q and the generating spot P_0 of the model over $\mathfrak{p}$. In §4, compatibility of ρ' with calculi of generalized cycles will be discussed. In §5, we shall show first that the notion of a $\mathfrak{p}$-variety in the sense of [8] is equivalent to that of a model over $\mathfrak{o}$, and that the operation ρ' is equivalent to the operation ρ defined in [8]. Theorem 5 will play an essential role in the proof.

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§ 0. Terminologies and notations

We use generally the terminologies and the notations in Nagata [4] but we summarize some basic notations for convenience of the readers.

A ring R is called *local* if it is a commutative Noetherian ring with unit and it has a unique maximal ideal $\mathfrak{m}$. To denote these facts we shall simply say that $(R, \mathfrak{m})$ is a local ring. Let $(R, \mathfrak{m})$ be a local ring and let $(S, \mathfrak{n})$ be a local ring contained in R, then we say that $(R, \mathfrak{m})$ *dominates* $(S, \mathfrak{n})$ if $\mathfrak{m} \cap S = \mathfrak{n}$.

Let I be a Dedekind domain or a field. Let L be a finitely generated field over the quotient field of I. A ring A is called an *affine ring* of L over I if A is a finitely generated subring of L over I and has L as the quotient field. A *spot* P of L is a quotient ring of an affine ring A of L with respect to a prime ideal of A. Then the set of all spots L which are rings of quotients of A is called an *affine model* defined by A. A *model* M of L is a set of spots of L satisfying the following conditions: M is a union of a finite number of affine models over I and, for any two spots P and P' of M, we have $P = P'$ if and only if they correspond, i.e., there exists a valuation ring of L which dominates both P and P'. L is called the *function field* of M. It is known that an affine model is a model.[1)]

A spot $(P', \mathfrak{m}')$ is called a *specialization* of a spot $(P, \mathfrak{m})$ if P contains P' and $P = P'_{\mathfrak{m} \cap P'}$. Let P be a spot of a model M. Then

1) See Lemma 1 of Chapter 2, § 11 in [4-I].

we denote by $M(P)$ the set of all spots of M which are specializations of P. $M(P)$ is called the *locus* of P in M. Let $\mathfrak{F}$ be the family of the sets which are unions of a finite number of loci of spots in M. Then we can introduce a topology in M such that $\mathfrak{F}$ is the family of the closed sets in this topology.[2)]

Let $(P, \mathfrak{m})$ be a spot of M and let ϕ_P be the natural homomorphism of P onto $P/\mathfrak{m}$. Let P' be a specialization of P in M. Then $\phi_P(P')$ is a spot of $P/\mathfrak{m}$ over $\phi_P(I)$, and it is known that the set of such spots $\phi_P(P')$ is a model in $P/\mathfrak{m}$ over $\phi_P(I)$. This model is called the *induced model* defined by P. Let I be a discrete valuation ring with the quotient field k. Let M_k be the set of spots of M which contain k. Then M_k is a model of L over k, and is called the *reduced model* of M over k.[3)] M_k is an open set of M.

Let L and K be two function fields contained in a field. If P and Q are spots of L and K respectively over I, the set of spots which are rings of quotients of $P[Q]$ and dominate both P and Q is called the *join* of P and Q; it will be denoted by $J(P, Q)$. Let M and N be models of L and K respectively. Then the union of all $J(P, Q)$, where P and Q run over all spots in M and N respectively, is called the *join* of M and N and will be denoted by $J(M, N)$. It is known that $J(M, N)$ is a model of $L(K)$ over I.[4)]

If the function field L of a model M over I is a regular extension over the quotient field of I, we say that M is an ***absolutely irreducible model*** over I. Let M and N be two absolutely irreducible models over I of the function fields L and K respectively. Then $L\otimes_I K$ has no zero-divisor. Let L^* be the field of quotients of $L\otimes_I K$ and we regard L and K as subfields of L^* in a natural way. Then the *join* of M and N in L^* is called the ***product model*** of M and N and is denoted by $M\otimes N$.[5)]

Let L be a function field over I, and we assume that L is separable over the quotient field of I. Let L' be a finite separable

2) See Chapter 2, §7 in [4-I].
3) See Chapter 2, §8 in [4-I].
4) See Chapter 2, §4 in [4-I].
5) See Chapter 5, §2 in [4-III].

extension of L. Let P be a spot of L and let $\bar{P}$ be the integral closure of P in L'. Then we denote by $N(P; L')$ the set of all the spots which are rings of quotients of $\bar{P}$ with respect to the maximal ideals of $\bar{P}$. If M is a model of L, the union of all $N(P; L')$ where P runs over all spots in M, will be called the *derived normal model of M in L'* and denoted by $N(M; L')$. Then it is known that $N(M; L')$ is a model of L' over I.[6]

Let M be an absolutely irreducible model of the function field L over I. Let M be the union of affine models with affine rings A_i $(i=1, 2, \cdots, n)$. Let I^* be a Dedekind domain or a field containing I and let L^* be the quotient field of $L \otimes_I I^*$. Let A_i^* be the affine ring $A_i \otimes_I I^*$ over I^* $(i=1, 2, \cdots, n)$. Then the union M^* of the affine models defined by A_i^* is a model of L^* over I^*. M^* is called the *extension of M over I^** and is denoted by $M \otimes I^*$.[7]

A valuation ring or a field which is a ring of quotients of I is called a *place* of I. When L is a function field over I, a valuation ring v of L is called a *place* of L if v dominates some place of I. Let M be a model of L. Let v be a place of L. Then if v dominates some spot P of M, P is uniquely determined and is called the *centre* of v in M. A model of L is called to be *complete* if every place of L has a centre in M.

Let $x_0=1, x_1, \cdots, x_n$ be elements of a function field L such that $I[x_1, \cdots, x_n]$ is an affine ring of L. Let M be the union of affine models defined by affine rings $I[x_0/x_i, \cdots, x_n/x_i]$ such that $x_i \neq 0$, respectively. Then M is a complete model of L over I and is called the *projective model* of L defined by the affine coordinates $(x_1, \cdots, x_n)$.[8]

§ 1. Algebraic preliminaries

LEMMA 1. *Let R and R' be two commutative rings with a common subring R''. Let S and S' be multiplicatively closed sets in*

6) See Chapter 2, § 5 in [4-I].
7) See Chapter 5, § 1 in [4-III].
8) See Theorem 5 of Chapter 2, § 2 in [4-I].

R and R' respectively. Moreover $S\otimes S'$ be the set of the elements $s\otimes s'$ in $R\otimes_{R''}R'$, where s and s' run over all the elements of S and S' respectively. Then, $R_S\otimes_{R''}R'_{S'}$ is canonically isomorphic to $(R\otimes_{R''}R')_{S\otimes S'}$.

The proof of this lemma has no essential difficulty. Therefore we omit the proof.

Let R be a ring. Then we denote by $l(R)$ the *length* of R as *R-module.* If S is an R-module, then we denote by $l(S; R)$ or $l(S)$ the *length* of S as *R-module.*

LEMMA 2. *Let $(R, \mathfrak{m})$ and $(R', \mathfrak{m}')$ be two local rings with a common subfield k and we assume that $A=R\otimes_k R'$ is Noetherian. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be primary ideals belonging to $\mathfrak{m}$ and $\mathfrak{m}'$ respectively and let $\mathfrak{P}$ be a minimal prime divisor of $(\mathfrak{m}, \mathfrak{m}')A$. Then we have*

$$l(A_{\mathfrak{P}}/(\mathfrak{q}, \mathfrak{q}')A_{\mathfrak{P}}) = l(A_{\mathfrak{P}}/\mathfrak{m}, \mathfrak{m}')A_{\mathfrak{P}})l(R/\mathfrak{q})l(R'/\mathfrak{q}').$$

PROOF. We shall prove by induction on $l(R/\mathfrak{q})+l(R'/\mathfrak{q}')$. If $l(R/\mathfrak{q})+l(R'/q')=2$, then $\mathfrak{m}=\mathfrak{q}$ and $\mathfrak{m}'=\mathfrak{q}'$ and hence we have nothing to prove. Let $l(R/\mathfrak{q})+l(R'/\mathfrak{q}')$ be larger than 2. Then we may assume $\mathfrak{m}\neq\mathfrak{q}$ and $\mathfrak{q}=\mathfrak{q}'=0$. Let $\bar{\mathfrak{q}}$ be a minimal $\mathfrak{m}$-primary ideal of R different from 0. Then if a is a non-zero element of $\bar{\mathfrak{q}}$, we have $aR=\bar{\mathfrak{q}}$ and $0:aR=\mathfrak{m}$, and hence aR is isomorphic to $R/\mathfrak{m}$ as R-module. Therefore, by induction hypothesis, we have as $A_{\mathfrak{P}}$-module

$$l(\bar{\mathfrak{q}}A_{\mathfrak{P}}) = l((\bar{\mathfrak{q}}R\otimes_k R')_{\mathfrak{P}}) = l((R/\mathfrak{m}\otimes_k R')_{\mathfrak{P}}) = l(A_{\mathfrak{P}}/(\mathfrak{m}, \mathfrak{m}')A_{\mathfrak{P}})l(R').$$

Again by induction hypothesis we have

$$\begin{aligned} l(A_{\mathfrak{P}}) &= l(A_{\mathfrak{P}}/\bar{\mathfrak{q}}A_{\mathfrak{P}})+l(\bar{\mathfrak{q}}A_{\mathfrak{P}}) \\ &= l(A_{\mathfrak{P}}/(\mathfrak{m}, \mathfrak{m}')A_{\mathfrak{P}})l(R')(l(R/\bar{\mathfrak{q}})+1) \\ &= l(A_{\mathfrak{P}}/(\mathfrak{m}, \mathfrak{m}')A_{\mathfrak{P}})l(R)l(R'). \end{aligned}$$

This means that our lemma is true. q.e.d.

LEMMA 3. *Let the notations and assumptions be the same as in Lemma 2. Then we have*

$$\operatorname{rank} A_{\mathfrak{P}} = \operatorname{rank} R+\operatorname{rank} R'.$$

PROOF. Since $(\mathfrak{m}^{2n}, \mathfrak{m}'^{2n})A_{\mathfrak{P}}\subseteq(\mathfrak{m}, \mathfrak{m}')^{2n} A_{\mathfrak{P}}\subseteq(m^n, m'^n)A_{\mathfrak{P}}$ for any positive integer n, we have by Lemma 2

$$l(A_{\mathfrak{P}}/(\mathfrak{m}, \mathfrak{m}')A_{\mathfrak{P}})l(R/\mathfrak{m}^{2n})l(R'/\mathfrak{m}'^{2n}) \geqq l(A_{\mathfrak{P}}/(\mathfrak{m}, \mathfrak{m}')^{2n}A_{\mathfrak{P}})$$
$$\geqq l(A_{\mathfrak{P}}/(\mathfrak{m}, \mathfrak{m}')A_{\mathfrak{P}})l(R/\mathfrak{m}^{n})l(R'/\mathfrak{m}'^{n}).$$

On the other hand it is known that there exist polynomials $f(X)$, $g(X)$ and $h(X)$ such that $f(n)=l(R/\mathfrak{m}^n)$, $g(n)=l(R'/\mathfrak{m}'^n)$ and $h(n)=l(A_{\mathfrak{P}}/(\mathfrak{m}, \mathfrak{m}')^n A_{\mathfrak{P}})$ for any sufficiently large n. It is also known that the degrees of f, g and h are rank R, rank R' and rank $A_{\mathfrak{P}}$ respectively.[9] Then the above inequality means that rank $A_{\mathfrak{P}}$ is equal to rank $R+$rank R'. q.e.d.

LEMMA 4. *Let the notations and assumptions be the same as in Lemma 2. Then* $(\mathfrak{m}, \mathfrak{m}')R\otimes_k R'$ *has no imbedded prime divisors.*

For the proof, see the lemma 2, Chap. 5 in Nagata [4 III].

LEMMA 5. *Let* $(R, \mathfrak{m})$ *and* $(R', \mathfrak{m}')$ *be two local rings with a common discrete valuation ring* $\mathfrak{o}$ *of rank* 1. *We assume that* $A=R\otimes_{\mathfrak{o}}R'$ *is Noetherian and let* $\mathfrak{P}$ *be a minimal prime divisor of* $(\mathfrak{m}, \mathfrak{m}')A$. *Let* π *be a prime element of* $\mathfrak{o}$ *and we assume that* πR *and* $\pi R'$ *are primary ideals belonging to maximal ideals of* R *and* R' *respectively. Then we have*

$$l(A_{\mathfrak{P}}/\pi A_{\mathfrak{P}}) = l(A_{\mathfrak{P}}/(\mathfrak{m}, \mathfrak{m}')A_{\mathfrak{P}})l(R/\pi R)l(R'/\pi R').$$

Proof. This is a direct consequence of Lemma 2, if we notice that $A/\pi A$ is isomorphic to $R/\pi R\otimes_{\kappa}R'/\pi R'$, where κ is the residue class field of $\mathfrak{o}$.

Next we shall list well known results on multiplicities of local rings.

Let $(R, \mathfrak{m})$ be a local ring of rank d and let $\mathfrak{q}$ be a primary ideal belonging to $\mathfrak{m}$. Then there exists a polynomial $\sigma(\mathfrak{q}\,;n)$ of degree d such that, for any sufficiently large n, $\sigma(\mathfrak{q}\,;n)=l(R/\mathfrak{q}^n)$. Let a be the coefficient of n^d in $\sigma(\mathfrak{q}\,;n)$. Then $(d!)a$ is called the *multiplicity* of $\mathfrak{q}$ and is denoted by $e(\mathfrak{q})$.[9]

Let R' be a semi-local ring with the maximal ideals $\mathfrak{m}'_1, \cdots, \mathfrak{m}'_r$ and let $(R, \mathfrak{m})$ be a local subring of R' such that 1) each $\mathfrak{m}'_i$ lies

9) See § 4 in [3].

ovor $\mathfrak{m}$, i.e., $\mathfrak{m}'_i \cap R = \mathfrak{m}$, and 2) each $R'/\mathfrak{m}'_i$ is a finite module over $R/\mathfrak{m}$. Let $\mathfrak{q}'_i$ be a primary ideal belonging to $\mathfrak{m}'_i$ for each i and set $\mathfrak{q}' = \mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_r$. Then $R'/\mathfrak{q}'$ can be regarded as an R-module, and the length $l(R'/\mathfrak{q}'^n; R)$ as R-module is defined. It is known that there exists a polynomial $\sigma(\mathfrak{q}', R; n)$ such that $\sigma(\mathfrak{q}', R; n) = l(R'/\mathfrak{q}'^n; R)$ for any sufficiently large n. Let d be the degree of $\sigma(\mathfrak{q}', R; n)$ and let a be the coefficient of n^d in $\sigma(\mathfrak{q}', R; n)$. The integer $(d!)a$ is called the ***relative multiplicity*** of $\mathfrak{q}'$ with respect to R and is denoted by $rm(\mathfrak{q}'; R)$.[9]

LEMMA 6. *Let $(R, \mathfrak{m})$ be a local ring of rank 1, let $\mathfrak{q}$ be a primary ideal belonging to $\mathfrak{m}$ and assume that x is a superficial element of $\mathfrak{q}$. Then if x is not a zero-divisor,* $e(\mathfrak{q}) = e(xR) = l(R/xR)$.

For the proof, see the lemma 5.3 in Nagata [3].

LEMMA 7. (*The Extension Formula*) *Let $(R, \mathfrak{m})$ be a local domain, and let R' be an integral domain containing R such that R' is a finite R-module. Let K and K' be the quotient fields of R and R' respectively. Then, for any primary ideal $\mathfrak{q}$ of R belonging to $\mathfrak{m}$, it holds that* $rm(\mathfrak{q}R'; R) = [K' : K]e(\mathfrak{q})$.

For the proof, see the corollary 2 of the theorem 2 in Nagata [3].

LEMMA 8. (*The Theorem of Additivity*) *Let $(R, \mathfrak{m})$ be a local ring and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_s$ be all of prime divisors of zero; we renumber them so that* co-rank $\mathfrak{p}_i = \operatorname{rank} R$ *if and only if $i \leqq r$. Let $\mathfrak{q}_1, \cdots, \mathfrak{q}_r$ be primary components of zero belonging to $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ respectively. Then, for any primary ideal $\mathfrak{q}$ belonging to $\mathfrak{m}$, it holds the equality* $e(\mathfrak{q}) = \sum_1^r e((\mathfrak{q} + \mathfrak{q}_i)/(\mathfrak{q}_i))$.

For the proof, see the theorem 3 in Nagata [3].

LEMMA 9. (*The Associativity Formula*) *Let $x_1, \cdots, x_d$ be a system of parameters of a local ring R and set $\mathfrak{q} = \sum_1^d x_i R$, $\mathfrak{a} = \sum_1^r x_i R$. Then we have the equality*

$$e(\mathfrak{q}) = \sum_{\mathfrak{p}} e(\mathfrak{a}R_{\mathfrak{p}}) e((\mathfrak{q} + \mathfrak{p})/\mathfrak{p}),$$

where $\mathfrak{p}$ runs over all (minimal) prime divisors of $\mathfrak{a}$ such that co-rank $\mathfrak{p} = d - s$ *and* rank $\mathfrak{p} = s$.

For the proof, see the theorem 8 in Nagata [3].

LEMMA 10. (*The Reduction Theorem*) *Let* $(R, \mathfrak{m})$ *be a local ring and assume that the zero ideal of* R *is primary. Let* $\mathfrak{p}$ *be the prime divisor of zero. Then, for any primary ideal* $\mathfrak{q}$ *belonging to* $\mathfrak{m}$, *we have* $e(\mathfrak{q}) = e((\mathfrak{q} + \mathfrak{p})/\mathfrak{p}) l(R_{\mathfrak{p}})$.

For the proof, see the theorem 9 in Nagata [3].

§ 2. Cycles on models over discrete valuation rings

Let k be a field, and let $(\mathfrak{o}, \mathfrak{p})$ be a discrete valuation ring of k of rank 1. We shall denote by κ the residue class field of $\mathfrak{o}$. For convenience we consider two "universal domains" Ω and Ω' which are algebraically closed fields of infinite degree of transcendency over k and κ, respectively. When we speak of a *ground ring extension* $\mathfrak{o}^*$ of $\mathfrak{o}$, we always assume that $\mathfrak{o}^*$ is a discrete valuation ring of rank 1 in Ω, which dominates $\mathfrak{o}$, and that the residue class field is a subfield of Ω'. We shall call an element of Ω (or Ω') a *quantity* of Ω (or Ω').

Let M be an absolutely irreducible model over $\mathfrak{o}$. Let $\mathfrak{o}^*$ be a ground ring extension of $\mathfrak{o}$. Then we can consider $M \otimes \mathfrak{o}^*$. Let $(P, \mathfrak{m})$ be a spot of M. Put $\mathfrak{m}^*$ be a minimal prime divisor of $\mathfrak{m}(P \otimes_{\mathfrak{o}} \mathfrak{o}^*)$ in $P \otimes_{\mathfrak{o}} \mathfrak{o}^*$. Then $P^* = (P \otimes_{\mathfrak{o}} \mathfrak{o}^*)_{\mathfrak{m}^*}$ is a spot over $\mathfrak{o}^*$, and we shall call such a spot P^* a *component of* P *over* $\mathfrak{o}^*$. Let π^* be a prime element of $\mathfrak{o}^*$. If P contains k, the length $l(P^*/\mathfrak{m}P^*)$ is denoted by $i(P/\mathfrak{o}\,;\, P^*/\mathfrak{o}^*)$, and if P dominates $\mathfrak{o}$, the symbol $i(P/\mathfrak{o}\,;\, P^*/\mathfrak{o}^*)$ stands for the length $l(P^*/(\pi^*, \mathfrak{m})P^*)$.

A spot $(P, \mathfrak{m})$ of M is called *simple* if P is a regular local ring, and moreover if a prime element π of $\mathfrak{o}$ is not in $\mathfrak{m}^2$, then P is called *unramified simple*. P is called *absolutely simple* if any component of P over $\mathfrak{o}^*$ is simple for any ground ring extension $\mathfrak{o}^*$ of $\mathfrak{o}$.

An element Z of the free module generated by all spots of M over the field of rational numbers is called a *generalized cycle* on M. For a generalized cycle $Z = \sum c_i P_i$ $(P_i \in M)$, each P_i whose coefficient c_i is different from zero is called a *component* of Z. If $c_i \geqq 0$ for all i, we say that Z is *positive*. A *cycle* is a generalized cycle whose components are absolutely simple and the coefficients are integers.

Let $\mathfrak{o}^*$ be a ground ring extension of $\mathfrak{o}$, and let $(P, \mathfrak{m})$ be a spot of M. Then we denote by $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P)$ the generalized cycle $\sum i(P/\mathfrak{o}\,;\, P^*/\mathfrak{o}^*)P^*$, where P^* runs over all components of P over $\mathfrak{o}^*$. By linearity, $\sigma_{\mathfrak{o}^*/\mathfrak{o}}$ defines a homomorphism of the group of the generalized cycles on M into that of $M \otimes \mathfrak{o}^*$.

Let M and N be two absolutely irreducible models over $\mathfrak{o}$ and let $(P, \mathfrak{m})$ and $(Q, \mathfrak{n})$ be spots of M and N respectively. We assume that $\mathfrak{m} \cap \mathfrak{o} = \mathfrak{n} \cap \mathfrak{o}$. Let $\mathfrak{a}$ be the ideal generated by $\mathfrak{m}$ and $\mathfrak{n}$ in $P \otimes_{\mathfrak{o}} Q$ and let $\mathfrak{l}_i$ $(i=1, \cdots, t)$ be the minimal prime ideals of $\mathfrak{a}$. Then $(P \otimes_{\mathfrak{o}} Q)_{\mathfrak{l}_i} = R_i$ is a spot of $M \otimes N$ for each i, and set $P \times Q = \sum_1^t l(R_i/\mathfrak{a}R_i)R_i$. If P and Q dominate k, $P \otimes_{\mathfrak{o}} Q = P \otimes_k Q$ and hence by Lemma 3 rank $P+$rank $Q=$rank R_i for each i. If P and Q dominate $\mathfrak{o}$, $P \otimes_{\mathfrak{o}} Q/(\pi)$ is isomorphic to $P/(\pi) \otimes_\kappa Q/(\pi)$ and hence it is easily seen, by Lemma 3, that rank $P+$rank Q $=$rank R_i+1 for each i.

Let $X = \sum a_i P_i$ and $Y = \sum b_j Q_j$ be generalized cycles in M and N respectively, such that all the components of X and Y dominate the same place of $\mathfrak{o}$. Then we say that $X \times Y$ is *well defined* and put $X \times Y = \sum a_i b_j (P_i \times Q_j)$.

Remark: 1) Let M' and N' be open subsets of M and N, and let P and Q be spots of M' and N' respectively. Then $P \times Q$ is invariant, whether $P \times Q$ is regarded as a generalized cycle in $M \otimes N$ or in $M' \otimes N'$.

2) Let $(P_0, \mathfrak{m}_0)$ and $(Q_0, \mathfrak{n}_0)$ be spots of M and N respectively, such that $\phi_{P_0}(M)$ and $\phi_{Q_0}(N)$ are absolutely irreducible models over $\mathfrak{o}/\mathfrak{m}_0 \cap \mathfrak{o} = \mathfrak{o}/\mathfrak{n}_0 \cap \mathfrak{o}$. Let P and Q be in $M(P_0)$ and $N(Q_0)$ respectively, such that $P \times Q$ is well defined. Let R be a component of $P \times Q$. On the other hand it is easily seen that $P_0 \times Q_0$ is a spot of $M \otimes N$. Then we have from definitions that the coefficient of R in $P \times Q$ is equal to that of $\phi_{P_0 \times Q_0}(R)$ in $\phi_{P_0}(P) \times \phi_{Q_0}(Q)$.

PROPOSITION 1. *Let M and N be two absolutely irreducible models over $\mathfrak{o}$, and let X and Y be generalized cycles on N and N respectively. Then if $X\times Y$ is well defined, we have*

$$\sigma_{\mathfrak{o}^*/\mathfrak{o}}(X\times Y) = \sigma_{\mathfrak{o}^*/\mathfrak{o}}(X)\times\sigma_{\mathfrak{o}^*/\mathfrak{o}}(Y)\,,$$

for any ground ring extension $\mathfrak{o}^*$ of $\mathfrak{o}$.

PROOF. It is sufficient to prove the case where X and Y are spots $(P, \mathfrak{m})$ and $(Q, \mathfrak{n})$ respectively. First we assume that P and Q contain k. Let $(R, \mathfrak{l})$ be a component of $P\times Q$, and let $(R^*, \mathfrak{l}^*)$ be a component of R over $\mathfrak{o}^*$. Then R is a quotient ring of $P\otimes_{\mathfrak{o}}Q=P\otimes_k Q$ with respect to the prime ideal $(P\otimes_{\mathfrak{o}}Q)\cap\mathfrak{l}$ and R^* is the quotient ring of $R\otimes_{\mathfrak{o}}\mathfrak{o}^*$ with respect to the prime ideal $(R\otimes_{\mathfrak{o}}\mathfrak{o}^*)\cap\mathfrak{l}^*$, and hence R^* is the quotient ring of $(P\otimes_k Q)\otimes_k k^*$ with respect to $(P\otimes_k Q)\otimes_k k^*\cap\mathfrak{l}^*$, where k^* is the quotient field of $\mathfrak{o}^*$. On the other hand if we regard $P\otimes_k k^*$ and $Q\otimes_k k^*$ as subrings of $(P\otimes_k Q)\otimes_k k^*=(P\otimes_k k^*)\otimes_{k^*}(Q\otimes_k k^*)$, we put $\mathfrak{m}^*=(P\otimes_k k^*)\cap\mathfrak{l}^*$ and $\mathfrak{n}^*=(Q\otimes_k k^*)\cap\mathfrak{l}^*$. Then we have $\mathfrak{m}^*\cap P=\mathfrak{m}$, $\mathfrak{n}^*\cap Q=\mathfrak{n}$, $\operatorname{rank}\mathfrak{m}=\operatorname{rank}\mathfrak{m}^*$ and $\operatorname{rank}\mathfrak{n}=\operatorname{rank}\mathfrak{n}^*$, since we have $\operatorname{rank}\mathfrak{m}^*+\operatorname{rank}\mathfrak{n}^*=\operatorname{rank}R^*=\operatorname{rank}R=\operatorname{rank}\mathfrak{m}+\operatorname{rank}\mathfrak{n}$ by Lemma 3. Therefore $P^*=(P\otimes_k k^*)_{\mathfrak{m}^*}$ and $Q^*=(Q\otimes_k k^*)_{\mathfrak{n}^*}$ are the only spots such that they are components of P and Q over $\mathfrak{o}^*$ respectively, and their product has R^* as a component.

Since R^* is a quotient ring of $R\otimes_k k^*$ and $(\mathfrak{m}, \mathfrak{n})R$ is a primary ideal belonging to $\mathfrak{l}$, we have, by Lemma 2,

$$l(R^*/(\mathfrak{m}, \mathfrak{n})R^*) = l(R^*/\mathfrak{l}R^*)l(R/(\mathfrak{m}, \mathfrak{n})R)\,.$$

Since we can easily see that R^* is also a quotient ring of $P^*\otimes_{k^*}Q^*$ by Lemma 1, we have

$$l(R^*/(\mathfrak{m}, \mathfrak{n})R^*) = l(R^*/(\mathfrak{m}^*, \mathfrak{n}^*)R^*)l(P^*/\mathfrak{m}P^*)\,l(Q^*/\mathfrak{n}Q^*)\,.$$

Therefore the coefficient of R^* in $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P\times Q)$ is equal to that of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P)\times\sigma_{\mathfrak{o}^*/\mathfrak{o}}(Q)$.

Next we consider the case where P and Q dominate $\mathfrak{o}$. Let R^*, R, P^* and Q^* be as above. Let π^* be a prime element of $\mathfrak{o}^*$. Then, calculating the length $l(R^*/(\pi^*, \mathfrak{m}, \mathfrak{n})R^*)=l(R^*/(\pi^*)/(\mathfrak{m}, \mathfrak{n})R^*/(\pi^*))$, we obtain the desired result. But the calculus is quite similar to the above case, and the details are omitted. q.e.d.

Let M be an absolutely irreducible model over $\mathfrak{o}$. Let P and Q be spots of M such that P and Q dominate the same place of $\mathfrak{o}$, and let $(R, \mathfrak{l})$ be a component of $M(P)\frown M(Q)$ such that R dominates the same place as P and Q. We put $\mathfrak{O}=R\otimes_{\mathfrak{o}}R$ and we understand by the diagonal ideal $\mathfrak{d}(\mathfrak{O})$ of $\mathfrak{O}$ the ideal generated by the elements $x\otimes 1-1\otimes x$ in $\mathfrak{O}$. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the prime ideals of R such that $P=R_{\mathfrak{m}}$ and $Q=R_{\mathfrak{n}}$. Then the ideal $\mathfrak{l}'=(\mathfrak{l}\otimes 1, \mathfrak{d}(\mathfrak{O}))\mathfrak{O}$ is a minimal prime divisor of $(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n}, \mathfrak{d}(\mathfrak{O}))$ in $\mathfrak{O}$. Then putting $\bar{\mathfrak{O}}=\mathfrak{O}_{\mathfrak{l}'}$, we denote by $i_{\mathfrak{o}}(R; P\cdot Q)$ the multiplicity $e(\mathfrak{d}(\mathfrak{O})\bar{\mathfrak{O}}/(\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n})\bar{\mathfrak{O}})$. We put $P\cdot R=\sum_R i_{\mathfrak{o}}(R; P\cdot Q)R$, where R runs over all components $M(P)\frown M(Q)$ dominating the same place as P and Q. Let $X=\sum a_iP_i$ and $Y=\sum b_jQ_j$ be two generalized cycles on M such that all the components of X and Y dominate the same place of $\mathfrak{o}$. Then we say that the intersection cycle $X\cdot Y$ of X and Y is *well defined* and we put $X\cdot Y=\sum a_ib_j(P_i\cdot Q_j)$.

Remark. Let $(P_0, \mathfrak{m}_0)$ be a spot of M such that $M(P_0)$ contains P and Q, and $\phi_{P_0}(M)$ is an absolutely irreducible model over $\mathfrak{o}/\mathfrak{o}\frown\mathfrak{m}_0$. Then it is easy to see $i_{\mathfrak{o}}(R; P\cdot Q)=i_{\mathfrak{o}/\mathfrak{o}\cap\mathfrak{m}_0}(\phi_P(R); \phi_{P_0}(P)\cdot\phi_{P_0}(Q))$.

PROPOSITION 2.[10] *Let M be an absolutely irreducible model over $\mathfrak{o}$. Let $\mathfrak{o}^*$ be a ground ring extension of $\mathfrak{o}$ and let X and Y be generalized cycles in M such that $X\cdot Y$ is well defined. Then we have*

$$\sigma_{\mathfrak{o}^*/\mathfrak{o}}(X\cdot Y) = \sigma_{\mathfrak{o}^*/\mathfrak{o}}(X)\cdot\sigma_{\mathfrak{o}^*/\mathfrak{o}}(Y)\,.$$

PROOF. It is sufficient to prove the case where $X=(P, \mathfrak{m})$ and $Y=(Q, \mathfrak{n})$ are spots. Let R be a component of $P\cdot Q$ and let $(R^*, \mathfrak{l}^*)$ be a component of R over $\mathfrak{o}^*$. Put $\mathfrak{O}^*=R^*\otimes_{\mathfrak{o}^*}R^*$, let $\mathfrak{d}^*$ be the diagonal ideal of $\mathfrak{O}^*$ and let $\bar{\mathfrak{O}}^*$ be the quotient ring of $\mathfrak{O}^*$ with respect to the prime ideal $(\mathfrak{l}^*\otimes 1, \mathfrak{d}^*)$. Then $\bar{\mathfrak{O}}^*$ is also a quotient ring of $(R\otimes_{\mathfrak{o}}\mathfrak{o}^*)\otimes_{\mathfrak{o}^*}(R\otimes_{\mathfrak{o}}\mathfrak{O}^*)$. First we assume that P, Q, and R contain k. Then it is easy to see by Lemma 3 thal all the ninimal

10) This result and proof are essentially shown in Theorem 5.4 in [5]. Notice that this result gives compatibility of our definition with that of Samuel [6].

prime divisors of $(\mathfrak{m}R^*\otimes 1,\ 1\otimes\mathfrak{n}R^*)\bar{\mathfrak{O}}^*$ have the same rank equal to $\operatorname{rank}\mathfrak{m}+\operatorname{rank}\mathfrak{n}$, and hence their coranks are equal to each others.[11] Therefore we have, by Lemma 8 and Lemma 10,

$$e(\mathfrak{d}^*\bar{\mathfrak{O}}^*/(\mathfrak{m}R^*\otimes 1,\ 1\otimes\mathfrak{n}R^*)\bar{\mathfrak{O}}^*)$$
$$=\sum_{h=1}^{t} e(\mathfrak{d}^*\bar{\mathfrak{O}}^*/\mathfrak{P}_h)\, l(\bar{\mathfrak{O}}^*_{\mathfrak{P}_h}/(\mathfrak{m}R^*\otimes 1,\ 1\otimes\mathfrak{n}R^*)),$$

where $\mathfrak{P}_h$ runs all minimal prime divisors of $(\mathfrak{m}R^*\otimes 1,\ 1\otimes\mathfrak{n}R^*)$.

Let $(P_1^*, \mathfrak{m}_1^*), \cdots, (P_r^*, \mathfrak{m}_r^*)$ be all the components of P over $\mathfrak{o}^*$ which have R^* as a specialization, and let $(Q_1^*, \mathfrak{n}_1^*), \cdots, (Q_s^*, \mathfrak{n}_s^*)$ be all the components of Q over $\mathfrak{o}^*$ which have R^* as a specialization. Then for each $\mathfrak{P}_h$ there exists only a pair (i, j) such that $\bar{\mathfrak{O}}^*_{\mathfrak{P}_h}$ dominates both P_i^* and Q_j^* (cf. the proof of Proposition 1), and we have, using Lemma 1 and Lemma 2,

$$l(\bar{\mathfrak{O}}^*_{\mathfrak{P}_h}/(\mathfrak{m}R^*\otimes 1,\ 1\otimes\mathfrak{n}R^*))$$
$$= l(\bar{\mathfrak{O}}^*_{\mathfrak{P}_h}/(\mathfrak{m}_i^*\otimes 1,\ 1\otimes\mathfrak{n}_j^*)\, l(P_i^*/\mathfrak{m}P_i^*)\, l(Q_j^*/\mathfrak{n}Q_j^*).$$

Therefore we have, using again Lemma 8 and Lemma 10,

$$e(\mathfrak{d}^*\bar{\mathfrak{O}}^*/(\mathfrak{m}R^*\otimes 1,\ 1\otimes\mathfrak{n}R^*)\bar{\mathfrak{O}}^*)$$
$$=\sum_{i,j} e(\mathfrak{d}^*\bar{\mathfrak{O}}^*/(\mathfrak{m}^*\otimes 1,\ 1\otimes\mathfrak{n}^*)\bar{\mathfrak{O}}^*)\, 1(P_i^*/\mathfrak{m}P_i^*)\, l(Q_j^*/\mathfrak{n}Q_i^*).$$

On the other hand, $\bar{\mathfrak{O}}^*$ is a quotient ring of $\bar{\mathfrak{O}}\otimes_{\mathfrak{o}}\mathfrak{o}^*$ and hence $\bar{\mathfrak{O}}^*$ is a quotient ring of $\bar{\mathfrak{O}}\otimes_k k^*$ in this case. Then because of $\mathfrak{d}^*=\mathfrak{d}\bar{\mathfrak{O}}^*$, we have, for any positive integer n,

$$l(\bar{\mathfrak{O}}^*/(\mathfrak{d}^{*n},\ \mathfrak{m}R^*\otimes 1,\ 1\otimes\mathfrak{n}R^*))$$
$$= l(\bar{\mathfrak{O}}^*/(\mathfrak{l}\otimes 1,\ \mathfrak{d},\ \mathfrak{m}R^*\otimes 1,\ 1\otimes\mathfrak{n}R^*))\, l(\bar{\mathfrak{O}}/\mathfrak{d}^n,\ \mathfrak{m}\otimes 1,\ 1\otimes\mathfrak{n}))$$
$$= l(R^*/\mathfrak{l}R^*)\, l(\bar{\mathfrak{O}}/(\mathfrak{d}^n,\ \mathfrak{m}\otimes 1,\ 1\otimes\mathfrak{n})),$$

applying Lemma 2 to $(\bar{\mathfrak{O}}/(\mathfrak{m}\otimes 1,\ 1\otimes\mathfrak{n}))\otimes_k k^*$. Hence we have

$$e(\mathfrak{d}\bar{\mathfrak{O}}^*/(\mathfrak{m}R^*\otimes 1,\ 1\otimes\mathfrak{n}R^*)) = l(R^*/\mathfrak{l}R^*)\, e(\mathfrak{d}\bar{\mathfrak{O}}/(\mathfrak{m}\otimes 1,\ 1\otimes\mathfrak{n})).$$

Therefore we have

$$(*)\quad i_{\mathfrak{o}}(R;\ P\cdot Q)\, i(R/\mathfrak{o};\ R^*/\mathfrak{o}^*)$$
$$=\sum_{i,j} i_{\mathfrak{o}^*}(R^*;\ P_i^*\cdot Q_j^*)\, i(P/\mathfrak{o};\ P_i^*/\mathfrak{o}^*)\, i(Q/\mathfrak{o};\ Q_j^*/\mathfrak{o}^*).$$

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11) *"If $\mathfrak{p}$ is a prime ideal of a spot P, then* $\operatorname{rank}\mathfrak{p}+\operatorname{corank}\mathfrak{p}=\operatorname{rank}P$" (Corollary 2 of Theorem 1 of Chapter 1 in [4-I].)

If P, Q and R dominate $\mathfrak{o}$, then we consider the multiplicity $e(\mathfrak{d}^*\bar{\mathfrak{L}}^*/(\pi^*, \mathfrak{m}R^*\otimes 1, 1\otimes\mathfrak{n}R^*))$ and we obtain the equality $(*)$ in the similar way as above, but we omit the calculus. q.e.d.

Let M and N be two absolutely irreducible models over $\mathfrak{o}$ such that M dominates N, i.e., any spot of M dominates a spot of N. Let P be a spot of M and let Q be the spot of N which is dominated by P; then Q is called the *projection* of P. The *algebraic projection* $\mathrm{Pr}^M{}_N(P)$ is defined to be $[\phi_P(P):\phi_Q(Q)]Q$. For any generalized cycle X, $\mathrm{Pr}^M{}_N(X)$ is defined by linearity.

Next we shall consider a generalized cycle attached to an element f of the function field L of an absolutely irreducible model M. For any spot P of rank 1 in the derived normal model $\bar{M}$ of M, let v_P be the normalized valuation defined by P. Then we put $(f)_{\bar{M}}=\sum v_P(f)P$, where P runs over all spots of rank 1 in in $\bar{M}$. We say that $(f)_{\bar{M}}$ is the *cycle of* f on $\bar{M}$.

Moreover when we put $\mathrm{Pr}^{\bar{M}}{}_M((f)_{\bar{M}})=(f)_M$, we say that $(f)_M$ is the *generalized cycle of* f.

In the above we defined some operations of generalized cycles *on absolutely irreducible models* over a valuation ring $\mathfrak{o}$. However, of course, these definitions are available for models defined over a field K. In this case any spot contains K. Therefore products and intersections of generalized cycles are always well defined, and we use symbols $i_K(\ ;\ \cdot\ )$ and $\sigma_{K^*/K}(\ )$ instead of $i_{\mathfrak{o}}(\ ;\ \cdot\ )$ and $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\ )$.

Let $\mathfrak{o}$ be a discrete valuation ring of a field k. Let M be an absolutely irreducible model over $\mathfrak{o}$. It should be noticed that each operation of generalized cycles on M_k is invariant whether M_k is considered as a model defined over a field k or as a model over a valuation ring $\mathfrak{o}$.

§3. Operations ρ and ρ'

Let M be an absolutely irreducible model over $\mathfrak{o}$. Then it is easy to see that the set of all the spots in M which dominate $\mathfrak{o}$ is a closed subset of M. Let $P_1, \cdots, P_n$ be the components of this closed subset. Then each P_i is called a *generating spot over* $\mathfrak{p}$. If M_k is the reduced model of M over k, M is the union of $M(P_1), \cdots, M(P_n)$ and M_k. Moreover M_k and $M(P_i)$ have no common spots for each i.

Let Q be a spot of M_k and put $M_0(Q) = M(Q) \cap (\bigcup_i M(P_i))$. Then $M_0(Q)$ is a closed subset of M. Let $Q'_1, \cdots, Q'_s$ be the generating spots of the irreducible components of $M_0(Q)$. Then the set $\phi_Q(Q'_1), \cdots, \phi_Q(Q'_s)$ is the set of the generating spots over $\mathfrak{p}$ of the induced model $\phi_Q(M)$. We shall call Q'_i an *induced spot* of Q over $\mathfrak{p}$ for each i. Now we shall show the following

PROPOSITION 3. *Let M be an absolutely irreducible medel over $\mathfrak{o}$, let $(Q, \mathfrak{n})$ be a spot of M_k and let $(Q', \mathfrak{n}')$ be an induced spot of Q over $\mathfrak{p}$. Then the transcendental degree of $Q/\mathfrak{n}$ over k is equal to that of $Q'/\mathfrak{n}'$ over $\kappa = \mathfrak{o}/\mathfrak{p}$.*

PROOF. Put $\bar{\mathfrak{n}} = Q' \cap \mathfrak{n}$ and let π be a prime element of $\mathfrak{o}$. Then $(\pi, \bar{\mathfrak{n}})Q'$ is a primary ideal belonging to $\mathfrak{n}'$. Since $\operatorname{rank} Q' = \operatorname{rank} \bar{\mathfrak{n}} + \operatorname{corank} \bar{\mathfrak{n}}$ and $\operatorname{corank} \bar{\mathfrak{n}} = 1$, we have $\operatorname{rank} Q' - 1 = \operatorname{rank} Q$. On the other hand it is known that (trans. deg. of L/k) = (trans. deg. of $(Q/\mathfrak{n})/k$) + $\operatorname{rank} Q$ = (trans. deg. of $(Q'/\mathfrak{n}')/\kappa$) $- 1 + \operatorname{rank} Q'$,[12] where L is the function field of M. Hence we have our proposition.

q.e.d.

Let $(Q', \mathfrak{n}')$ be an induced spot of a spot $(Q, \mathfrak{n})$ of M_k. Now we define a *multiplicity* $\mu(Q\,;\,Q')$ of Q' as an induced spot of Q. Let π be a prime element of $\mathfrak{o}$. Then $(\pi)Q'/Q' \cap \mathfrak{n}$ is a primary ideal belonging to $\mathfrak{n}'/Q' \cap \mathfrak{n}$ and hence the multiplicity $e(\pi Q'/Q' \cap \mathfrak{n})$ is well defined. Then we put $\mu(Q\,;\,Q') = e(\pi Q'/Q' \cap \mathfrak{n})$. In particular if Q is the function field L of M, we write $\mu(Q')$ instead of $\mu(L\,;\,Q')$. The following proposition is a direct consequence of definitions.

PROPOSITION 4. *Let M be an absolutely irreducible model over $\mathfrak{o}$ and let Q be a spot of M_k. Let P be a spot of M_k such that $M(P)$ contains Q. Then spot Q' in M is an induced spot of Q if and only if $\phi_P(Q')$ is that of $\phi_P(Q)$, and moreover we have $\mu(Q\,;\,Q') = \mu(\phi_P(Q)\,;\,\phi_P(Q'))$.*

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12) See Corollary 3 of Theorem 1 of Chapter 1 in [4-I].

Remark. In the above we treat the case where M is an absolutely irreducible model over $\mathfrak{o}$. But it is easy to see that the notion of induced spots can also be defined on any model over $\mathfrak{o}$, and Propositions 3 and 4 are true even if we replace absolutely irreducible models by models over $\mathfrak{o}$.

Let M be an absolutely irreducible model over $\mathfrak{o}$. Now we shall define an *operation* ρ, which is a homomorphism of the group of generalized cycles in M_k into that of $M-M_k$. If Q is a spot of M_k, then we put $\rho(Q)=\sum_{Q'}\mu(Q\,;\,Q')Q'$, where Q' runs over all the induced spots of Q over $\mathfrak{p}$. If $X=\sum a_iQ_i$ is the generalized cycle on M_k, then we put $\rho(X)=\sum a_i\rho(Q_i)$.

Now it should be shown that the operation ρ does not depend on ground rings. In fact we have the following

PROPOSITION 5. *Let M be an irreducible model over $\mathfrak{o}$ and let $\mathfrak{o}^*$ be a ground ring extension of $\mathfrak{o}$. Then for any generalized cycle X in M_k we have $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(X))=\rho(\sigma_{\mathfrak{o}^*/\mathfrak{o}}(X))$.*

PROOF. We may assume that X is a spot $(Q, \mathfrak{n})$. Let $(Q', \mathfrak{n}')$ be an induced spot of Q and let S be a component of Q' over $\mathfrak{o}^*$. Moreover let $R_1, \cdots, R_t$ be all the components of Q over $\mathfrak{o}^*$ which have S as an induced spot over $\mathfrak{p}^*$. Then S is a quotient ring of $Q'\otimes_{\mathfrak{o}}\mathfrak{o}^*$ with respect to a prime ideal $\mathfrak{l}$. Let $\bar{\mathfrak{n}}$ be the prime ideal $\mathfrak{n}\otimes Q'$ and put $S'=S/(\bar{\mathfrak{n}}\otimes 1)S$. It is clear that S' is also a quotient ring of $\phi_Q(Q')\otimes_{\mathfrak{o}}\mathfrak{o}^*$ with respect to a prime ideal $\mathfrak{l}'$.

Let π and π^* be prime elements of $\mathfrak{o}$ and $\mathfrak{o}^*$ respectively. Then we have by Lemma 5

$$\begin{aligned} l(S'/\pi S') &= l(S'/(\phi_Q(\mathfrak{n}'),\ \pi^*)S')\,l(\phi_Q(Q')/\pi\phi_Q(Q'))\,l(\mathfrak{o}^*/\pi\mathfrak{o}^*) \\ &= l(S/(\pi^*,\ \mathfrak{n}'\otimes 1)S)\,l(Q'/(\pi,\ \mathfrak{n})Q')\,l(\mathfrak{o}^*/\pi\mathfrak{o}^*)\,. \end{aligned}$$

Since S' is of rank 1 and π is not a zero divisor in S', we have, by Lemma 6, $l(S'/\pi S')=e(\pi S')$. Similarly we have $l(\mathfrak{o}^*/\pi\mathfrak{o}^*)=e(\pi\mathfrak{o}^*)$. Therefore we have

$$e(\pi S') = i(Q'/\mathfrak{o}\,;\ S/\mathfrak{o}^*)\,\mu(Q\,;\ Q')\,e(\pi\mathfrak{o}^*)\,.$$

On the other hand let $\mathfrak{q}_1, \cdots, \mathfrak{q}_t$ be the minimal prime divisors of zero in $\phi_Q(Q')\otimes_{\mathfrak{o}}\mathfrak{o}^*$ which are contained in $\mathfrak{l}'$. $\mathfrak{q}_1, \cdots, \mathfrak{q}_t$ correspond naturally to $R_1, \cdots, R_t$. Therefore corank $\mathfrak{q}_i$ in S' is one for each i, and hence we have by Lemma 8 and Lemma 10

$$\begin{aligned} e(\pi S') &= \sum_{i=1}^{t} e(\pi S'/\mathfrak{q}_i) l(S'_{\mathfrak{q}_i}) \\ &= \sum_{i=1}^{t} e(\pi\phi_{R_i}(S)) l(R_i/\mathfrak{n}R_i) \\ &= \sum_{i=1}^{t} e(\pi\phi_{R_i}(S)) i(Q/\mathfrak{o}\,;\, R_i/\mathfrak{o}^*)\,. \end{aligned}$$

If we put $l(\pi\mathfrak{o}^*) = l(\pi^{*u}\mathfrak{o}^*) = u$, we have $e(\pi\phi_{R_i}(S)) = e(\pi^{*u}\phi_{R_i}(S)) = ue(\pi^*\phi_{R_i}(S)) = u\mu(R_i\,;\, S)$. Therefore we have

$$i(Q'/\mathfrak{o}\,;\, S/\mathfrak{o}^*)\mu(Q\,;\, Q') = \sum_{i=1}^{t} \mu(R_i\,;\, S)\ i(Q/\mathfrak{o}\,;\, R_i/\mathfrak{o}^*)\,. \quad \text{q.e.d.}$$

Next we shall give a criterion for unramified simplicity. The next lemma is necessary in the proof.

LEMMA 11. *Let L be a function field over a ground ring $\mathfrak{o}$ such that L is a separable extension of the quotient field of $\mathfrak{o}$. Let P be a spot of L and let π be a non-unit of P. Suppose that*

(i) *πP has only one minimal prime divisor $\mathfrak{m}$ and $\pi P_{\mathfrak{m}}$ is the maximal ideal of $P_{\mathfrak{m}}$.*

(ii) *$P/\mathfrak{m}$ is normal.*

Then $\mathfrak{m} = \pi P$ and P is normal itself.

For the proof, see Lemma 4 in [2]. Although in Lemma 4 of [2] $\mathfrak{o}$ is assumed to satisfy the finiteness condition for integral extensions, the proof is also available for our Lemma 11. For the derived normal ring of P is a finite P-module, since L is a separable extension over $\mathfrak{o}$.

PROPOSITION 6. *Let M be an absolutely irreducible model over $\mathfrak{o}$, and let R be a spot of M which dominates $\mathfrak{o}$. Then R is unramified simple if and only if the following conditions are satisfied:*

(i) *There exists only one generating spot P over $\mathfrak{p}$ such that $M(P)$ contains R. Moreover $\mu(P) = 1$.*

(ii) *$\phi_P(R)$ is a regular local ring.*

PROOF. First let R be unramified simple and let $(\pi, t_1, \cdots, t_r)$

be a regular system of parameters containing a prime element π of $\mathfrak{o}$. Then $\mathfrak{m}=\pi R$ is a prime ideal and $R_{\mathfrak{m}}$ is the unique generating spot over $\mathfrak{p}$ satisfying (i). Then (ii) is also evidently satisfied. Conversely we assume that the conditions are satisfied. By (i), πR has only one minimal prime ideal $\mathfrak{m}$ and $\pi R_{\mathfrak{m}}=\mathfrak{m}R_{\mathfrak{m}}$. Since R satisfies also the conditions in Lemma 11, R is normal and hence πR has no imbedded prime ideal. Therefore $\pi R=\mathfrak{m}$, and hence R is unramified simple. q.e.d.

Let M be an absolutely irreducible model over $\mathfrak{o}$. Then we shall call M to be *absolutely irreducible modulo* $\mathfrak{p}$, if M has only one generating spot P_0 over $\mathfrak{p}$ and if $\phi_{P_0}(M)$ is absolutely irreducible over $\kappa=\mathfrak{o}/\mathfrak{p}$. Moreover if $\mu(P_0)=1$, we shall call M to be a $\mathfrak{p}$*-simple model over* $\mathfrak{o}$.

Let M be an absolutely irreducible model modulo $\mathfrak{p}$ and let P_0 be the unique generating spot over $\mathfrak{p}$. Then M is a disjoint sum of M_k and $M(P_0)$, and $M(P_0)$ has a one to one correspondence ϕ_{P_0} with $\phi_{P_0}(M)$. Therefore M can be considered as a complex notion of two models M_k and $\phi_{P_0}(M)$, which are models defined over different fields k and $\kappa=\mathfrak{o}/\mathfrak{p}$ respectively.

Now we shall define an *operation* ρ' which is obtained naturally from ρ. Let M be an absolutely irreducible model modulo $\mathfrak{p}$ with the unique generating spot P_0 over $\mathfrak{p}$. For any spot P of M_k, we put $\rho'(P)=\phi_{P_0}(\rho(P))$. Then $\rho'(P)$ is a generalized cycle on $\phi_{P_0}(M)$. By linearity we define ρ' for any generalized cycle on M_k.

We shall terminate this section by showing that the multiplicity $\mu(Q\,;\,Q')$ on a $\mathfrak{p}$-simple model M can be interpreted as an intersection multiplicity.

Let A be an open subset of M such that A is an affine model over $\mathfrak{o}$ and contains Q and Q'. Let $\mathfrak{o}[x_1, \cdots, x_n]$ be the affine ring of A. Then A is an induced model of a model B defined by a polynomial ring $\mathfrak{o}[X_1, \cdots, X_n]$. Let R be the spot of B such that $A=\phi_R(B)$, and let $(Q_1, \mathfrak{n}_1)$ and $(Q_1', \mathfrak{n}_1')$ be the spots of B such that $\phi_R(Q_1)=Q$ and $\phi_R(Q_1')=Q'$. Then we know that $\mu(Q_1\,;\,Q_1')=\mu(Q\,;\,Q')$.

We put $\mathfrak{O}_1=Q_1'\otimes_0 Q_1'$ and we denote by $\mathfrak{d}_1$ the diagonal ideal of $\mathfrak{O}_1$. Then $\mathfrak{d}_1$ is generated by n elements $d_1=X_1\otimes 1-1\otimes X_1, \cdots,$

$d_n = X_n \otimes 1 - 1 \otimes X_n$. Let $\bar{\mathfrak{O}}_1$ be the quotient ring of $\mathfrak{O}_1$ with respect to the prime ideal $(\mathfrak{n}' \otimes 1, \mathfrak{d}_1)$ and $\bar{\mathfrak{n}}_1 = Q_1' \cap \mathfrak{n}_1$. If π is a prime element of $\mathfrak{o}$, $(\pi)\mathfrak{o}[X_1, \cdots, X_n]$ is the prime ideal corresponding to the unique generating spot P_1 of B. Since Q_1' is a component of $B(Q_1) \cap B(P_1)$, $(\mathfrak{d}_1, \pi, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_1$ is a primary ideal belonging to $(\mathfrak{d}_1, \mathfrak{n}_1' \otimes 1)\bar{\mathfrak{O}}_1$. Therefore $e((\mathfrak{d}_1, \pi, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_1/(\mathfrak{n}_1 \otimes 1)\bar{\mathfrak{O}}_1)$ is well defined. On the other hand we easily see by Proposition 3 and the relations between ranks and coranks of spots that rank $\bar{\mathfrak{O}}_1/(\mathfrak{n}_1 \otimes 1)$ is equal to $n+1$. Hence $(d_1, \cdots, d_n, \pi)$ is a system of parameters of $\bar{\mathfrak{O}}_1/(\bar{\mathfrak{n}}_1 \otimes 1)$. By Lemma 9 we have

$$e((\mathfrak{d}_1, \pi, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_1/(\bar{\mathfrak{n}}_1 \otimes 1)) = e((\mathfrak{d}_1, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_{1(\mathfrak{d}_1, \bar{\mathfrak{n}}_1 \otimes 1)}/(\bar{\mathfrak{n}}_1 \otimes 1)) \times e((\mathfrak{d}_1, \pi, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_1/(\mathfrak{d}_1, \bar{\mathfrak{n}}_1 \otimes 1)).$$

Since $\bar{\mathfrak{O}}_{1(\mathfrak{d}_1, \bar{\mathfrak{n}}_1 \otimes 1)}/(\bar{\mathfrak{n}}_1 \otimes 1)$ is a regular local ring, the first factor of the right hand side of the above equality is one. On the other hand since $(\mathfrak{d}_1, \pi, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_1/(\mathfrak{d}_1, \bar{\mathfrak{n}}_1 \otimes 1)$ is isomorphic to $\pi Q_1'/\bar{\mathfrak{n}}_1 Q_1'$, the second factor of the right hand side of the above equality is equal to $\mu(Q_1; Q_1')$.

Let $\mathfrak{m}_1, \cdots, \mathfrak{m}_s$ be all the minimal prime divisors of $(\pi, \mathfrak{n}_1 \otimes 1)\bar{\mathfrak{O}}_1$. Then it is evident that the coranks of $\mathfrak{m}_i$ are equal to each other. By Lemma 9 we have

$$\begin{aligned} &e((\mathfrak{d}_1, \pi, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_1/(\bar{\mathfrak{n}}_1 \otimes 1)) \\ &\quad = \sum_{i=1}^{s} e((\pi, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_{1\mathfrak{m}_i}/(\bar{\mathfrak{n}}_1 \otimes 1))\, e((\mathfrak{d}_1, \mathfrak{m}_i)\bar{\mathfrak{O}}_1/\mathfrak{m}_i). \end{aligned}$$

Since $\bar{\mathfrak{O}}_{1\mathfrak{m}_i}/(\bar{\mathfrak{n}}_1 \otimes 1)$ is an integral domain of rank 1, we have by Lemma 6 $e((\pi, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_{1\mathfrak{m}_i}/(\bar{\mathfrak{n}}_1 \otimes 1)) = l(\bar{\mathfrak{O}}_{1\mathfrak{m}_i}/(\pi, \bar{\mathfrak{n}}_1 \otimes 1))$. Therefore we have, by Lemma 8 and Lemma 10,

$$\begin{aligned} &e((\mathfrak{d}_1, \pi, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_1/(\bar{\mathfrak{n}}_1 \otimes 1)) \\ &\quad = \sum_{i=1}^{s} e((\mathfrak{d}_1, \mathfrak{m}_i)\bar{\mathfrak{O}}_1/\mathfrak{m}_i)\, l(\bar{\mathfrak{O}}_{1\mathfrak{m}_i}/(\pi, \bar{\mathfrak{n}}_1 \otimes 1)) \\ &\quad = e((\mathfrak{d}_1, \pi, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_1/(\pi, \bar{\mathfrak{n}}_1 \otimes 1)) \end{aligned}$$

and hence $\mu(Q_1; Q_1') = e((\mathfrak{d}_1, \pi, \bar{\mathfrak{n}}_1 \otimes 1)\bar{\mathfrak{O}}_1/(\pi, \bar{\mathfrak{n}}_1 \otimes 1))$.

Now we put $\mathfrak{O} = Q' \otimes_{\mathfrak{o}} Q'$ and let $\mathfrak{d}$ be the diagonal ideal of $\mathfrak{O}$. Let $\mathfrak{n}$ and $\mathfrak{n}'$ be the prime ideals of Q' corresponding to Q

and Q'. If we put $\bar{\mathfrak{O}}=(Q'\otimes_{\mathfrak{o}}Q')_{(\mathfrak{d},\mathfrak{n}'\otimes 1)}$, $(\mathfrak{d}_1, \pi, \bar{\mathfrak{n}}_1\otimes 1)\bar{\mathfrak{O}}_1/(\pi, \mathfrak{n}_1\otimes 1)$ is isomorphic to $(\mathfrak{d}, \pi, \mathfrak{n}\otimes 1)\bar{\mathfrak{O}}/(\pi, \mathfrak{n}\otimes 1)$ and hence we have $\mu(Q\,;\,Q')=e((\mathfrak{d}, \pi, \mathfrak{n}\otimes 1)\bar{\mathfrak{O}}/(\pi, \mathfrak{n}\otimes 1))$.

The right hand side of this equality may be denoted by $i_{\mathfrak{o}}(Q'\,;\,P_0\cdot Q)$ if we abuse the symbol. Therefore we may consider $\mu(Q\,;\,Q')$ as an intersection multiplicity of Q' of two spots P_0 and Q.

§4. Properties of ρ'

In this section we shall show relations between ρ' and operations of cycles defined in §2.

PROPOSITION 7. *Let M and N be two absolutely irreducible models modulo $\mathfrak{p}$. Then $M\otimes N$ is also an absolutely irreducible model modulo $\mathfrak{p}$. Moreover if P, Q and R are generating spots over $\mathfrak{p}$ of M, N and $M\otimes N$ respectively, then we have $\mu(R)=\mu(P)\mu(Q)$.*

PROOF. Let $(P, \mathfrak{m})$ and $(Q, \mathfrak{n})$ be generating spots of M and N over $\mathfrak{p}$ respectively. By assumptions $P/\mathfrak{m}$ and $Q/\mathfrak{n}$ are regular extensions over $\kappa=\mathfrak{o}/\mathfrak{p}$. Therefore $P/\mathfrak{m}\otimes_{\kappa}Q/\mathfrak{n}$ is an integral domain and hence $(\mathfrak{m}, \mathfrak{n})$ is a prime ideal of $P\otimes_{\mathfrak{o}}Q$. This means that $M\otimes N$ has only one generating spot over $\mathfrak{p}$, which will be denoted by $(R, \mathfrak{l})$. Since $R/\mathfrak{l}$ is the quotient field of $P/\mathfrak{m}\otimes_{\mathfrak{o}_\kappa}Q/\mathfrak{n}$, $R/\mathfrak{l}$ is also regular over κ. Therefore $\phi_R(M\otimes N)$ is an absolutely irreducible model over κ. On the other hand we have by Lemma 5 and definitions

$$\mu(R) = l(R/\pi R) = l(R/(\mathfrak{m}, \mathfrak{n})R)l(P/\pi P)l(Q/\pi Q) = \mu(P)\mu(Q)\,.$$

q.e.d.

COROLLARY. *If M and N are $\mathfrak{p}$-simple models over $\mathfrak{o}$, then $M\otimes N$ is also a $\mathfrak{p}$-simple model over $\mathfrak{o}$.*

THEOREM 1. *Let M and N be two absolutely irreducible models modulo $\mathfrak{p}$, and let X and Y be generalized cycles belonging to M_k and N_k resppectively. Then we have $\rho'(X\times Y)=\rho'(X)\times\rho'(Y)$.*

PROOF. It is sufficient to show $\rho(P)\times\rho(Q)=\rho(P\times Q)$ for a spot $(P, \mathfrak{m})$ of M_k and a spot $(Q, \mathfrak{n})$ of N_k. Let S be a component of

$\rho(P\times Q)$, and let $R_1, \cdots, R_s$ be all the components of $P\times Q$, which have S as an induced spot over $\mathfrak{p}$. Then R_i is a quotient ring of $P\otimes_{\mathfrak{o}}Q$ with respect to a prime ideal and is also a quotient ring of S with respect to a prime ideal $\mathfrak{l}_i$ of S. It is easy to see that $S\cap(\mathfrak{m}\otimes 1)R_1=\cdots=S\cap(\mathfrak{m}\otimes 1)R_s$. This ideal of S will be denoted by $\bar{\mathfrak{m}}$. Similarly we shall denote $S\cap(1\otimes\mathfrak{n})R_i$ by $\bar{\mathfrak{n}}$. Let $(P', \mathfrak{m}')$ and $(Q', \mathfrak{n}')$ be the projections of S on M and on N respectively. Since S is a specialization of R_i and the projections of R_i on M and on N are P and Q respectively, it is easy to see that P' and Q' are specializations of P and Q respectively. Moreover we have $\operatorname{rank} P=\operatorname{rank} P'$ and $\operatorname{rank} Q=\operatorname{rank} Q'$. Therefore P' and Q' are only induced spots of P and Q respectively, whose product cycle has S as a component. Put $S^*=S/(\bar{\mathfrak{m}}, \bar{\mathfrak{n}})$. Then S^* is of rank 1 and πS^* is a primary ideal belonging to the maximal ideal. Since $\mathfrak{l}_1, \cdots, \mathfrak{l}_s$ are all the minimal prime divisors of $(\bar{\mathfrak{m}}, \mathfrak{n})$, we have by Lemma 8 and Lemma 10

$$\begin{aligned} e(\pi S^*) &= \sum_{i=1}^{s} e((\pi, \mathfrak{l}_i)S^*/\mathfrak{l}_i S^*)\, l(S^*_{\mathfrak{l}_i/(\mathfrak{m}, \mathfrak{n})}) \\ &= \sum_{i=1}^{s} \mu(R_i\,;\ S)\, l(R_i/\mathfrak{m}\otimes 1, 1\otimes\mathfrak{n}))\,. \end{aligned}$$

On the other hand S^* is also a quotient ring of $(P'/\mathfrak{m}\cap P')\otimes_{\mathfrak{o}}(Q'/\mathfrak{n}\cap Q')$ with respect to a prime ideal. Therefore we have, by Lemma 5 and Lemma 6,

$$\begin{aligned} e(\pi S^*)=l(S^*/\pi S^*) &= l(S^*/(\mathfrak{m}', \mathfrak{n}')S^*)\, l(P'/(\pi, \mathfrak{m}\cap P'))\, l(Q'/(\pi, \mathfrak{n}\cap Q')) \\ &= l(S/(\mathfrak{m}', \mathfrak{n}'))\, \mu(P\,;\ P')\, \mu(Q\,;\ Q')\,. \end{aligned}$$

Therefore the coefficient of S in $\rho(P\times Q)$ is equal to that of $\rho(P)\times\rho(Q)$. q.e.d.

Let M be an absolutely irreducible model over $\mathfrak{o}$ of the function field L. Then M is called to be *n-dimensional* if the transcendental degree of L over $\mathfrak{o}$ is n. A spot $(P, \mathfrak{m})$ of M is called to be *r dimensional* if the transcendental degree of $P/\mathfrak{m}$ over $\mathfrak{o}/\mathfrak{m}\cap\mathfrak{o}$ is r. Let P and Q be two spots of M such that $P\cdot Q$ is well defined, and let n, r and s be the dimensions of M, P and Q respectively. Then a component R of $P\cdot Q$ is called to be *proper* if R is absolutely simple and $(r+s-n)$-dimensional. Moreover let $X=\sum a_iP_i$ and $Y=\sum b_jQ_j$ be two positive generalized cycles of M that $X\cdot Y$ is well defined. Then a component R of $X\cdot Y$ is called to be a *proper component* of $X\cdot Y$ if R is a proper component of $P_i\cdot Q_j$ such that $M(P_i)\frown M(Q_j)$ contains R.

PROPOSITION 8. *Let M be an n-dimensional absolutely irreducible model over $\mathfrak{o}$, and let $(P, \mathfrak{m})$ and $(Q, \mathfrak{n})$ be two spots in M_k of dimensions r and s respectively. Let $(R, \mathfrak{l})$ be a component of $\rho(P)\cdot\rho(Q)$ such that R is $(r+s-n)$-dimensional and, if $\mathfrak{d}$ is the diagonal ideal of $R\otimes_{\mathfrak{o}}R$, $\mathfrak{d}(R\otimes_{\mathfrak{o}}R)_{(\mathfrak{d},\mathfrak{l}\otimes 1)}$ is generated by n elements $d_1, \cdots, d_n$. Then the coefficient of R in $\rho(P)\cdot\rho(Q)$ is equal to that of $\rho(P\cdot Q)$.*

PROOF. Put $S=(R/\mathfrak{m}\frown R\otimes_{\mathfrak{o}}R/\mathfrak{n}\frown R)_{(\mathfrak{d},\mathfrak{l}\otimes 1)/(\mathfrak{m}\cap R,\, \mathfrak{n}\cap R)}$. By assumptions it is easy to see that rank S is equal to $n+1$, and hence $(\pi, d_1, \cdots, d_n)$ is a system of parameters of S. If $\mathfrak{l}_1, \cdots, \mathfrak{l}_\lambda$ are all the minimal prime divisors of $\mathfrak{d}S$, they correspond naturally to the components $(R_1, \mathfrak{l}'_1), \cdots, (R_\lambda, \mathfrak{l}'_\lambda)$ of $P\cdot Q$, which have R as a specialization. In fact we have $\mathfrak{l}_i=(\mathfrak{d}, (\mathfrak{l}'_i\frown R)\otimes 1)S$. Each R_i has evidently S as an induced spot over $\mathfrak{p}$, and hence $\mathfrak{l}_1, \cdots, \mathfrak{l}_\lambda$ have the same rank and the corank. Therefore we have by Lemma 9

$$\begin{aligned} e((\pi, \mathfrak{d})S) &= \sum_{h=1}^{\lambda} e(\mathfrak{d}S_{\mathfrak{l}_h})\, e((\pi, \mathfrak{l}_h)S/\mathfrak{l}_h) \\ &= \sum_{h=1}^{\lambda} i_{\mathfrak{o}}(R_h : P\cdot Q)\mu(R_h : R). \end{aligned}$$

On the other hand let $\mathfrak{q}_1, \cdots, \mathfrak{q}_t$ be all the minimal prime divisors of πS. Then we have again by Lemma 9

$$e((\pi, \mathfrak{d})S) = \sum_{u=1}^{t} e(\pi S_{\mathfrak{q}_u})\, e((\mathfrak{d}, \mathfrak{q}_u)S/\mathfrak{q}_u).$$

Let $(P'_1, \mathfrak{m}'_1), \cdots, (P'_\mu, \mathfrak{m}'_\mu)$ be all the induced spots of P over $\mathfrak{p}$, which have R as a specialization, and let $(Q'_1, \mathfrak{n}'_1), \cdots, (Q'_\nu, \mathfrak{n}'_\nu)$ be all the induced spots of Q over $\mathfrak{p}$, which have R as a specialization. Then for each u, there exists only one pair (i, j) such that $S_{\mathfrak{q}_u}$ is a component of $P'_i\times Q'_j$ on $M\otimes N$. In fact we easily see that $\mathfrak{q}_u\frown(R\otimes 1)$, and $\mathfrak{q}_u\frown(1\otimes R)$ correspond to P'_i and Q'_j (cf. the proof of Proposition 1). Then $S_{\mathfrak{q}_u}$ is a quotient ring of $(P'_i/\mathfrak{m}\cap P'_i)$ $\otimes_{\mathfrak{o}}(Q'_j/\mathfrak{n}\cap Q'_j)$, and hence by Lemma 5 and Lemma 6 we have

$$e(\pi S_{\mathfrak{q}_u}) = l(S_{\mathfrak{q}_u}/(\pi)) = l(S_{\mathfrak{q}_u}/(\mathfrak{m}'_i,\ \mathfrak{n}'_j))l(P'_i/(\pi,\ P'_i\cap\mathfrak{m}))l(Q'_j/\pi,\ Q'_j\cap\mathfrak{n}))\,.$$

We have also by Lemma 8 and Lemma 10, for a fixed (i, j),

$$\begin{aligned} i_0(R\,;\ P'_i\cdot Q'_j) &= e(\mathfrak{d}S/(\mathfrak{m}'_i,\ \mathfrak{n}'_j)) \\ &= \sum_{\mathfrak{q}_u \supset (\mathfrak{m}_i',\ \mathfrak{n}_j')} e((\mathfrak{d},\ \mathfrak{q}'_u)S/\mathfrak{q}_u)l(S_{\mathfrak{q}_u}/(\mathfrak{m}'_i,\ \mathfrak{n}'_j))\,. \end{aligned}$$

Therefore we have

$$e((\pi,\ \mathfrak{d})S) = \sum_{i,\,j} i_0(R\,;\ P'_i\cdot Q'_j)\mu(P\,;\ P'_i)\mu(Q\,;\ Q'_j)$$

and hence

$$\sum_{h=1}^{\lambda} i_0(R_h\,;\ P\cdot Q)\mu(R_h\,;\ R) = \sum_{i,\,j} i_0(R\,;\ P'_i\cdot Q'_j)\mu(P\,;\ P'_i)\mu(Q\,;\ Q'_j)\,.$$

q.e.d.

THEOREM 2. *Let M be a $\mathfrak{p}$-simple model ouer $\mathfrak{o}$ and let P_0 be the unique generating spot over $\mathfrak{p}$. Let X and Y be two positive generalized cycles on M_k and let R be a spot of $M(P_0)$ such that $\phi_{P_0}(R)$ is a proper component of $\rho'(X)\cdot\rho'(Y)$. Then the coefficient of $\phi_{P_0}(R)$ in $\rho'(X)\cdot\rho'(Y)$ is equal to that of $\rho'(X\cdot Y)$.*

PROOF. Since $\phi_{P_0}(R)$ is absolutely simple on $\phi_{P_0}(M)$, there exists a specialization R' of R such that $\phi_{P_0}(R')$ is absolutely simple on $\phi_{P_0}(M)$ and zero-dimensional. Then the residue class field κ^* of R' is a finite algebraic extension over $\kappa=\mathfrak{o}/\mathfrak{p}$. κ^* may be contained in the universal domain Ω' of κ. Then there exists a ground ring $(\mathfrak{o}^*,\ \mathfrak{p}^*)$ extended over $\mathfrak{o}$, whose residue class field is κ^*. It is easy to see that there exists a component $(R'^*,\ \mathfrak{l}'^*)$ of R' over $\mathfrak{o}^*$ whose residue class field is κ^*. Put $M^*=M\otimes\mathfrak{o}^*$ and let P_0^* be the unique extension of P_0 over $\mathfrak{o}^*$. Then M^* is a $\mathfrak{p}^*$-simple model over $\mathfrak{o}^*$ by Proposition 5. Since $\phi_{P_0^*}(R'^*)$ is simple, R'^* is unramified simple by Proposition 6. Therefore there exists a system $(\pi^*,\ d_1,\cdots,d_n)$ of parameters containing a prime element π^* of $\mathfrak{o}^*$.

Let $(R^*,\ \mathfrak{l}^*)$ be a component of R over $\mathfrak{o}^*$ such that R^* has R'^* as a specialization. Let $\mathfrak{d}$ and $\mathfrak{d}'$ be the diagonal ideals of

$R^*\otimes_{\mathfrak{o}^*}R^*$ and $R'\otimes_{\mathfrak{o}^*}R'^*$ respectively. The residue class field of R'^* is $\kappa^*=\mathfrak{o}^*/\mathfrak{p}^*$, we have $\mathfrak{d}'=(d_1\otimes 1-1\otimes d_1, \cdots, d_n\otimes 1-1\otimes d_n)+(\mathfrak{l}'^*\otimes 1, 1\otimes\mathfrak{l}'^*)^m$ for any positive integer m, and hence we see easily that $\mathfrak{d}'(R'^*\otimes_{\mathfrak{o}^*}R'^*)_{(\mathfrak{d}',\mathfrak{l}'^*\otimes 1)}$ is generated by n elements $d_1\otimes 1-1\otimes d_1, \cdots, d_n\otimes 1-1\otimes d_n$.

On the other hand, by Lemma 1, we see that $(R^*\otimes_{\mathfrak{o}^*}R^*)_{(\mathfrak{d},\mathfrak{l}\otimes 1)}$ is a quotient ring of $(R'^*\otimes_{\mathfrak{o}^*}R'^*)_{(\mathfrak{d}',\mathfrak{l}'^*\otimes 1)}$ and $(\mathfrak{d})$ is generated by n elements $d_1\otimes 1-1\otimes d_1, \cdots, d_n\otimes 1-1\otimes d_n$ in $(R^*\otimes_{\mathfrak{o}^*}R^*)_{(\mathfrak{d},\mathfrak{l}^*\otimes 1)}$.

Let P and Q be components of X and Y respectively such that R is a component of $P\cdot Q$. If P^* and Q^* are components of P and Q over $\mathfrak{o}^*$ respectively such that $\rho(P^*)\cdot\rho(Q^*)$ has R^* as a component, the coefficient of R^* in $\rho(P^*)\cdot\rho(Q^*)$ is equal to that of $\rho(P^*\cdot Q^*)$ by Proposition 8. This fact means that the coefficient of R^* in $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(P)\cdot\rho(Q))$ is equal to that of $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(P\cdot Q))$, since we have $\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(P)\cdot\rho(Q))=\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(P))\cdot\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(Q))=\rho(\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P))\cdot\rho(\sigma_{\mathfrak{o}^*/\mathfrak{o}}(Q))$ and $\rho(\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P)\cdot\sigma_{\mathfrak{o}^*/\mathfrak{o}}(Q))=\rho(\sigma_{\mathfrak{o}^*/\mathfrak{o}}(P\cdot Q))=\sigma_{\mathfrak{o}^*/\mathfrak{o}}(\rho(P\cdot Q))$. Therefore the coefficient of R in $\rho(P)\cdot\rho(Q)$ is equal to that of $\rho(P\cdot Q)$, since R is the only spot of M whose components over $\mathfrak{o}^*$ contain R^*. From this we can deduce our theorem. q.e.d.

Remark. Let M be an absolutely irreducible model over $\mathfrak{o}$. Then even if M is not $\mathfrak{p}$-simple, a similar result as in Theorem 2 is obtained. Let P and Q be spots of M_k. Let R be an unramified simple spot of $M-M_k$, and put $P_0=R_{(\pi)R}$, where π is a prime element of $\mathfrak{o}$. If $\phi_{P_0}(M)$ is absolutely irreducible over $\kappa=\mathfrak{o}/\mathfrak{p}$ and if $\phi_{P_0}(R)$ is a proper component of $\phi_{P_0}(\rho(P))\cdot\phi_{P_0}(\rho(Q))$, the coefficient of R in $\rho(P)\cdot\rho(Q)$ is equal to that of $\rho(P\cdot Q)$.

In fact let $P_1, \cdots, P_s$ be all the generating spots of M over $\mathfrak{p}$, which are different from P_0. Then $M-(\bigcup_{i=1}^{s} M(P_i))$ is an open subset of M containing R, P and Q, and a $\mathfrak{p}$-simple model over $\mathfrak{p}$. Therefore our assertion is obtained if we apply Theorem 2 to this open subset.

PROPOSITION 9. *Let M and N be two absolutely irreducible models over $\mathfrak{o}$ such that M dominates N. Suppose that M is an affine model defined by $\mathfrak{o}[x]$. Let P and Q be a spot of M_k and* *its projection on N_k respectively, such that $[\phi_P(P):\phi_Q(Q)]<\infty$. If Q' is an induced spot of Q over $\mathfrak{p}$ such that $\phi_P(\mathfrak{o}[x])$ is integral over $\phi_Q(Q')$. Then the coefficient of Q' in $\rho(\mathrm{Pr}^M{}_N(P))$ is equal to that of* $\mathrm{Pr}^M{}_N(\rho(P))$.

PROOF. Put $\phi_P(\mathfrak{o}[x])=\mathfrak{o}[\bar{x}]$ and $S=\phi_Q(Q')[\bar{x}]$. Then S is a finite $\phi_Q(Q')$-module and the maximal ideals $\mathfrak{q}_1,\cdots,\mathfrak{q}_r$ of S correspond to the spots $P_1,\cdots,P_r$ of M projected to Q'. Then we have by Lemma 7

$$e(\pi\phi_Q(Q'))[\phi_P(P):\phi_Q(Q)] = rm(\pi S\,;\ \phi_Q(Q'))$$
$$= \sum_{i=1}^{r} e(\pi S_{\mathfrak{q}_i})[\phi_{P_i}(P_i):\phi_{Q'}(Q')] = \sum_{i=1}^{r} e(\pi\phi_P(P_i))[\phi_{P_i}(P_i):\phi_{Q'}(Q')]$$

and hence

$$\mu(Q\,;\ Q')[\phi_P(P):\phi_Q(Q)] = \sum_{i=1}^{r}\mu(P\,;\ P_i)[\phi_{P_i}(P_i):\phi_{Q'}(Q')].$$

q.e.d.

THEOREM 3. *Let M and N be two absolutely irreducible models modulo $\mathfrak{p}$ such that M is complete and dominates N. Let P be a spot of M_k and let Q be its projection on N. Let Q' be an induced spot of Q over $\mathfrak{p}$ such that if $P_1,\cdots,P_n$ are all the induced spots of P over $\mathfrak{p}$ whose projections on N are Q'. Suppose that there exists an open subset A of M which is an affine model and contains $P_1,\cdots,P_n$. Then the coefficient of $\phi_{Q_0}(Q')$ in $\mathrm{Pr}^{M'}{}_{N'}(\rho'(P))$ is equal to that of $\rho'(\mathrm{Pr}^M{}_N(P))$, where Q_0 is the unique generating spot of N over $\mathfrak{p}$ and where M' and N' are the induced models of M and N with respect to the generating spots.*

PROOF. We may assume that $[\phi_P(P):\phi_Q(Q)]<\infty$. Let $\mathfrak{o}[x]$ be the affine ring of A. Then it is sufficient by Proposition 9 to show that $\phi_P(\mathfrak{o}[x])$ is integral over $\phi_Q(Q')$. If it is not so, there exists a valuation ring v of $\phi_P(P)$ which dominates $\phi_Q(Q')$ and does not contain $\phi_P(\mathfrak{o}[x])$. Since M is complete, there exists a spot P' of M such that P' is contained in $M(P)$ and $\phi_P(P')$ is dominated by v. Since P' dominates Q', the dimension of P' is not less than that of Q'. On the other hand since P' is a specialization of P, the dimension of P' is not more than that of P, and hence they are equal to each other. Therefore P' is an induced spot of P over $\mathfrak{p}$. Since P' does not contain $\mathfrak{o}[x]$, P' is different from any P_i $(i=1, \cdots, n)$. This is a contradiction. q.e.d.

COROLLARY. *Let M and N be two absolutely irreducible models modulo $\mathfrak{p}$ such that M dominates N. If M is a projective model, then for any spot of M_k, we have $\mathrm{Pr}^{M'}{}_{N'}(\rho'(P))=\rho'(\mathrm{Pr}^{M}{}_{N}(P))$, where M' and N' are the induced models of M and N with respect to the generating spots over $\mathfrak{p}$ respectively.*

Let M be an absolutely irreducible model modulo $\mathfrak{p}$ with the generating spot P_0 over $\mathfrak{p}$. Now we assume that P_0 is normal. Let f be an element of the function field L of M. Then P_0 contains f or $1/f$, since P_0 is a valuation ring. Therefore we can define a generalized quantity $\bar{f}$ of the function field of $\phi_{P_0}(M)$ over κ. In this situation we have the following

THEOREM 4. *Let M be an absolutely irreducible model modulo $\mathfrak{p}$ with the generating spot P_0 over $\mathfrak{p}$. Suppose that P_0 is normal. Let f be an element of the function field of M, such that $\bar{f}$ is an element of the function field of $\phi_{P_0}(M)$ other than zero. Then we have $\rho'((f)_M)=\mu(P_0)(\bar{f})_{\phi_{P_0}(M)}$.*

Proof. Let R be a spot of M corresponding to a spot $\bar{R}$ of rank 1 in $\phi_{P_0}(M)$. Then R is of rank 2. First we assume that f is contained in R. By assumptions f is not contained in the minimal prime divisor $\mathfrak{m}$ of $(\pi)R$, since $R_{\mathfrak{m}}$ is P_0. Therefore if f is not a unit of R, (π, f) is a system of parameters of R and hence we have, by Lemma 9,

$$\begin{aligned} e((\pi, f))R) &= e((\pi)R_{\mathfrak{m}})\, e((\pi, f)R/\mathfrak{m}) \\ &= \sum_{i=1}^{n} e((f)R_{\mathfrak{q}_i})\, e((\pi, \mathfrak{q}_i)R/\mathfrak{q}_i)\,, \end{aligned}$$

where $\mathfrak{q}_1, \cdots, \mathfrak{q}_n$ are all the minimal prime divisors of $(f)R$. If we denote by $v_{\bar{R}}(\bar{f})$ (resp. $v_{R_i}(f)$) the coefficient of $\bar{R}$ in $(\bar{f})$ (resp. that of R_i in (f)), the above relation means that

$$\mu(P_0)v_{\bar{R}}(\bar{f}) = \sum_{i=1}^{n} v_{R_i}(f)\mu(R_i\,;\ R)\,,$$

if we put $R_{\mathfrak{q}_i}=R_i$. Therefore we easily see that the coefficient of of $\bar{R}$ in $\rho'((f)_M)$ is equal to that of $\mu(P_0)(\bar{f})_{\phi_{P_0}(M)}$.

If f is a unit of R, then the coefficients in both generalized cycles are zero.

If f is not contained in R, we may put $f=t_1/t_2$, where t_1 and t_2 are elements of R not contained in $\mathfrak{m}$. In fact f is contained in $P_0=R_{\mathfrak{m}}$ and not in $\mathfrak{m}R_{\mathfrak{m}}$. Therefore we have

$$\mu(P_0)v_{\bar{R}}(\bar{t}_1) = \sum_{j=1}^{r} v_{R'_j}(t_1)\mu(R'_j\,;\,R)$$

and

$$\mu(P_0)v_{\bar{R}}(\bar{t}_2) = \sum_{j=1}^{r} v_{R'_j}(t_2)\mu(R'_j\,;\,R),$$

where R'_j $(j=1,\cdots,r)$ are all the spots corresponding to the minimal prime divisors of $(t_1)R$ or $(t_2)R$.

On the other hand it is easy to see that

$$v_{\bar{R}}(\bar{t}_1) = v_{\bar{R}}(\bar{f})+v_{\bar{R}}(\bar{t}_2)$$

and

$$v_{R'_j}(t_1) = v_{R'_j}(f)+v_{R'_j}(t_2)\,.$$

Therefore we have

$$\mu(P_0)v_{\bar{R}}(\bar{f}) = \sum_{j=1}^{r} v_{R'_j}(f)\mu(R'_j\,;\,R)\,.$$

This means our theorem. q.e.d.

COROLLARY. *Let M be a $\mathfrak{p}$ simple model over $\mathfrak{o}$ with the generating spot P_0 over $\mathfrak{p}$. Let f be an element of the function field of M. Then if $\bar{f}$ is an element of the function field of $\phi_{P_0}(M)$ different from zero, we have $\rho'((f)_M)=(\bar{f})_{\phi_{P_0}(M)}$.*

§ 5. Relation between models over $\mathfrak{o}$ and $\mathfrak{p}$-varieties

Let $(\mathfrak{o},\mathfrak{p})$ be a discrete valuation ring of rank 1 with the quotient field k. Let V be an affine variety defined over k and let (x) be a generic point of V over k. Then it is easy to see that the set of all the specialization rings of points of V in $k(x)$ is an affine model defined by $k[x]$, which will be denoted by $M_k(V)$.

Now we consider the affine ring $\mathfrak{o}[x]$ over $\mathfrak{o}$. Let $M(V)$ be the affine model defined by $\mathfrak{o}[x]$. Then the reduced model of $M(V)$ over k is obviously $M_k(V)$. If $\bar{V}$ is the bunch obtained from V by the reduction with respect of $\mathfrak{p}$ in the sense of [8], then it is easily seen that $M(V)-M_k(V)$ is the set of all the specialization rings of points of $\bar{V}$ in $k(x)$. Moreover the generating spots of $M(V)$ over $\mathfrak{p}$ correspond to the generic points of the prime rational components of $\bar{V}$ over $\kappa=\mathfrak{o}/\mathfrak{p}$. Conversely let M be an affine model defined by $\mathfrak{o}[x]$, which is assumed to be absolutely irreducible over $\mathfrak{o}$. Then if we denote by V the locus of (x) over k, then M_k is $M_k(V)$, and $M(V)$ is nothing else than M. We shall call a complex notion of V and $\bar{V}$ an *affine* $\mathfrak{p}$-*variety*, and we denote it by $(V, \bar{V})$.

Let V' be an affine variety defined over k and let (x') be a generic point of V' over k. Then it should be noticed that even if $k[x]=k[x']$, we have not always $\mathfrak{o}[x]=\mathfrak{o}[x']$ and hence $M(V)$ is not always equal to $M(V')$. We have $M(V)=M(V')$ if and only if $(V, \bar{V})$ corresponds to $(V', \bar{V}')$ biregularly everywhere on $\bar{V}$ and $\bar{V}'$.

Let (a) be a point of V or $\bar{V}$. Then we denote by $M(V)_{(a)}$ the locus of the spot corresponding to (a).

A $\mathfrak{p}$-*variety* in the sense of [8] is defined as follows:

(1) There are given a finite number of affine $\mathfrak{p}$-varieties $(V_1, \bar{V}_1), \cdots, (V_n, \bar{V}_n)$ satisfying the following conditions:

(i) There exist generic points $(x_1), \cdots, (x_n)$ of $V_1, \cdots, V_n$ over k respectively such that $k(x_1)= \cdots =k(x_n)=L$. We shall denote by M_i the affine model defined by $\mathfrak{o}[x_i]$ for each i.

(ii) For each i, there exist a finite number of points (a_{ij}) of $(V_i, \bar{V}_i)$ such that the union of n models $M^{(i)}=M_i-(\bigcup_j M_{i(a_{ij})})$ is also a model M in L defined over $\mathfrak{o}$.

(2) Let M_k be the reduced model of M over k, and we denote by V an abstract variety defined over k corresponding to M_k in the sense of [4]. (V is an abstract variety defined by affine representatives $V_1, \cdots, V_n$ with some frontiers).

(3) Let $P_1, \cdots, P_m$ be the generating spots of M over $\mathfrak{p}$, and let

W_i be an algebraic set, i.e., an algebraic variety in the sense of of Serre [7], which corresponds to the induced model $\phi_{P_i}(M)$ for each i. We shall denote by $\bar{V}$ the union $W_1, \cdots, W_m$. $\bar{V}$ is also an algebraic set defined over $\mathfrak{o}/\mathfrak{p}$.

(4) A $\mathfrak{p}$-variety is defined as a complex notion of V and $\bar{V}$.[13]

We shall denote henceforce a $\mathfrak{p}$-variety thus defined by $(V, \bar{V})$. When a $\mathfrak{p}$-variety $(V, \bar{V})$ is given, we shall denote by $M(V, \bar{V})$ the model obtained in (1)-(ii).

Conversely let M be an absolutely irreducible model over $\mathfrak{o}$, and let $A_1, \cdots, A_n$ be affine models which cover M. Let $(V_i, \bar{V}_i)$ be the affine $\mathfrak{p}$-variety corresponding to A_i for each i. Then it is easy to see that there exists a $\mathfrak{p}$-variety $(V, \bar{V})$ with an affine open covering $(V_i, \bar{V}_i)$ such that M is equal to the model $M(V, \bar{V})$.

It is also easily seen that for two $\mathfrak{p}$-varieties $(V, \bar{V})$ and $(V', \bar{V}')$, which have the same function field over k, we have $M(V, \bar{V}) = M(V', \bar{V}')$ if and only if there exists an everywhere biregular birational correspondence between $(V, \bar{V})$ and $(V', \bar{V}')$.

Next we shall show that the operation ρ' defined in §3 is equivalent to the operation ρ defined in [8]. In order to do so, we shall consider to represent the multiplicity of a proper specialization[14] over $\mathfrak{o}$ by that of a primary ideal of its specialization ring.

Theorem 5.[15] *Let (t) be a set of quantities in Ω and let (τ) be a finite specialization of (t) over $\mathfrak{o}$, which is a set of quantities in Ω'. Let (s) be a set of quantities in Ω, algebraic over $k(t)$ and let (σ) be a proper specialization of (s) over the specialization ring $[(t) \xrightarrow{\mathfrak{o}} (\tau)]$. Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be the specialization rings $[(t) \xrightarrow{\mathfrak{o}} (\tau)]$ and $[(s, t) \xrightarrow{\mathfrak{o}} (\sigma, \tau)]$ respectively. Then the multiplicity of (σ) as a proper specialization of (s) over R is equal to $e(\mathfrak{m}S)[\kappa(\tau, \sigma) : \kappa(\tau)]_i$ if the following conditions are satisfied:*

13) This definition is not apparently the same one as given in [8], but attentive readers will find easily that they are essentially the same.

14) As for the definition, see §2 in [8].

15) This theorem gives more precise result than the theorem 2 in [8], and the theorem 3 in [8] will be obtained from this theorem using Lemma 7.

(i) $\mathfrak{o}[t]$ *is integrally closed and* $\mathfrak{o}[t, s]$ *is integral over* $\mathfrak{o}[t]$.

(ii) R *is a simple spot.*

(iii) $k(t)$ *is separably generated over* k *and* $k(t, s)$ *is separable algebraic over* $k(t)$.

Moreover (iii) *is not necessary if* $\mathfrak{o}$ *satisfies the finiteness condition for integral extensions.*[16]

PROOF.[17] First we assume that $k(t, s)$ is separable extension of $k(t)$. Let (t) be $(t_1, \cdots, t_n)$ and let (s) be $(s_1, \cdots, s_m)$. Let $u_1, \cdots, u_{n-m}$ be independent variables over $k(t)$. Then if we put $s_{m+1}=\sum_{i=1}^{n} u_i t_i+\sum_{j=1}^{m} u_{n+j} s_j$, it is easy to see that $k(u, t, s_1, \cdots, s_m)=k(u, t, s_{m+1})$. On the other hand let $\bar{u}_1, \cdots, \bar{u}_{n-m}$ be independent variables over $\kappa(\tau)$ and let $\mathfrak{o}'$ be the specialization ring $[(u) \xrightarrow{\mathfrak{o}} (\bar{u})]$. Then (τ, σ) is also a specialization of (t, s) over $\mathfrak{o}'$. Let $(R', \mathfrak{m}')$ be the specialization ring $[(t) \xrightarrow{\mathfrak{o}'} (\tau)]$. Now we notice that if $(\sigma'_1, \cdots, \sigma'_{m+1})$ and $(\sigma''_1, \cdots, \sigma''_{m+1})$ are specializations of $(s_1, \cdots, s_{m+1})$ over R', $(\sigma')=(\sigma'')$ if and only if $\sigma'_{m+1}=\sigma''_{m+1}$.

Let $f(X)=X^d+c_1X^{d-1}+\cdots+c_d$ be the irreducible equation for s_{m+1} over $k(u, t)$. Then by assumptions all the c_i are in $\mathfrak{o}[t, u]$. Therefore we can consider the equation $\bar{f}(X)$ in $\kappa(\tau, \bar{u})[X]$ obained from $f(X)$ by reduction of coefficients modulo $\mathfrak{p}$.

Let $(s^{(1)}), \cdots, (s^{(d)})$ be the complete set of conjugates of (s) over $k(t)$. If we put $s_{m+1}^{(\alpha)}=\sum_{i=1}^{n} u_i t_i+\sum_{j=1}^{m} u_{n+j} s_j^{(\alpha)}$, then $(s_{m+1}^{(1)}, \cdots, s_{m+1}^{(d)})$ is the complete set of conjugates of s_{m+1} over $k(u, t)$. Therefore the multiplicity λ of (σ) in the specialization $(\sigma^{(1)}), \cdots, (\sigma^{(d)})$ of $(s^{(1)}), \cdots, (s^{(d)})$ is equal to that of σ_{m+1} as a root of $\bar{f}(X)$, where $\sigma_{m+1}=\sum_{i=1}^{n} \bar{u}_i \tau_i+\sum_{j=1}^{m} \bar{u}_{j+n} \sigma_j$. On the other hand let $g(X)$ be the equation for σ_{m+1} over $\kappa(\tau, \bar{u})$. Then $\bar{f}(X)=g(X)^{\lambda'}h(X)$, where $h(X)$ is an equation in $\kappa(\tau, \bar{u})[X]$ such that $h(\sigma_{m+1}) \neq 0$. Therefore if σ_{m+1} is a root of $g(X)$ of multiplicity μ, we have $\lambda=\lambda'\mu$. Let S' and S'_1 be the specialization rings $[(t, s) \xrightarrow{\mathfrak{o}'} (\tau, \sigma)]$ and $[(t, s_{m+1}) \xrightarrow{\mathfrak{o}'}$

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[16] As for the definition, see the introduction of [4-I].

[17] The original idea of this proof is due to the proof of Theorem 5. 16 in [5].

$(\tau, \sigma_{m+1})]$ respectively. Then S'_1 is a quotient of $R'[s_{m+1}]$, which is isomorphic to $R'[X]/(f(X))$. Since R' is simple, the unmixedness theorem holds in R'[18] and hence in S'_1.[19] Since S'_1 is a local ring, any system of parameters in S'_1 is a distinct system of parameters.[20] Since R' is regular and rank $R' = S'_1$, the maximal ideal $\mathfrak{m}'$ of R' is generated by rank S'_1 elements and hence we have $e(\mathfrak{m}'S'_1) = l(S'_1/\mathfrak{m}'S'_1)$. Moreover it is easily seen that $S'_1/\mathfrak{m}'S'_1$ is isomorphic to $(\kappa(\bar{u}, \tau)[X]/(g(X)^{\lambda'}))_{(g(X))}$. Therefore we have $e(\mathfrak{m}'S'_1) = \lambda'$ and hence $\lambda = \lambda'\mu = e(\mathfrak{m}'S'_1)[\kappa(\bar{u}, \tau, \sigma_{m+1}) : \kappa(\bar{u}, \tau)]_i$. Since (s) has the unique specialization (σ) over S'_1, S' is integral over S'_1. The quotient field of S' is that of S'_1 and hence, by assumptions, S' is a finite S'_1-module. Therefore, by Lemma 7, it is easy to see that

$$e(\mathfrak{m}'S'_1) = rm(\mathfrak{m}'S' ; S'_1) = e(\mathfrak{m}'S')[\kappa(\bar{u}, \tau, \sigma) : \kappa(\bar{u}, \tau, \sigma_{m+1})] .$$

Since $\kappa(\bar{u}, \tau, \sigma)$ is purely inseparable over $\kappa(\bar{u}, \tau, \sigma_{m+1})$, λ is equal to $e(\mathfrak{m}'S')[\kappa(\bar{u}, \tau, \sigma) : \kappa(\bar{u}, \tau)]_i$. It is evident that $e(\mathfrak{m}S) = e(\mathfrak{m}'S')$ and $[\kappa(\tau, \sigma) : \kappa(\tau)]_i = [\kappa(\bar{u}, \tau, \sigma) : \kappa(\bar{u}, \tau)]_i$, and hence we obtain $\lambda = e(\mathfrak{m}S)[\kappa(\tau, \sigma) : \kappa(\tau)]_i$.

If $k(t, s)$ is not separable over $k(t)$, let L be the separable closure of $k(t)$ in $k(t, s)$ and A the intersection of L and $\mathfrak{o}[t, s]$. Then A is an affine ring $\mathfrak{o}[t, r]$ over $\mathfrak{o}$ and $\mathfrak{o}[t, s]$ is integral over A. Therefore a unique specialization (τ, ρ) of (t, r) is determined by a specialization (τ, σ) of (t, s) over $\mathfrak{o}$ and conversely any specialization (τ, ρ) of (t, r) over $\mathfrak{o}$ is extended to (τ, σ') of (t, s) uniquely. If λ_0 is the multiplicity of (ρ) in a specialization of the complete set of conjugates of (r) over $k(t)$, the mutliplicity of (σ) in a complete set of conjugates of (s) over $k(t)$ is equal to $\lambda_0[k(t, s) : k(t, r)]$. If S_0 is the specialization ring $[(t, r) \xrightarrow{\mathfrak{o}} (\tau, \rho)]$, λ_0 is equal to $e(\mathfrak{m}S_0)[\kappa(\tau, \rho) : \kappa(\tau)]_i$ by the above investigation. On the other hand it is easy to see that S is integral over S_0. Since now $\mathfrak{o}$ satisfies the finiteness condition for integral extensions and S_0 is

18) See Theorem 6 in [3].
19) See Propositions 8 and 9, and Remark 1 of p. 211 in [3].
20) See Theorems 4 and 5 in [3].

a spot over $\mathfrak{o}$, S is a finite S_0-module. Therefore, by Lemma 7, we have

$$\lambda = \lambda_0[k(t, s) : k(t, r)] = e(\mathfrak{m}S_0)[\kappa(\tau, \rho) : \kappa(\tau)]_i[k(t, s) : k(t, r)]$$
$$= e(\mathfrak{m}S)[\kappa(\tau,\sigma) : \kappa(\tau, \rho)]_i[\kappa(\tau, \rho) : \kappa(\tau)]_i = e(\mathfrak{m}S)[\kappa(\tau, \sigma) : \kappa(\tau)]_i .$$

q.e.d.

Now we shall show that our multiplicity $\mu(\ ;\ )$ is equivalent to $\mu(\ ;\ )$ defined in [8].[21] It is sufficient to treat only affine varieties.

Let V^r be a prime rational cycle over k in an affine space A^n, where a system of coordinates is fixed, and let C be a prime component of the bunch of varieties obtained from V by the canonical reduction with respect to this system of coordinates. Let P^n be a projective space containing A^n and let V_0 be the closure of V in P^n. Let $\bar{V}_0$ be the bunch of varieties obtained from V_0 by the reduction with respect to $\mathfrak{p}$. Now we may assume that the residue class field κ of $\mathfrak{o}$ is not finite. In fact if κ is finite, let t and τ be independent variables over k and over κ respectively. Then $\mathfrak{o}$ is extended to the functional valuation of $k(t)$ having the residue class field $\kappa(\tau)$. It is evident that we may replace k by $k(t)$.

Therefore there exists a hyperplane H in P^n such that $\bar{H}$ in $\bar{P}^n$ does not contain any component of $\bar{V}_0$. Let A' and V' be an affine space $P^n - H$ and a prime rational cycle $V_0 - H$ in A' respectively. Let C' be the prime rational cycle over κ in $\bar{P}^n - \bar{H}$ corresponding to C. Then it is sufficient to investigate V' and C' in place of V and C, since the treated properties are local.

Let $(P, \mathfrak{m})$ and $(Q, \mathfrak{n})$ be the spots in $M(A')$ corresponding to V' and C' respectively. Then we have to show $\mu(P ; Q) = \mu(V' ; C')$, where the right hand side is the multiplicity defined in [8].

First we assume that V' has no multiple components. Let (x) and (ξ) be generic points of V' and C' respectively. Let t_{ij} $(i=1, \cdots, r ; j=1, \cdots, n)$ be independent variables over $k(x)$ and τ_{ij} $(i=1, \cdots, r ; j=1, \cdots, n)$ independent variables over $\kappa(\xi)$.

21) This is shown in [2] in the case when $\mathfrak{o}$ satisfies the finiteness condition for integral extensions.

Put $t_i=\sum_{j=1}^{n} t_{ij}x_j$ and $\tau_i=\sum_{j=1}^{n}\tau_{ij}\xi_j$ for each i. Moreover we put $\mathfrak{o}^*=[(t_{ij})\xrightarrow{\mathfrak{o}}(\tau_{ij})]$. Then it is easy to see that (x) is integral and separable over $\mathfrak{o}^*[t_1,\cdots,t_r]$. Let R be the specialization ring $[(t)\xrightarrow{\mathfrak{o}^*}(\tau)]$ and let S be the specialization ring $[(t,x)\xrightarrow{\mathfrak{o}^*}(\tau,\xi)]$. Then $(\pi)R$ is the maximal ideal of R and S is equal to $[(x)\xrightarrow{\mathfrak{o}^*}(\xi)]$. If λ is the multiplicity of (ξ) as a specialization of (x) over R, then by Theorem 5 we have $\lambda=e(\pi S)[\kappa(\tau_{ij},\tau,\xi):\kappa(\tau_{ij},\tau)]_i$. On the other hand it is easy to see that $e(\pi S)$ is equal to the multiplicity of the ideal generated by π in the specialization ring $[(x)\xrightarrow{\mathfrak{o}}(\xi)]$ and hence to $\mu(P;Q)$ in our sense. Moreover it is easy to see $[\kappa(\xi):\kappa]_i=[\kappa(\tau_{ij},\tau,\xi):\kappa(\tau_{ij},\tau)]_i$ and hence we have $e(\pi S)=\lambda/[\kappa(\xi):\kappa]_i$.

However $\lambda/[\kappa(\xi):\kappa]_i$ is, by definition, equal to the multiplicity $\mu(V';C')$ in the sense of [8].

Next we consider the case where V' has multiple components. Let (x) and (ξ) be as above, and let k^* be a purely inseparable finite extension of k such that $k^*(x)$ is separable over k^*. Let $\mathfrak{o}^*$ be the unique extension of $\mathfrak{o}$ in k^* and let κ^* be the residue class field of $\mathfrak{o}^*$. Then κ^* is also purely inseparable over κ. Let V_1' be the locus of (x) over k^* and let C_1' be that of (ξ) over κ^*. Let P_1 and Q_1 be the spots over $\mathfrak{o}^*$ corresponding to V_1' and C_1' respectively. Then we have $\mu(P_1;Q_1)=\mu(V_1';C_1')$.

On the other hand $\rho(V')$ and $\rho'(P)$ are compatible with extension of ground rings. Therefore the equality $\mu(P_1;Q_1)=\mu(V_1';C_1')$ means the equality $\mu(P;Q)=\mu(V';C')$.

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