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On soudures of differentiable fibre bundles

Dedicated to Prof. Y. Akizuki on his 60th birthday

By

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Introduction

In the differential geometry, the notion of a tangent space bundle which may be defined by a soudure structure palys an important rôle.

In this paper, we consider some geometric structures closely related to the soudure, and research into their existences. In §2, considering extensions of linear maps of vector bundles, we make preparations for the later sections. Moreover, we need some results on connections and on extensions of tensorial forms described respectively in §4 and in §5. We introduce in §6 the notion of a (G, ρ)-structure and its structure tensor. A soudure may be regarded as a special case of (G, ρ)-structures. Combining a connection and a soudure under a suitable conditon, we get the notion of a Cartan connection. In the last section, we make remarks on the Cartan structure tensor of a soudure.

It will be shown that the obstruction classes of the existences of such structures in the complex analytic case may be represented by differential forms through the theorem of Dolbeault.

§1. Fibre bundles

Throughout this paper, we assume that any differentiable manifold is paracompact, and that any fibre bundle is of class C^∞ or complex analytic.

Let $P(M, G)$ be a *principal G-bundle* over M with projecton

$\pi: P \to M$. Then, by definition, it satisfies the following conditions.

(i) The structural group G is a right transformation group on P which operates simply transitive on each fibre $G_x = \pi^{-1}(x)$ of P over $x \in M$.

(ii) There exists a local section $s: U \to P$, $\pi \circ s = \mathbf{1}$, on a neighborhood U of any point $x \in M$, where $\mathbf{1}$ denotes the identical map.

The operation of G will be expressed as multiplication

$$P \times G \to P, \quad (p, g) \to pg.$$

Then, taking a point $p \in P$ and an element $g \in G$, we have diffeomorphisms

$$p: G \to G_{\pi(p)}, \quad g \to pg, \text{ and } R_g: P \to P, \quad p \to pg,$$

called respectively an *admissible map* and a *right translation*. The right translation gives a diffeomorphism $R_g: G_x \to G_x$ on each fibre G_x for $x \in M$. Moreover, there exists an open covering $\{U_i\}_{i \in I}$ of M and a system of local sections

$$\{s_i\}_{i \in I}, \quad s_i: U_i \to P, \quad \pi \circ s_i = \mathbf{1}.$$

Then, we have uniquely a system of maps

$$\{g_{ij}\}_{i,j \in I}, \quad g_{ij}: U_i \cap U_j \to G, \quad s_j(x) = s_i(x) g_{ij}(x), \quad x \in U_i \cap U_j,$$

clearly satisfying the conditions

$$g_{ii}(x) = e, \quad x \in U_i, \quad \text{and} \quad g_{ij}(x) g_{jk}(x) = g_{ik}(x), \quad x \in U_i \cap U_j \cap U_k,$$

where e denotes the unit element of the group G. Let $\boldsymbol{G}$ denote the sheaf of germs of local G-valued functions on M. Then, the 1-cocycle $\{g_{ij}\}$ determines a cohomology class $\xi \in H^1(M, \boldsymbol{G})$, and it is well-known that the cohomology set $H^1(M, \boldsymbol{G})$ can be regarded as the set of all G-bundle structures on M.

Now, let G be a left transformation group on a manifold F, and express its operation as multiplication

$$G \times F \to F, \quad (g, y) \to gy.$$

An *associated bundle* $B = P \times_G F$ of $P(M, G)$ with fibre F is defined as a quotient space of $P \times F$ by the equivalent relation

$$(pg, y) \sim (p, gy), \quad p \in P, \quad g \in G, \quad y \in F,$$

Namely, its natural projection being also expressed as multiplication

$$P\times F\to B, \quad (p, y)\to py,$$

is characterized by the relation $(pg)y=p(gy)$ for $p\in P$, $g\in G$ and $y\in F$. Moreover, the projection $\pi': B\to M$ of the bundle B is defined by $\pi'(py)=\pi(p)$ for $p\in P$ and $y\in F$. Then, we can regard a point $p\in P$ as a diffeomorphism

$$p: F\to F_x, \quad y\to py,$$

where $F_x=\pi'^{-1}(x)$ is a fibre of B over $x=\pi(p)\in M$.

Let $\rho: G\to\tilde{G}$ be a homomorphism of Lie groups, and let $P(M, G)$, $\tilde{P}(M, \tilde{G})$ be principal bundles over M. A differentiable map $\tilde{\rho}: P\to\tilde{P}$ is called a *homomorphism* of bundles, if it satisfies the relation

$$\tilde{\rho}(pg) = \tilde{\rho}(p)\rho(g), \quad p\in P, \quad g\in G.$$

When such a map $\tilde{\rho}$ is given, we call $\tilde{P}$ an *extension* of P, and P a *restriction* of $\tilde{P}$. The homomorphism of groups $\rho: G\to\tilde{G}$ induces a homomomorphism of sheaves $\rho: \boldsymbol{G}\to\tilde{\boldsymbol{G}}$ and hence a map of cohomology sets

$$\tilde{\rho}: H^1(M, \boldsymbol{G})\to H^1(M, \tilde{\boldsymbol{G}}),$$

which maps each G-bundle structure to its extension. Thus, to any G-bundle P, corresponds a unique extension $\tilde{P}$ by ρ given by an associated bundle $\tilde{P}=P\times_{\rho(G)}\tilde{G}$. However, a $\tilde{G}$-bundle $\tilde{P}$ has not in general its restriction by ρ. When $\tilde{P}$ has a restriction P by ρ, we may suppose that the structural group $\tilde{G}$ of $\tilde{P}$ can be reduced to its subgroup $\rho(G)$.

Let $P(M, G)$ be a principal bundle with projection π, and let $\varphi: M'\to M$ be a differentiable map. Then, we obtain uniquely a principal bundle $P'(M', G)$ with projection π' and a map $\tilde{\varphi}: P'\to P$ such that

$$\pi\circ\tilde{\varphi} = \varphi\circ\pi', \quad \tilde{\varphi}(p'g) = \tilde{\varphi}(p')g, \quad p'\in P', \quad g\in G.$$

The bundle P' is called an *induced bundle* of P by φ. It is easy to see that the map φ induces a map

$$\varphi^* : H^1(M, \boldsymbol{G}) \to H^1(M', \boldsymbol{G}),$$

which maps each bundle structure on M to its induced bundle structure on M' by φ. Moreover, let $B(M, F, G)$ be a fibre bundle, and let $\varphi : M' \to M$ be a differentiable map. Taking an associated principal bundle $P(M, G)$ of B, we obtain an induced bundle $P'(M', G)$ of P. Then the induced bundle $B'(M', F, G)$ of B is defined as an associated bundle $B' = P' \times_G F$.

§2. Vector bundles

A fibre bundle $B(M, E, G)$ over M is called a *vector bundle*, if the fibre E is a vector space and the group G operates on E as a linear transformation group. Then, each fibre E_x over $x \in M$ becomes also a vector space. Let $P(M, G)$ be a principal bundle, and let $\rho : G \to GL(E)$ be a representaion of G on a vecter space E. Then, an associated bundle $B = P \times_{\rho(G)} E$ becomes a vector bundle. Moreover, denoting by $\rho^* = {}^t\rho^{-1} : G \to GL(E^*)$ the dual representation of ρ on the dual space E^* of E, we have an associated vector bundle $B^* = P \times_{\rho^*(G)} E^*$ called the *dual vector bundle* of B. Let $B(M, E, G)$, $B'(M, E', G')$ be two vector bundles over M with projections π, π' respectively. A differentiable map $\alpha : B \to B'$ is called a *linear map* of vector bundles, if $\pi' \circ \alpha = \pi$ and α induces a linear map $\alpha : E_x \to E'_x$ of fibres at each point $x \in M$. Moreover, we have vector bundles over M

$$B \oplus B' = \bigcup_{x \in M} E_x \oplus E'_x, \quad B \otimes B' = \bigcup_{x \in M} E_x \otimes E'_x,$$

called respectively the *Whitney sum* and the *tensor product* of vector bundles B and B'. Let $Hom\,(E, E')$ denote the module of all linear maps of E into E'. Then, we have a vector bundle over M

$$Hom\,(B, B') = \bigcup_{x \in M} Hom\,(E_x, E'_x) \cong B^* \otimes B'.$$

A linear map of vector bundles $\alpha : B \to B'$ can be regarded as a global section of the vector bundle $\alpha : M \to Hom\,(B, B')$. Let us denote by $E^{[k]}$, $E_{[k]}$ respectively the exterior k-vector spaces over E and over E^*. For a vector bundle $B(M, E, G)$, we have associated k-vector bundles

$$B^{[k]}(M)=\bigcup_{x\in M}E_x^{[k]},\quad B_{[k]}(M)=\bigcup_{x\in M}E_{x[k]}.$$

Let $\theta: W(M)\rightarrow V(M)$ be a linear map of vector bundles over M, and let $B(M)$ be a vector bundle over M. Then, the map θ induces linear maps of vector bundles over M

$$\theta: Hom\,(B,\ W)\rightarrow Hom\,(B,\ V),\quad \varphi\rightarrow\theta\circ\varphi\,,$$
$$\theta^*: Hom\,(V,\ B)\rightarrow Hou\,(W,\ B),\quad \psi\rightarrow\psi\circ\theta\,.$$

Let $V(M)$ be a vector bundle over M. We denote by $\Gamma_\infty(M,\ V)$, $\Gamma_h(M,\ V)$ respectively, the module of all C^∞ sections of $V(M)$, and the module of all complex analytic sections of $V(M)$, when they are well defined. One of them will be denoted simply by $\Gamma(M,\ V)$. Moreover, we denote by $\boldsymbol{V}_\infty$, $\boldsymbol{V}_h$ respectively, the sheaf of germs of local C^∞ sections of $V(M)$, and the sheaf of germs of local holomorphic sections of $V(M)$. One of them will be denoted simply by $\boldsymbol{V}$. As well-known the 0-dimensional cohomology groups with coefficients in $\boldsymbol{V}_\infty$, $\boldsymbol{V}_h$ are given by respectively

$$H^0(M,\ \boldsymbol{V}_\infty)\cong\Gamma(M,\ \boldsymbol{V}_\infty)\cong\Gamma_\infty(M,\ V)\,,$$
$$H^0(M,\ \boldsymbol{V}_h)\cong\Gamma(M,\ \boldsymbol{V}_h)\cong\Gamma_h(M,\ V)\,.$$

Now, we consider an exact sequence of vector bundles over M

$$0\rightarrow K(M)\xrightarrow{\iota}W(M)\xrightarrow{\tau}V(M)\rightarrow 0\,.$$

For a given linear map of vector bundles $\varphi: B(M)\rightarrow V(M)$, a linear map of vector bundles $\psi: B(M)\rightarrow W(M)$ is called an *extension* of φ over $W(M)$, if $\tau\circ\psi=\varphi$. Applying a functor $Hom\,(B,\ *)$ to the sequence, we have an exact sequence of vector bundles over M

$$0\rightarrow Hom\,(B,\ K)\xrightarrow{\iota}Hom\,(B,\ W)\xrightarrow{\tau}Hom\,(B,\ V)\rightarrow 0\,.$$

Taking the sheaves of germs of their local sections, we obtain an exact sequence of sheaves on M

$$0\rightarrow \boldsymbol{Hom}\,(B,\ K)\xrightarrow{\iota}\boldsymbol{Hom}\,(B,\ W)\xrightarrow{\tau}\boldsymbol{Hom}\,(B,\ V)\rightarrow 0$$

and its cohomology sequence

$$\xrightarrow{\iota} \Gamma(M, Hom(B, W)) \xrightarrow{\tau} \Gamma(M, Hom(B, V)) \xrightarrow{\delta} H^1(M, \boldsymbol{Hom}(B, K)) \rightarrow$$
$$\psi \longrightarrow \varphi \longrightarrow \delta\varphi.$$

Then, by the exactness of this sequence, there exists an extension $\psi \in \Gamma(M, Hom(B, W))$ of φ if and only if $\delta\varphi = 0$. Therefore, the class $\delta\varphi \in H^1(M, \boldsymbol{Hom}(B, K))$ may be regarded as the obstruction class of extension of φ over $W(M)$. In the case of C^∞ vector bundles, since the sheaf $\boldsymbol{Hom}(B, K)_\infty$ is fine, the class $\delta\varphi$ always vanishes, and so there exists a C^∞ extension ψ of φ. In the case of complex analytic vector bundles, the obstruction class $\delta\varphi$ of analytic extension of an analytic linear map φ appears in generel.

Let us take a linear map of vector bundles $\theta: T(M) \rightarrow B(M)$. Then, it induces an exact and commutative diagram of sheaves

$$\begin{array}{ccccccccc} 0 & \rightarrow & \boldsymbol{Hom}(B, K) & \xrightarrow{\iota} & \boldsymbol{Hom}(B, W) & \xrightarrow{\tau} & \boldsymbol{Hom}(B, V) & \rightarrow & 0 \\ & & \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle \theta^*} & & \\ 0 & \rightarrow & \boldsymbol{Hom}(T, K) & \xrightarrow{\iota} & \boldsymbol{Hom}(T, W) & \xrightarrow{\tau} & \boldsymbol{Hom}(T, V) & \rightarrow & 0 \end{array}$$

and their cohomology sequences

$$\begin{array}{ccccccc} \xrightarrow{\iota} & \Gamma(M, Hom(B, W)) & \xrightarrow{\tau} & \Gamma(M, Hom(B, V)) & \xrightarrow{\delta} & H^1(M, \boldsymbol{Hom}(B, K)) & \rightarrow \\ & \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle \theta^*} & \\ \xrightarrow{\iota} & \Gamma(M, Hom(T, W)) & \xrightarrow{\tau} & \Gamma(M, Hom(T, V)) & \xrightarrow{\delta} & H^1(M, \boldsymbol{Hom}(T, K)) & \rightarrow \\ & & & \varphi\circ\theta & \longrightarrow & \theta^*\delta\varphi. & \end{array}$$

Thus, the following relation holds.

Proposition 2.1. $\delta(\varphi\circ\theta) = \theta^*\delta\varphi, \quad \varphi \in \Gamma(M, Hom(B, V)),$ $\theta \in \Gamma(M, Hom(T, B))$.

Again, we consider the exact sequence of vector bundles

$$0 \rightarrow K(M) \xrightarrow{\iota} W(M) \xrightarrow{\tau} V(M) \rightarrow 0.$$

For a given linear map of vector bundles $\mu: K(M) \rightarrow L(M)$, a linear map of vector bundles $\nu: W(M) \rightarrow L(M)$ is also called an *extension* of μ over $W(M)$, if $\nu\circ\iota = \mu$. Applying the functor $Hom(*, L)$ to the sequence, we have an exact sequence of vector bundles

$$0 \to Hom(V, L) \xrightarrow{\tau^*} Hom(W, L) \xrightarrow{\iota^*} Hom(K, L) \to 0 .$$

Taking the exact sequence of sheaves of germs of their local sections, we obtain its cohomology sequence

$$\begin{array}{ccccccc} \xrightarrow{\tau^*} \Gamma(M, Hom(W, L)) & \xrightarrow{\iota^*} & \Gamma(M, Hom(K, L)) & \xrightarrow{\delta^*} & H^1(M, \mathbf{Hom}(V, L)) \to \\ \nu & \longrightarrow & \mu & \longrightarrow & \delta^*\mu. \end{array}$$

Then, there exists an extension $\nu \in \Gamma(M, Hom(W, L))$ of μ, if and only if $\delta^*\mu=0$. Thus, the class $\delta^*\mu \in H^1(M, \mathbf{Hom}(V, L))$ may be regarded as the obstruction class of extension of μ over $W(M)$.

Let us take a linear map of vector bundles $\kappa: L(M) \to J(M)$. Then, it induces a commutative diagram of cohomology sequences

$$\begin{array}{ccccc} \xrightarrow{\tau^*} \Gamma(M, Hom(W, L)) & \xrightarrow{\iota^*} & \Gamma(M, Hom(K, L)) & \xrightarrow{\delta^*} & H^1(M, \mathbf{Hom}(V, L)) \to \\ \downarrow \kappa & & \downarrow \kappa & & \downarrow \kappa \\ \xrightarrow{\tau^*} \Gamma(M, Hom(W, J)) & \xrightarrow{\iota^*} & \Gamma(M, Hom(K, J)) & \xrightarrow{\delta^*} & H^1(M, \mathbf{Hom}(V, J)) \to \\ & & \kappa\circ\mu & \longrightarrow & \kappa\delta^*\mu. \end{array}$$

Thus, the following relation holds.

Proposition 2. 2. $\delta^*(\kappa\circ\mu) = \kappa\delta^*\mu, \qquad \mu \in \Gamma(M, Hom(K, L)),$ $\kappa \in \Gamma(M, Hom(L, J))$.

Consider an exact sequence of vector bundles

$$\mathfrak{S}: 0 \to K(M) \xrightarrow{\iota} W(M) \xrightarrow{\tau} V(M) \to 0 .$$

A linear map of vector bundles $\gamma: V(M) \to W(M)$ is called a *splitting* of $\mathfrak{S}$, if $\tau\circ\gamma=\mathbf{1}$. And a linear map $\omega: W(M) \to K(M)$ is also called a splitting of $\mathfrak{S}$, if $\omega\circ\iota=\mathbf{1}$. Such splittings γ and ω may be regarded as the same one, if they satisfy the relation

$$\iota\circ\omega+\gamma\circ\tau = \mathbf{1} ,$$

which determines a one-to-one correspondence between γ and ω. It is clear that a splitting of $\mathfrak{S}$ means to give a bijection $W(M) \cong K(M) \oplus V(M)$.

Applying a functor $Hom(V, *)$ to $\mathfrak{S}$, we have an exact sequence of vector bundles

$$0 \to Hom(V, K) \xrightarrow{\iota} Hom(V, W) \xrightarrow{\tau} Hom(V, V) \to 0 .$$

Taking the exact sequence of sheaves of germs of their local sections, we obtain its cohomology sequence

$$\begin{array}{ccccccc} \xrightarrow{\iota} \Gamma(M, Hom(V, W)) & \xrightarrow{\tau} & \Gamma(M, Hom(V, V)) & \xrightarrow{\delta} & H^1(M, \boldsymbol{Hom}(V, K)) \rightarrow \\ \gamma & \longrightarrow & \mathbf{1} & \longrightarrow & \delta\mathbf{1}. \end{array}$$

Then, there exists a splitting γ of $\mathfrak{S}$, if and only if $\delta\mathbf{1}=0$. Set

$$a(\mathfrak{S}) = \delta\mathbf{1} \in H^1(M, \boldsymbol{Hom}(V, K)),$$

and we may regard $a(\mathfrak{S})$ as the obstruction class of splittng of $\mathfrak{S}$.

Moreover, applying a functor $H(*, K)$ to $\mathfrak{S}$, we have an exact sequence of vector bundles

$$0 \rightarrow Hom(V, K) \xrightarrow{\tau^*} Hom(W, K) \xrightarrow{\iota^*} Hom(K, K) \rightarrow 0.$$

Taking the exact sequence of sheaves of germs of their local sections, we obtain its cohomology sequence

$$\begin{array}{ccccccc} \xrightarrow{\tau^*} \Gamma(M, Hom(W, K)) & \xrightarrow{\iota^*} & \Gamma(M, Hom(K, K)) & \xrightarrow{\delta^*} & H^1(M, \boldsymbol{Hom}(V, K)) \rightarrow \\ \omega & \longrightarrow & \mathbf{1} & \longrightarrow & \delta^*\mathbf{1}. \end{array}$$

Then, we get another obstruction class $\delta^*\mathbf{1} \in H^1(M, \boldsymbol{Hom}(V, K))$ of splitting of $\mathfrak{S}$.

Proposition 2.3. $\delta\mathbf{1}+\delta^*\mathbf{1}=0$, *that is*, $\delta^*\mathbf{1}=-a(\mathfrak{S})$.

Proof. There exists always a local splitting over a neighborhood of any point $x \in M$. Hence, we can take an open covering $\mathfrak{U}=\{U_i\}_{i\in I}$ of M and a 0-cochain

$$\{\gamma_i\} \in C^0(\mathfrak{U}, \boldsymbol{Hom}(V, W)) \quad \text{such that} \quad \tau\circ\gamma_i = \mathbf{1}.$$

Then, by definition of the coboundary homomorphism δ, we have a 1-cocycle

$$\{a_{ij}\} \in C^1(\mathfrak{U}, \boldsymbol{Hom}(V, K)), \qquad a_{ij} = \iota^{-1}\circ(\gamma_i-\gamma_j),$$

which represents the class $\delta\mathbf{1} \in H^1(M, \boldsymbol{Hom}(V, K))$. On the other hand, we can take a 0-cochain

$$\{\omega_i\} \in C^0(\mathfrak{U}, \boldsymbol{Hom}(W, K)), \qquad \omega_i\circ\iota = \mathbf{1},$$

determined by the relation $\iota\circ\omega_i+\gamma_i\circ\tau=\mathbf{1}$, and we obtain a 1-cocycle

$$\{a^*_{ij}\} \in C^1(\mathfrak{U}, \boldsymbol{Hom}(V, K)), \qquad a^*_{ij} = (\omega_i - \omega_j) \circ \tau^{-1},$$

which represents the class $\delta^*\mathbf{1} \in H^1(M, \boldsymbol{Hom}(V, K))$. Then, we have

$$\begin{aligned} a_{ij} + a^*_{ij} &= \iota^{-1} \circ \{(\gamma_i - \gamma_j) \circ \tau + \iota \circ (\omega_i - \omega_j)\} \circ \tau^{-1} \\ &= \iota^{-1} \circ (\mathbf{1} - \mathbf{1}) \circ \tau^{-1} = 0 . \end{aligned}$$

This proves that $\delta\mathbf{1} + \delta^*\mathbf{1} = 0$.

Moreover, in the Propositions 2.1 and 2.2, setting $\varphi = \mathbf{1}$ and $\mu = \mathbf{1}$ respectively, we get the followings.

Proposition 2.4. *Let $a(\mathfrak{S})$ be the obstruction class of splitting of the exact sequence $\mathfrak{S}$.*

1° $\delta\theta = \theta^* a(\mathfrak{S})$, $\quad \theta \in \Gamma(M, Hom(T, V))$.

2° $\delta^*\kappa = -\kappa a(\mathfrak{S})$, $\quad \kappa \in \Gamma(M, Hom(K, J))$.

Now, we consider two exact sequences of vector bundles

$$\begin{array}{l} \mathfrak{S}:\ 0 \to K(M) \xrightarrow{\iota} W(M) \xrightarrow{\tau} V(M) \to 0 \\ \qquad\qquad \Vert \qquad\qquad\qquad\qquad \Vert \\ \tilde{\mathfrak{S}}:\ 0 \to K(M) \xrightarrow{\tilde{\iota}} \tilde{W}(M) \xrightarrow{\tilde{\tau}} V(M) \to 0 . \end{array}$$

A bijection $\beta:\ W(M) \to \tilde{W}(M)$ is called an *isomorphism* of $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$, if $\beta \circ \iota = \tilde{\iota}$ and $\tilde{\tau} \circ \beta = \tau$.

Proposition 2.5. *There exists an isomorphism of $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$, if and only if $a(\mathfrak{S}) = a(\tilde{\mathfrak{S}})$.*

Proof. If $\beta:\ W(M) \to \tilde{W}(M)$ is an isomorphism of $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$, then it induces a commutative diagram of cohomology sequences

$$\begin{array}{l} \xrightarrow{\iota} \Gamma(M, Hom(V, W)) \xrightarrow{\tau} \Gamma(M, Hom(V, V)) \xrightarrow{\delta} H^1(M, \boldsymbol{Hom}(V, K)) \to \\ \qquad \Big\downarrow \beta \qquad\qquad\qquad\qquad\quad \Vert \qquad\qquad\qquad\qquad \Vert \\ \xrightarrow{\tilde{\iota}} \Gamma(M, Hom(V, \tilde{W})) \xrightarrow{\tilde{\tau}} \Gamma(M, Hom(V, V)) \xrightarrow{\tilde{\delta}} H^1(M, \boldsymbol{Hom}(V, K)) \to . \end{array}$$

Hence, $a(\mathfrak{S}) = \delta\mathbf{1} = \tilde{\delta}\mathbf{1} = a(\tilde{\mathfrak{S}})$. Conversely, assume that $a(\mathfrak{S}) = a(\tilde{\mathfrak{S}})$. We can take an open covering $\{U_i\}_{i \in I}$ of M and local splittings

$$\begin{aligned} &\{\omega_i\} \in C^0(\mathfrak{U}, \boldsymbol{Hom}(W, K)), \qquad \omega_i \circ \iota = \mathbf{1}, \\ &\{\tilde{\gamma}_i\} \in C^0(\mathfrak{U}, \boldsymbol{Hom}(V, \tilde{W})), \qquad \tilde{\tau} \circ \tilde{\gamma}_i = \mathbf{1}, \end{aligned}$$

of $\mathfrak{S}$, $\tilde{\mathfrak{S}}$ respectively, and we have 1-cocycles

$$\{a^*_{ij}\},\ \{\tilde{a}_{ij}\} \in C^1(\mathfrak{U},\ \boldsymbol{Hom}(V,\ K))\ ,$$
$$a^*_{ij} = (\omega_i - \omega_j)\circ\tau^{-1},\quad \tilde{a}_{ij} = \tilde{\iota}^{-1}\circ(\tilde{\gamma}_i - \tilde{\gamma}_j)\ ,$$

which represent the classes $-a(\mathfrak{S})$, $a(\tilde{\mathfrak{S}})$ respectively. Since $a(\mathfrak{S}) = a(\tilde{\mathfrak{S}})$, we can choose the 0-cochains $\{\omega_i\}$, $\{\tilde{\gamma}_i\}$ such that $a^*_{ij} + \tilde{a}_{ij} = 0$. Then, taking a 0-cochain

$$\{\beta_i\} \in C^0(\mathfrak{U},\ \boldsymbol{Hom}(W,\ \tilde{W}))\ ,\quad \beta_i = \tilde{\iota}\circ\omega_i + \tilde{\gamma}_i\circ\tau\ ,$$

we see that

$$\begin{aligned}\beta_i - \beta_j &= \tilde{\iota}\circ(\omega_i - \omega_j) + (\tilde{\gamma}_i - \tilde{\gamma}_j)\circ\tau \\ &= \tilde{\iota}\circ(a^*_{ij} + \tilde{a}_{ij})\circ\tau = 0\ .\end{aligned}$$

Hence, we obtain a bijection β: $W(M) \to \tilde{W}(M)$ such that $\beta = \beta_i$ in U_i, which gives an isomorphism of $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$.

Thus, we can regard the class $a(\mathfrak{S}) - a(\tilde{\mathfrak{S}}) \in H^1(M,\ \boldsymbol{Hom}(V,\ K))$ as the obstruction class of isomorphism of $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$.

§ 3. Tangent vector bundles

Let M be a differentiable manifold. We denote by $A(U)$ and by $\boldsymbol{A}_x$ respectively, the ring of all differentiable functions on an open set $U \subset M$, and the ring of germs of $A(U)$ on a point $x \in M$. A *tangent vector* X of M at $x \in M$ is by definition a map X: $\boldsymbol{A}_x \to R$ satisfying the conditions:

(i) $X(c_1 f_1 + c_2 f_2) = c_1 X f_1 + c_2 X f_2$, $\quad c_1, c_2 \in R$, $\quad f_1, f_2 \in \boldsymbol{A}_x$,
(ii) $X(fg) = (Xf)g(x) + f(x)(Xg)$, $\quad f, g \in \boldsymbol{A}_x$.

When M is complex analytic, we must take the complex number field C in stead of the real number field R in the above definition.

Let $T_x(M)$, $T(M)$ denote respectively the tangent vector space at a point $x \in M$ and the tangent vector bundle over M. For two manifolds M, N, we have natural bijections

$$T(M\times N) \cong T(M)\times T(N)\ ,\quad T_{(x,y)}(M\times N) \cong T_x(M)\oplus T_y(N)\ .$$

As well-known, a differentiable map α: $M \to N$ induces a map

$$\alpha:\ T(M) \to T(N)\ ,\quad (\alpha X)f = X(f\circ\alpha)\ ,\quad f \in \boldsymbol{A}_y\ ,\quad y \in N\ ,$$

which we shall denote by the same letter α. In particular, for a differentiable map expressed as multiplication

$$\mu: M\times N \to K, \quad (x, y) \to xy,$$

taking a point $a\in M$ or $b\in N$, we get a map

$$a: N\to K, \quad y\to ay, \text{ or } b^*: M\to K, \quad x\to xb,$$

whose induced map may be also expressed as multiplication

$$a: T(N)\to T(K), \quad Y\to aY, \text{ or } b^*: T(M)\to T(K), \quad X\to Xb.$$

Then, the induced map of the multiplication μ is given by

$$\mu: T(M)\times T(N)\to T(K), \quad (X, Y)\to Xy+xY,$$

where $X\in T_x(M)$, $Y\in T_y(N)$ and $x\in M$, $y\in N$. Now, we set

$$T^k(M) = \bigcup_{x\in M} T^k_x(M), \quad T^k_x(M) = T_x(M)\oplus\cdots\oplus T_x(M).$$

Then, $T^k(M)$ becomes an associated bundle of $T(M)$. Let E be a vector space. An *E-valued k-form* φ on M is by definition a differentiable map

$$\varphi: T^k(M)\to E, \quad (X_1, \cdots, X_k)\to\varphi(X_1, \cdots, X_k),$$

being multilinear and alternate with respect to vectors $X_1, \cdots, X_k\in T_x(M)$ for $x\in M$.

Let U, V and W be vector spaces, and let $F: U\times V\to W$ be a bilinear map. Two forms

$$\varphi: T^r(M)\to U \quad\text{and}\quad \psi: T^s(M)\to V$$

on M can be substituted into F, and we have a form

$$F(\varphi, \psi): T^{r+s}(M)\to W$$

on M defined by

$$F(\varphi, \psi)(X_1, \cdots, X_{r+s}) = \frac{1}{(r+s)!}\sum_\sigma \operatorname{sign}(\sigma)F(\varphi(X_{\sigma(1)}, \cdots, X_{\sigma(r)}), \psi(\cdots, X_{\sigma(r+s)}))$$

for $X_1, \cdots, X_{r+s}\in T_x(M)$, where the summation is extended over all permutations of a set of numbers $\{1, 2, \cdots, r+s\}$. We remark that the exterior product of real or complex valued forms is a special case when $F(x, y)=xy$.

Moreover, we can define the *exterior derivation d* which maps each E-valued k-form θ to an E-valued $(k+1)$-form $d\theta$, satisfying the following conditions.

(i) $d(\theta_1+\theta_2) = d\theta_1+d\theta_2$.

(ii) $dF(\varphi, \psi) = F(d\varphi, \psi)+(-1)^r F(\varphi, d\psi)$.

(iii) $d\circ d = 0$.

(iv) $df(X) = Xf$, for $f\in A_x$, $X\in T_x(M)$.

Let G be a Lie group, and let $\mathfrak{g}$ denote its Lie algebra being identified with the tangent vector space $T_e(G)$ at the unit element $e\in G$. Any point $g\in G$ defines diffeomorphisms of G

$$L_g: G\to G, \quad x\to gx, \text{ and } R_g: G\to G, \quad x\to xg,$$

called respectively *left* and *right translations.* It is notable that the tangent vector bundle of G becomes a product bundle such as

$$G\times\mathfrak{g} \cong T(G), \quad (x, A)\to xA, \quad x\in G, \quad A\in\mathfrak{g}.$$

Thereby, we have a $\mathfrak{g}$-valued 1-form on G

$$\omega: T(G)\to\mathfrak{g}, \quad xA\to A, \quad x\in G, \quad A\in\mathfrak{g},$$

called the *Maurer-Cartan form* of G, which satisfies clearly the relations

$$\omega\circ L_g = \omega, \quad \omega\circ R_g = ad(g^{-1})\omega, \quad g\in G,$$

where we denote by $ad: G\to GL(\mathfrak{g})$ the *adjoint representation* of G, that is,

$$ad(x): \mathfrak{g}\to\mathfrak{g}, \quad A\to xAx^{-1}, \quad x\in G.$$

A homomorphism of Lie groups $\rho: G\to G'$ induces a homomorphism of their Lie algebras

$$\bar{\rho}: \mathfrak{g}\to\mathfrak{g}', \quad A\to\rho A.$$

In particular, the adjoint representation $ad: G\to GL(\mathfrak{g})$ induces a representation $\overline{ad}: \mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ given by

$$\overline{ad}(A)B = [A, B], \quad A, B\in\mathfrak{g}.$$

It is well-known that the exterior derivative $d\omega$ of the Maurer-Cartan form ω of G is given by

$$d\omega = -\frac{1}{2}[\omega, \omega],$$

so called the *structure equation* of Cartan.

Let G be a closed subgroup of a Lie group $\tilde{G}$. Take a homogeneous space $F=\tilde{G}/G$, and denote by $\tau: \tilde{G}\rightarrow F$ its natural projection. Then, we have a principal bundle $\tilde{G}(F, G)$ with projection τ, whose right translation is given by

$$R_g\tilde{x} = \tilde{x}g\,, \quad g\in G\,, \quad \tilde{x}\in\tilde{G}\,.$$

Taking the Lie algebras $\tilde{\mathfrak{g}}=T_e(\tilde{G})$, $\mathfrak{g}=T_e(G)$, and the tangent vector space $\mathfrak{f}=T_{\tau e}(F)$, we get an exact and commutative diagram of vector spaces

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathfrak{g} & \xrightarrow{\iota} & \tilde{\mathfrak{g}} & \xrightarrow{\tau} & \mathfrak{f} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle ad(g)} & & \downarrow{\scriptstyle ad(g)} & & \downarrow{\scriptstyle is(g)} & & \\ 0 & \rightarrow & \mathfrak{g} & \xrightarrow{\iota} & \tilde{\mathfrak{g}} & \xrightarrow{\tau} & \mathfrak{f} & \longrightarrow & 0 \end{array}$$

for $g\in G$, where we denote by $is: G\rightarrow GL(\mathfrak{f})$ the *isotropy representation.*

Let $B(M, F, G)$ be a fibre bundle with projection π. A tangent vector $X\in T(B)$ of B is said to be *vertical*, if $\pi X=0$. Let $P(M, G)$ be a principal bundle. Then the vertical vector bundle $V(P)$ of P becomes a product bundle such as

$$\lambda: P\times\mathfrak{g}\approxeq V(P)\,, \quad (p, A)\rightarrow pA\,, \quad p\in P\,, \quad A\in\mathfrak{g}\,,$$

and hence we have an injection $\lambda: P\times\mathfrak{g}\rightarrow T(P)$. Since $pg(g^{-1}Ag)=(pA)g$ for $g\in G$, dividing the spaces $P\times\mathfrak{g}$ and $T(P)$ by G, we obtain an exact sequence of vector bundles over M

$$\mathfrak{S}: 0\rightarrow L(M)\xrightarrow{\lambda} Q(M)\xrightarrow{\pi} T(M)\rightarrow 0\,,$$

where $L(M)=P\times_{ad(G)}\mathfrak{g}$ is an associated bundle of P determined by the adjoint representation of G, and $Q(M)=T(P)/G$ denotes a quotient space of $T(P)$ by the equivalent relation $X\sim Xg$ for $g\in G$. The sequence $\mathfrak{S}$ is called the *fundamental sequence* of $P(M, G)$. In particular, let us consider a homogeneous space $F=\tilde{G}/G$ and the principal bundle $\tilde{G}(F, G)$. Since $T(\tilde{G})=\tilde{G}\times\tilde{\mathfrak{g}}$, dividing the sequence

$$0\rightarrow\tilde{G}\times\mathfrak{g}\xrightarrow{\iota}\tilde{G}\times\tilde{\mathfrak{g}}\xrightarrow{\tau}\tilde{G}\times\mathfrak{f}\rightarrow 0$$

by G, we obtain the fundamental sequence of $\tilde{G}(F, G)$

$$0 \to L(F) \xrightarrow{\iota} Q(F) \xrightarrow{\tau} T(F) \to 0 .$$

This proves that $Q(F) \simeq \tilde{G} \times_{ad(G)} \tilde{\mathfrak{g}}$ and $T(F) \simeq \tilde{G} \times_{is(G)} \mathfrak{f}$. Hence, the structural group of the tangent bundle of a homogeneous space $F = \tilde{G}/G$ can be reduced to its isotropy group $is(G)$.

Let $\tilde{P}(M, \tilde{G})$ be a principal bundle, and let G be a closed subgroup of $\tilde{G}$. Then, we have an associated bundle $B = \tilde{P}/G = \tilde{P} \times_{\tilde{G}} F$ of $\tilde{P}(M, \tilde{G})$ with fibre $F = \tilde{G}/G$, and a principal bundle $\tilde{P}(B, G)$ over B. Let $W(\tilde{P})$, $\tilde{W}(\tilde{P})$ and $V(B)$ denote respectively the vertical vector bundles of $\tilde{P}(B, G)$, $\tilde{P}(M, \tilde{G})$ and $B(M, F, \tilde{G})$. Since $W(\tilde{P}) = \tilde{P} \times \mathfrak{g}$ and $\tilde{W}(\tilde{P}) = \tilde{P} \times \tilde{\mathfrak{g}}$, dividing the sequence

$$0 \to \tilde{P} \times \mathfrak{g} \xrightarrow{\iota} \tilde{P} \times \tilde{\mathfrak{g}} \xrightarrow{\tau} \tilde{P} \times \mathfrak{f} \to 0$$

by G, we obtain an exact sequence of vector bundles over B

$$0 \to W(\tilde{P})/G \xrightarrow{\iota} \tilde{W}(\tilde{P})/G \xrightarrow{\tau} V(B) \to 0 .$$

This shows that $V(B) \simeq \tilde{P} \times_{is(G)} \mathfrak{f}$. Hence, the structural group of the vertical vector bundle $V(B)$ of $B = \tilde{P}/G$ can be reduced to the isotropy group $is(G)$.

Let $P(M, G)$ be a principal bundle with projection π, and let $\rho : G \to GL(E)$ be a representation of G on a vector space E. An E-valued k-form $\theta : T^k(P) \to E$ on P is said to be a *contravariant k-form* on $P(M, G)$ of type (ρ, E), if it satisfies the condition:

(i) $\theta \circ R_g = \rho(g^{-1})\theta , \quad g \in G .$

Moreover, a contravariant k-form θ is said to be *tensorial*, if it satisfies the condition:

(ii) $\theta(X_1, \cdots, X_k) = 0$ for $X_1, \cdots, X_k \in T_p(P)$, if $\pi X_1 = 0$.

In particular, a tensorial 0-form is called a *tensor*. In this case, the condition (ii) is not necessary.

Let us take an associated vector bundle $V(M) = P \times_{\rho(G)} E$ of $P(M, G)$, and let $\tilde{A}^k(M, V)$, $A^k(M, V)$ denote respectively the module of all contravariant k-forms on $P(M, G)$ of type (ρ, E) and the module of all tensorial k-forms on $P(M, G)$ of type (ρ, E). Let

$$0 \to L(M) \xrightarrow{\lambda} Q(M) \xrightarrow{\pi} T(M) \to 0$$

be the fundamental sequence of $P(M, G)$. In general, an E-valued k-form $\theta: T^k(P) \to E$ on P can be regarded as a linear map of vector bundles over P such as

$$\theta: T^{[k]}(P) \to P \times E, \quad u \to (p, \theta(u)), \quad u = \sum X_1 \wedge \cdots \wedge X_k \in T_p^{[k]}(P).$$

If θ is a contravariant form on $P(M, G)$ of type (ρ, E), then dividing $T^{[k]}(P)$ and $P \times E$ by G, we can regard θ as a linear map of vector bundles over M

$$\theta: Q^{[k]}(M) \to V(M) = P \times_{\rho(G)} E.$$

Moreover, if θ is tensorial, then it can be regarded as a linear map of vector bundles

$$\theta: T^{[k]}(M) \to V(M).$$

Thus, we obtain the following.

Proposition 3. 1. *There exist natural bijections*:

$$\tilde{A}^k(M, V) \cong \Gamma(M, Hom\,(Q^{[k]}, V)), \quad A^k(M, V) \cong \Gamma(M, Hom\,(T^{[k]}, V)).$$

In the sense of this proposition, a tensorial k-form on $P(M, G)$ of type (ρ, E) is called sometimes a *$V(M)$-valued k-form* on M.

We remark that the exterior derivative $d\theta$ of a contravariant form θ is clearly contravariant, but $d\theta$ is not tensorial in general even though θ is tensorial.

Here, let us consider a complex analytic vector bundles $V(M)$ over a complex analytic manifold M. In this case, we have modules of holomorphic r-forms or C^∞ (r, s)-forms such as

$$\begin{aligned} &\tilde{A}^r(M, V) \cong \Gamma_h(M, Hom\,(Q^{[r]}, V)), \\ &A^r(M, V) \cong \Gamma_h(M, Hom\,(T^{[r]}, V)), \\ &\tilde{A}^{rs}(M, V) \cong \Gamma_\infty(M, Hom\,(Q^{[r]} \otimes \bar{Q}^{[s]}, V)), \\ &A^{rs}(M, V) = \Gamma_\infty(M, Hom\,(T^{[r]} \otimes \bar{T}^{[s]}, V)), \end{aligned}$$

where $\bar{Q}(M)$ and $\bar{T}(M)$ denote respectively the conjugate vector bundles of $Q(M)$ and $T(M)$.

The exterior derivative $d\theta$ of a contravariant (r, s)-form $\theta \in \tilde{A}^{rs}(M, V)$ is decomposed uniquely as

$$d\theta = \theta' + \theta'', \quad \theta' \in \tilde{A}^{r+1,s}(M, V), \quad \theta'' \in \tilde{A}^{r,s+1}(M, V).$$

Thereby, setting $d'\theta = \theta'$ and $d''\theta = \theta''$, we have differentiations

$$d' : \tilde{A}^{rs}(M, V) \to \tilde{A}^{r+1,s}(M, V),$$
$$d'' : \tilde{A}^{rs}(M, V) \to \tilde{A}^{r,s+1}(M, V),$$

which satisfy the relations

$$d' \circ d' = 0, \quad d'' \circ d'' = 0, \quad d' \circ d'' + d'' \circ d' = 0.$$

Moreover, we can see that the derivative $d''\theta$ of a tensorial form $\theta \in A^{rs}(M, V)$ is also tensorial, and we have a cochain complex

$$A^{r*}(M, V) = \sum_s A^{rs}(M, V), \; d'' : A^{rs}(M, V) \to A^{r,s+1}(M, V).$$

Let us take the sheaves on M

$$\boldsymbol{A}^r = \boldsymbol{Hom}\,(T^{[r]}, V)_h, \quad \boldsymbol{A}^{rs} = \boldsymbol{Hom}\,(T^{[r]} \otimes \bar{T}^{[s]}, V)_\infty,$$

which denote respectively the sheaf of germs of local holomophic tensorial r-forms and the sheaf of germs of local C^∞ tensorial (r, s)-forms. Then, we have a fine resolution of $\boldsymbol{A}^r$

$$0 \to \boldsymbol{A}^r \xrightarrow{j} \boldsymbol{A}^{r0} \xrightarrow{d''} \boldsymbol{A}^{r1} \xrightarrow{d''} \cdots \xrightarrow{d''} \boldsymbol{A}^{rs} \xrightarrow{d''} \cdots,$$

and hence the Dolbeault isomorphism

$$H^s(M, \boldsymbol{A}^r) \cong H^s(A^{r*}(M, V)) = Z^{rs}(M, V)/d''A^{r,s-1}(M, V), \quad s \geq 1,$$

where $Z^{rs}(M, V)$ denotes the module of all (r, s)-cocycles of $A^{r*}(M, V)$ by d''-cohomology.

Let us show explicitly this isomorphism when $s=1$. Take an exact sequence of sheaves

$$0 \to \boldsymbol{A}^r \xrightarrow{j} \boldsymbol{A}^{r0} \xrightarrow{d''} \boldsymbol{Z}^{r1} \to 0,$$

and its cohomology sequence

$$\xrightarrow{j} A^{r0}(M, V) \xrightarrow{d''} Z^{r1}(M, V) \xrightarrow{\delta''} H^1(M, \boldsymbol{A}^r) \to 0,$$

where $H^1(M, \boldsymbol{A}^{r0}) = 0$, since $\boldsymbol{A}^{r0}$ is fine. Then, certainly we obtain the Dolbeault isomorphism

$$\delta'' : H^1(A^{r*}(M, V)) = Z^{r1}(M, V)/d''A^{r0}(M, V) \cong H^1(M, \boldsymbol{A}^r).$$

§4. Connections

Let $P(M, G)$ be a principal bundle, and let us take its fundamental sequence

$$\mathfrak{S}:\ 0 \to L(M) \xrightarrow{\lambda} Q(M) \xrightarrow{\pi} T(M) \to 0\,.$$

A differentiable splitting $\omega:\ Q(M) \to L(M)$, $\omega\circ\lambda = \mathbf{1}$, is called a *connection* on $P(M, G)$. Then, the linear map of vector bundles ω can be regarded as a contravariant 1-form $\omega:\ T(P) \to \mathfrak{g}$ on $P(M, G)$ of type $(ad,\ \mathfrak{g})$ called the *connection form.*

Proposition 4.1. *The connection form* $\omega:\ T(P) \to \mathfrak{g}$ *is characterized by the following properties.*

(i) $\omega\circ R_g = ad(g^{-1})\omega\,,\quad g\in G\,.$

(ii) $\omega(pA) = A\,,\quad p\in P\,,\quad A\in\mathfrak{g}\,.$

Proof. The property (i) shows that ω is a contravariant 1-form of type $(ad,\ \mathfrak{g})$. Hence, we can regard ω as a linear map of vector bundles $\omega:\ Q(M) \to L(M)$. Then, the property (ii) means that $\omega\circ\lambda = \mathbf{1}$.

The *curvature form* Ω of the connection ω is a $\mathfrak{g}$-valued 2-form on P given by the *structure equation*

$$\Omega = d\omega + \frac{1}{2}[\omega,\ \omega]\,.$$

It is known that Ω becomes a tensorial 2-form of type $(ad,\ \mathfrak{g})$.

Let $\theta:\ T^k(P) \to E$ be a tensorial k-form on $P(M, G)$ of type (ρ, E). The *covariant derivative* $D\theta$ with respect to the connection ω is an E-valued $(k+1)$-form given by the formula

$$D\theta = d\theta + \bar{\rho}(\omega)\theta\,,$$

where $\bar{\rho}:\ \mathfrak{g} \to \mathfrak{gl}(E)$ denotes the representation of Lie algebra induced by ρ. With use of some formulas related to the Lie derivative and the exterior derivative, we can see directly that $D\theta$ becomes a tensorial $(k+1)$-form of type (ρ, E). Applying a functor $Hom\,(*,\ L)$ to the fundamental sequence $\mathfrak{S}$ of $P(M, G)$, we have an exact sequence of vector bundles

$$0 \to Hom\,(T, L) \xrightarrow{\pi^*} Hom\,(Q, L) \xrightarrow{\lambda^*} Hom\,(L, L) \to 0\,.$$

Taking the sheaves of germs of their local sections, we obtain an exact sequence of sheaves

$$0 \to \boldsymbol{Hom}\,(T, L) \xrightarrow{\pi^*} \boldsymbol{Hom}\,(Q, L) \xrightarrow{\lambda^*} \boldsymbol{Hom}\,(L, L) \to 0\,,$$

and its cohomology sequence

$$\begin{array}{l} \xrightarrow{\pi^*} \Gamma(M, Hom\,(Q, L)) \xrightarrow{\lambda^*} \Gamma(M, Hom\,(L, L)) \xrightarrow{\delta^*} H^1(M, \boldsymbol{Hom}\,(T, L)) \to \\ \qquad \omega \longrightarrow \mathbf{1} \longrightarrow \delta^*\mathbf{1}. \end{array}$$

Hence, there exists a connection ω on $P(M, G)$, if and only if $\delta^*\mathbf{1} = -a(\mathfrak{S}) = 0$. Since $\boldsymbol{Hom}\,(T, L)_\infty$ is fine, we obtain the following result.

On a C^∞ principal bundle there exists a C^∞ connection, and on a complex analytic principal bundle there exists a (1, 0)*-connection, which means that its connection form ω becomes a* (1, 0)*-form.*

In the case of a complex analytic principal bundle $P(M, G)$, the obstruction class $a(\mathfrak{S})$ of the existence of analytic connection is represented by a (1, 1)-form through the Dolbeault isomorphism, and we have the following result derived by Atiyah.

Theorem 4.1. *Let ω be any* (1, 0)*-connection on a complex analytic principal bundle, and let Ω^{11} be the* (1, 1)*-component of its curvature form Ω. Then, the class $[\Omega^{11}]$ corresponds to the obstruction class $a(\mathfrak{S})$ of the existence of analytic connection under the Dolbeault isomorphism, that is,*

$$H^1(M, \boldsymbol{Hom}\,(T, L)_h) \cong H^1(A^{1*}(M, L))\,, \quad a(\mathfrak{S}) \to [\Omega^{11}]\,.$$

Proof. Setting $\tilde{\boldsymbol{A}}^1 = \boldsymbol{Hom}\,(Q, L)_h$, $\boldsymbol{A}^1 = \boldsymbol{Hom}\,(T, L)_h$, and $\boldsymbol{A}^{10} = \boldsymbol{Hom}\,(T, L)_\infty$, we have exact sequences of sheaves on M

$$0 \to \boldsymbol{A}^1 \xrightarrow{\pi^*} \tilde{\boldsymbol{A}}^1 \xrightarrow{\lambda^*} \boldsymbol{Hom}\,(L, L)_h \to 0\,,$$

$$0 \to \boldsymbol{A}^1 \xrightarrow{j} \boldsymbol{A}^{10} \xrightarrow{d''} \boldsymbol{Z}^{11} \to 0\,,$$

and their cohomology sequences

$$\xrightarrow{\pi^*} \tilde{A}^1(M, L) \xrightarrow{\lambda^*} \Gamma_h(M, Hom\,(L, L)) \xrightarrow{\delta^*} H^1(M, \boldsymbol{A}^1) \xrightarrow{\pi^*} \cdots ,$$
$$\xrightarrow{j} A^{10}(M, L) \xrightarrow{d''} Z^{11}(M, L) \xrightarrow{\delta''} H^1(M, \boldsymbol{A}^1) \to 0 .$$

For a (1, 0)-connection $\omega \in \tilde{A}^{10}(M, L)$, it holds that

$$\Omega^{11} = \left(d\omega + \frac{1}{2}[\omega, \omega]\right)^{11} = d''\omega \in Z^{11}(M, L) .$$

If $\omega, \omega_1 \in \tilde{A}^{10}(M, L)$ are two (1, 0)-connections, then the form $\varphi = \omega_1 - \omega$ becomes a tensorial (1, 0)-form of type $(ad, \mathfrak{g})$, and we see that

$$\Omega_1^{11} - \Omega^{11} = d''\omega_1 - d''\omega = d''\varphi, \quad \varphi \in A^{10}(M, L) .$$

This shows that the class $[\Omega^{11}] \in H^1(A^{1*}(M, L))$ does not depend on the choice of (1, 0)-connection ω. Take an open covering $\{U_i\}_{i\in I}$ of M and local holomorphic splittings of $\mathfrak{S}$

$$\{\omega_i\} \in C^0(\mathfrak{U}, \tilde{\boldsymbol{A}}^1), \quad \omega_i \circ \lambda = \mathbf{1} ,$$

and we obtain a 1-cocycle

$$\{a_{ij}^*\} \in C^1(\mathfrak{U}, \boldsymbol{A}^1) , \quad a_{ij}^* = \omega_i - \omega_j ,$$

which represents the obstruction class $\delta^*\mathbf{1} = -a(\mathfrak{S})$. On the other hand, by definition of the coboundary homomorphism δ'', we can take a tensorial (1, 1)-form $\Phi \in Z^{11}(M, L)$ and a 0-cochain $\{\varphi_i\} \in C^0(\mathfrak{U}, \boldsymbol{A}^{10})$ such that $d''\varphi_i = \Phi$ and $a_{ij}^* = \varphi_i - \varphi_j$. Then, the class $[\Phi] \in H^1(A^{1*}(M, L))$ corresponds to the class $-a(\mathfrak{S})$ under the Dolbeault isomorphism. Moreover, since $\omega_i - \omega_j = \varphi_i - \varphi_j$, we obtain a (1, 0)-connection $\omega \in \tilde{A}^{10}(M, L)$ such that $\omega = \omega_i - \varphi_i$ in U_i, and we see that

$$\Omega^{11} = d''\omega = d''\omega_i - d''\varphi_i = -\Phi ,$$

since ω_i is holomorphic. This proves that the class $[\Omega^{11}] \in H^1(A^{1*}(M, L))$ corresponds to the class $a(\mathfrak{S}) \in H^1(M, \boldsymbol{A}^1)$ under the Dolbeault isomorphism. The theorem has been thus proved.

Moreover, we can see directly the class $[\Omega^{11}] \in H^1(A^{1*}(M, L))$ express the obstruction of the existence of analytic connection on $P(M, G)$. Assume that ω is an analytic connection. Then, obviously $\Omega^{11} = d''\omega = 0$, and hence $[\Omega^{11}] = 0$. Conversely, assume that $[\Omega^{11}] = 0$, Then, there exists a tensorial (1, 0)-form $\varphi \in A^{10}(M, L)$ such that $d''\varphi = \Omega^{11}$. Setting $\tilde{\omega} = \omega - \varphi \in \tilde{A}^{10}(M, L)$, we obtain an analytic connection $\tilde{\omega}$, since $d''\tilde{\omega} = 0$.

§ 5. Extensions of tensorial forms

Let $P(M, G)$ be a principal bundle, and let us consider an exact sequence of associated vector bundles of $P(M, G)$

$$\tilde{\mathfrak{S}}:\ 0 \to K(M) \xrightarrow{\iota} W(M) \xrightarrow{\tau} V(M) \to 0 .$$

An extension of a tensorial k-form $\theta \in A^k(M, V)$ is by definition a tensorial k-form $\varphi \in A^k(M, W)$ such that $\tau \circ \varphi = \theta$. Applying a functor $Hom\,(T^{[k]}, *)$ to $\tilde{\mathfrak{S}}$, we have an exact sequence of vector bundles

$$0 \to Hom\,(T^{[k]}, K) \xrightarrow{\iota} Hom\,(T^{[k]}, W) \xrightarrow{\tau} Hom\,(T^{[k]}, V) \to 0 .$$

Taking the sheaves of germs of their local sections, we obtain an exact sequence of sheaves

$$0 \to \boldsymbol{Hom}\,(T^{[k]}, K) \xrightarrow{\iota} \boldsymbol{Hom}\,(T^{[k]}, W) \xrightarrow{\tau} \boldsymbol{Hom}\,(T^{[k]}, V) \to 0 ,$$

and its cohomology sequence

$$\begin{array}{l} \xrightarrow{\iota} A^k(M, W) \xrightarrow{\tau} A^k(M, V) \xrightarrow{\delta} H^1(M, \boldsymbol{Hom}\,(T^{[k]}, K)) \xrightarrow{\iota} \\ \qquad \varphi \longrightarrow \theta \longrightarrow \delta\theta. \end{array}$$

Hence, there exists an extension φ of θ, if and only if $\delta\theta = 0$. Since $\boldsymbol{Hom}\,(T^{[k]}, K)_\infty$ is fine, we get the following result.

For an exact sequence $\tilde{\mathfrak{S}}$ *of* C^∞ *vector bundles there exists a* C^∞ *extension* $\varphi \in A^k(M, W)$ *of a given* C^∞ *tensorial k-form* $\theta \in A^k(M, V)$, *and for an exact sequence* $\tilde{\mathfrak{S}}$ *of complex analytic vector bundles there exists a* C^∞ *(r, 0)-extension* $\varphi \in A^{r0}(M, W)$ *of a given* C^∞ *tensorial (r, 0)-form* $\theta \in A^{r0}(M, V)$.

Let $P(M, G)$ be a complex analytic principal bundle, and let $\rho:\ G \to GL(E)$ be a complex analytic representation. Take an associated vector bundle $V(M) = P \times_{\rho(G)} E$ and a (1, 0)-connection ω on $P(M, G)$. The covariant derivative $D\theta$ of an (r, s)-tensorial form

$\theta \in A^{rs}(M, V)$ is also becomes a tensorial form, which is decomposed uniquely as

$$D\theta = D'\theta + D''\theta\,, \quad D'\theta \in A^{r+1,s}(M, V), \quad D''\theta \in A^{r,s+1}(M, V)\,.$$

Since $D\theta$ is given by the formula $D\theta = d\theta + \bar{\rho}(\omega)\theta$, we have

$$D'\theta = d'\theta + \bar{\rho}(\omega)\theta\,, \quad D''\theta = d''\theta\,.$$

This shows that the derivation

$$D'' = d'' : A^{rs}(M, V) \to A^{r,s+1}(M, V)$$

does not depend on the choice of (1, 0)-connection, and we obtain a cochain complex

$$A^{r*}(M, V) = \sum_s A^{rs}(M, V), \; d'' : A^{rs}(M, V) \to A^{r,s+1}(M, V)\,,$$

on which we have remarked in § 3.

Let us consider an exact sequence of complex analytic associated vector bundles of $P(M, G)$

$$\tilde{\mathfrak{S}} :\; 0 \to K(M) \xrightarrow{\iota} W(M) \xrightarrow{\tau} V(M) \to 0\,.$$

Then, for a given analytic tensorial r-form $\theta \in A^r(M, V)$, the obstruction class

$$\delta\theta = \theta^* a(\tilde{\mathfrak{S}}) \in H^1(M, \boldsymbol{Hom}\,(T^{[r]}, K)_h)$$

of analytic extension of θ may be represented by an $(r, 1)$-form through the Dolbeault isomorphism.

Theorem 5.1. *Let $P(M, G)$ be a complex analytic principal bundle, and let*

$$\tilde{\mathfrak{S}} :\; 0 \to K(M) \xrightarrow{\iota} W(M) \xrightarrow{\tau} V(M) \to 0$$

be an exact sequence of complex analytic vector bundles associated with $P(M, G)$. For a given analytic tensorial r-form $\theta \in A^r(M, V)$, take an $(r, 0)$-extension $\varphi \in A^{r0}(M, W)$ of θ. Then the class $[-d''\varphi]$ corresponds to the obstruction class $\delta\theta$ of analytic extension of θ under the Dolbeault isomorphism, that is,

$$H^1(M, \boldsymbol{Hom}\,(T^{[r]}, K)_h) \cong H^1(A^{r*}(M, K))\,, \quad \delta\theta \to [-d''\varphi]\,.$$

Proof. Setting $\boldsymbol{A}^r = \boldsymbol{Hom}(T^{[r]}, K)_h$ and $\boldsymbol{A}^{r0} = \boldsymbol{Hom}(T^{[r]}, K)_\infty$, we have exact sequences of sheaves on M

$$0 \to \boldsymbol{A}^r \xrightarrow{\iota} \boldsymbol{Hom}(T^{[r]}, W)_h \xrightarrow{\tau} \boldsymbol{Hom}(T^{[r]}, V)_h \to 0\,,$$
$$0 \to \boldsymbol{A}^r \xrightarrow{j} \boldsymbol{A}^{r0} \xrightarrow{d''} \boldsymbol{Z}^{r1} \to 0\,,$$

and their cohomology sequences

$$\xrightarrow{\iota} \boldsymbol{A}^r(M, W) \xrightarrow{\tau} \boldsymbol{A}^r(M, V) \xrightarrow{\delta} H^1(M, \boldsymbol{A}^r) \xrightarrow{\iota} \cdots\,,$$
$$\xrightarrow{j} A^{r0}(M, K) \xrightarrow{d''} Z^{r1}(M, K) \xrightarrow{\delta''} H^1(M, \boldsymbol{A}^r) \xrightarrow{j} 0\,.$$

For an $(r, 0)$-extension $\varphi \in A^{r0}(M, W)$ of θ, it holds that

$$\tau d''\varphi = d''\tau\varphi = d''\theta = 0\,,$$

since θ is analytic. This shows that $d''\varphi \in Z^{r1}(M, K)$. If $\varphi, \varphi_1 \in A^{r0}(M, W)$ are two $(r, 0)$-extensions of θ. Then, we have a form $\kappa = \varphi_1 - \varphi \in A^{r0}(M, K)$ since $\tau\kappa = \theta - \theta = 0$, and we see that

$$d''\varphi_1 - d''\varphi = d''\kappa\,, \quad \kappa \in A^{r0}(M, K)\,.$$

This shows that the class $[d''\varphi] \in H^1(A^{r*}(M, K))$ does not depend on the choice of $(r, 0)$-extension φ of θ. Take an open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ and local holomorphic extensions of θ

$$\{\varphi_i\} \in C^0(\mathfrak{U}, \boldsymbol{Hom}(T^{[r]}, W)_h)\,, \quad \tau\varphi_i = \theta\,,$$

and we obtain a 1-cocycle

$$\{b_{ij}\} \in C^1(\mathfrak{U}, \boldsymbol{A}^r)\,, \quad b_{ij} = \varphi_i - \varphi_j\,,$$

which represents the obstruction class $\delta\theta = \theta^* a(\tilde{\mathfrak{S}})$. On the other hand, by definition of δ'', we can take an $(r, 1)$-form $\Psi \in Z^{r1}(M, K)$ and a 0-cochain $\{\psi_i\} \in C^0(\mathfrak{U}, \boldsymbol{A}^{r0})$ such that $d''\,\psi_i = \Psi$ and $b_{ij} = \psi_i - \psi_j$. Then, the class $[\Psi] \in H^1(A^{r*}(M, K))$ corresponds to the class $\delta\theta$ under the Dolbeault isomorphism. Moreover, since $\varphi_i - \varphi_j = \psi_i - \psi_j$, we obtain an $(r, 0)$-extension $\varphi \in A^{r0}(M, W)$ of θ such that $\varphi = \varphi_i - \psi_i$ in U_i, and we see that

$$d''\varphi = d''\varphi_i - d''\psi_i = -\Psi\,.$$

This proves that the class $[-d''\varphi] \in H^1(A^{r*}(M, K))$ corresponds to the obstruction class $\delta\theta \in H^1(M, A^r)$ under the Dolbeault isomorphism. The theorem has been thus proved.

Moreover, we can see directly the class $[d''\varphi] \in H^1(A^{r*}(M, K))$ express the obstruction of analytic extension of θ. Assume that φ is an analytic extension of θ. Then, obviously $d''\varphi = 0$, and so $[d''\varphi] = 0$. Conversely, assume that $[d''\varphi] = 0$. Then, there exists a tensorial $(r, 0)$-form $\psi \in A^{r0}(M, K)$ such that $d''\psi = d''\varphi$. Setting $\tilde{\varphi} = \varphi - \psi \in A^{r0}(M, W)$, we obtain an analytic extension $\tilde{\varphi}$ of θ, since $d''\tilde{\varphi} = 0$ and $\tau\tilde{\varphi} = \theta$.

§ 6. (G, ρ)-structures

Let $\rho: G \to GL(E)$ be a representation of a Lie group G on an n-dimensional vector space E. Then, it induces a representation of the Lie algebra of G

$$\bar{\rho}: \mathfrak{g} \to \mathfrak{gl}(E) \cong Hom(E, E) \cong E^* \otimes E, \; A \to \rho A, \; A \in \mathfrak{g} = T_e(G),$$

where E^* denotes the dual space of E. Now, we define a linear map α as follows:

$$\begin{aligned} \alpha: E^* \otimes \mathfrak{g} &\to E^1_{[2]} = (E^* \wedge E^*) \otimes E, \\ \sum u \otimes A &\to -\sum (u \wedge v) \otimes y, \quad A \in \mathfrak{g}, \quad u, v \in E^*, \quad y \in E, \end{aligned}$$

where we set $\bar{\rho}(A) = \sum v \otimes y \in E^* \otimes E$. Taking the dual representation $\rho^* = {}^t\rho^{-1}: G \to GL(E^*)$ of ρ, we have representations

$$\begin{aligned} \rho_1 &= \rho^* \otimes ad: G \to GL(E^* \otimes \mathfrak{g}), \\ \rho_2 &= (\rho^* \wedge \rho^*) \otimes \rho: G \to GL(E^1_{[2]}), \end{aligned}$$

of G. Then, we get an exact and commutative diagram of vector spaces

$$\begin{tikzcd} 0 \arrow[r] & Ker\,\alpha \arrow[r, "\lambda"] \arrow[d, "\rho_1(g)"] & E^* \otimes \mathfrak{g} \arrow[r, "\alpha"] \arrow[d, "\rho_1(g)"] & E^1_{[2]} \arrow[r, "\kappa"] \arrow[d, "\rho_2(g)"] & Coker\,\alpha \arrow[r] \arrow[d, "\rho_2(g)"] & 0 \\ 0 \arrow[r] & Ker\,\alpha \arrow[r, "\lambda"] & E^* \otimes \mathfrak{g} \arrow[r, "\alpha"] & E^1_{[2]} \arrow[r, "\kappa"] & Coker\,\alpha \arrow[r] & 0 \end{tikzcd}$$

for $g \in G$. Moreover, for a principal bundle $P(M, G)$, taking its associated vector bundles

$$K(M) = P\times_{\rho_1(G)} Ker\,\alpha\,,\quad L_1(M) = P\times_{\rho_1(G)}(E^*\otimes\mathfrak{g})\,,$$
$$V_2(M) = P\times_{\rho_2(G)}E^1_{(2)}\,,\quad J(M) = P\times_{\rho_2(G)} Coker\,\alpha\,,$$

we obtain an exact sequence of vector bundles.

$$0 \to K(M) \xrightarrow{\lambda} L_1(M) \xrightarrow{\alpha} V_2(M) \xrightarrow{\kappa} J(M) \to 0\,.$$

Now, let us consider the tangent frame bundle $P'(M, GL(E))$ of an n-dimensional manifold M, and let $\rho: G\to GL(E)$ be a representation. A restriction $P(M, G)$ of P' by ρ is called a *(G, ρ)-structure* on M. Namely, a (G, ρ)-structure on M is defined, if a homomorphism of bundles

$$\begin{aligned}\tilde\rho:\ & P(M, G) \to P'(M, GL(E))\,,\\ & \tilde\rho(pg) = \tilde\rho(p)\rho(g)\,,\quad p\in P\,,\quad g\in G\,,\end{aligned}$$

is given. We remark that, if there exists a (G, ρ)-structure on M, then the structural group $GL(E)$ of the tangent vector bundle $T(M)$ over M must be reduced to its subgroup $\rho(G)$. Assume that $P(M, G)$ defines a (G, ρ)-structure on M. Then, we obtain a natural bijection of vector bundles

$$\theta:\ T(M) \cong P\times_{\rho(G)}E\,,\quad \tilde\rho(p)y \to py\,,\quad p\in P\,,\quad y\in E\,,$$

which can be regarded as a tensorial 1-form $\theta:\ T(P)\to E$ on $P(M, G)$ of type (ρ, E). The form θ is called the *basic form* of the (G, ρ)-structure.

Proposition 6.1. *The basic form $\theta:\ T(P)\to E$ of a (G, ρ)-structure on M is characterized by the following properties.*

(i) $\theta\circ R_g = \rho(g^{-1})\theta\,,\quad g\in G\,.$

(ii) $\theta(X) = 0$ *for* $X\in T(P)$, *if and only if X is vertical.*

Proof. The properties (i), (ii) mean that the form θ is a tensorial 1-form on $P(M, G)$ of type (ρ, E). Hence, we can regard θ as a linear map of vector bundles $\theta:\ T(M)\to P\times_{\rho(G)}E$. Then, the property (ii) shows that θ is a bijection. This proves that the principal bundle $P(M, G)$ defines a (G, ρ)-structure on M with basic form θ.

Let $\theta:\ T(P)\to E$ be the basic form of a (G, ρ)-structure on M.

Then, to any tensorial k-form $\varphi: T^k(P)\to W$ on $P(M, G)$ of type (σ, W), corresponds uniquely a tensor

$$f: P\to Hom(E^{[k]}, W) = E_{[k]}\otimes W$$

on $P(M, G)$ of type $((\rho^*\wedge\cdots\wedge\rho^*)\otimes\sigma, E_{[k]}\otimes W)$ such that

$$\varphi = f\theta\wedge\cdots\wedge\theta\,.$$

Because, the form φ can be regarded as a linear map of vector bundles

$$\varphi:\ T^{[k]}(M)\to P\times_{\sigma(G)}W\,,$$

and by the Proposition 6.1 the k-form

$$\theta^{[k]} = \theta\wedge\cdots\wedge\theta:\ T^{[k]}(P)\to E^{[k]}$$

can be regarded as a bijection of vector bundles over M

$$\begin{tikzcd} \theta^{[k]}:\ T^{[k]}(M) \arrow[r, phantom, "\cong"] \arrow[dr, "\varphi"'] & P\times_{\rho^{[k]}(G)}E^{[k]} \arrow[d, "f"] \\ & P\times_{\sigma(G)}W\,. \end{tikzcd}$$

Hence, we have a one-to-one correspondence between φ and f.

From the definition of the linear map of vector bundles $\alpha: L_1(M)\to V_2(M)$, we can see easily the following.

Proposition 6.2. *Assume that a principal bundle $P(M, G)$ gives a (G, ρ)-structure on M with basic form $\theta: T(P)\to E$. Let $\varphi: T(P)\to\mathfrak{g}$ be a tensorial 1-form on $P(M, G)$ of type $(ad, \mathfrak{g})$. Then, the 2-form $\bar{\rho}(\varphi)\theta$ becomes a tensorial 2-form on $P(M, G)$ of type (ρ, E), and is determined by the relations:*

$$t = \alpha f\,,\quad \varphi = f\theta\,,\quad \bar{\rho}(\varphi)\theta = t\theta\wedge\theta\,,$$

where f, t are tensors on $P(M, G)$ of types $(\rho_1, E^*\otimes\mathfrak{g})$, $(\rho_2, E^1_{[2]})$ respectively.

Assume that $P(M, G)$ gives a (G, ρ)-structure θ on M, and take a connection ω on $P(M, G)$. The covariant derivative of the basic form

$$\Theta = D\theta = d\theta + \bar{\rho}(\omega)\theta$$

is called the (G, ρ)-*torsion form* of the connection ω. Then, there exists a unique tensor $T: P \to E^1_{(2)}$ on $P(M, G)$ of type $(\rho_2, E^1_{(2)})$ such that $\Theta = T\theta \wedge \theta$. The tensor T is called the *(G, ρ)-torsion* of the connection ω. Moreover, we obtain a tensor

$$S = \kappa T : P \to Coker\, \alpha$$

on $P(M, G)$ of type $(\rho_2, Coker\, \alpha)$ called the *structure tensor* of the (G, ρ)-structure.

Theorem 6. 1. *Assume that a principal bundle $P(M, G)$ gives a (G, ρ)-structure on M. Then, its structure tensor S does not depend on the choice of connection on $P(M, G)$.*

Proof. Let ω, ω_1 be two connections on $P(M, G)$, and set $\varphi = \omega_1 - \omega$. Then, φ becomes a tensorial 1-form on $P(M, G)$ of type $(ad, \mathfrak{g})$, and there exists a unique tensor $f: P \to E^* \otimes \mathfrak{g}$ on $P(M, G)$ of type $(\rho_1, E^* \otimes \mathfrak{g})$ such that $\varphi = f\theta$. Let $\Theta = T\theta \wedge \theta$, $\Theta_1 = T_1\theta \wedge \theta$ denote respectively the (G, ρ)-torsion forms of ω, ω_1. Then,

$$\Theta_1 - \Theta = (d\theta + \bar{\rho}(\omega_1)\theta) - (d\theta + \bar{\rho}(\omega)\theta) = \bar{\rho}(\varphi)\theta .$$

It follows from the Proposition 6. 2 that $T_1 - T = \alpha f$ and hence

$$\kappa T_1 - \kappa T = \kappa \circ \alpha f = 0 .$$

This proves that $\kappa T_1 = \kappa T = S$.

Here, we make some remarks on the existence of (G, ρ)-structure on a manifold M. Let $\rho : G \to GL(E)$ be a representation, and set $N = Ker\, \rho$, $G' = Im\, \rho$. Taking sheaves of germs of local functions on M with values in N, G, G' respectively, we have an exact sequence of sheaves on M

$$0 \to \boldsymbol{N} \xrightarrow{\lambda} \boldsymbol{G} \xrightarrow{\rho} \boldsymbol{G}' \to 0 ,$$

and its cohomology sequence

$$\begin{array}{l} \to H^1(M, \boldsymbol{N}) \xrightarrow{\lambda} H^1(M, \boldsymbol{G}) \xrightarrow{\rho} H^1(M, \boldsymbol{G}') \\ \qquad\qquad\qquad\qquad\quad \xi \longrightarrow \xi'. \end{array}$$

Assume that the structural group $GL(E)$ of the tangent vector bundle $T(M)$ of M is reduced to its subgroup G', and let $\xi' \in H^1(M, \boldsymbol{G}')$ be its bundle structure. Then, there exists a (G, ρ)-

structure on M, if $\xi' \in Im\, \rho$. For instance, if there exists a homomorphism of groups $j: G' \to G$ such that $\rho \circ j = \mathbf{1}$, then the bundle structure $\xi = j\xi' \in H^1(M, \boldsymbol{G})$ gives a (G, ρ)-structure on M. If N is contained in the center of G, the cohomology sequence can be defined as far as the next term

$$\xrightarrow{\rho} H^1(M, \boldsymbol{G}) \xrightarrow{\delta} H^2(M, \boldsymbol{N}),$$

and we obtain the obstruction class $\delta\xi' \in H^2(M, \boldsymbol{N})$ of the existence of (G, ρ)-structure on M.

Now, we research into connections with a given (G, ρ)-torsion.

Assume that a principal bundle $P(M, G)$ defines a (G, ρ)-structure θ on M. Take a connection ω_0 on $P(M, G)$, and then any connection ω on $P(M, G)$ is uniquely given by $\omega = \omega_0 + f\theta$, where f is any tensor on $P(M, G)$ of type $(\rho_1, E^* \otimes \mathfrak{g})$. In fact, for two connections ω_0, ω on $P(M, G)$, the form $\varphi = \omega - \omega_0$ becomes a tensorial 1-form on $P(M, G)$ of type $(ad, \mathfrak{g})$ which is uniquely given by $\varphi = f\theta$.

In the case of a complex analytic (G, ρ)-structure, take a $(1, 0)$-connection ω_0 on $P(M, G)$, and then any $(1, 0)$-connection ω on $P(M, G)$ is uniquely given by $\omega = \omega_0 + f\theta$, where f is any C^∞ tensor on $P(M, G)$ of type $(\rho_1, E^* \otimes \mathfrak{g})$. Moreover, since the form θ is analytic, the (G, ρ)-torsion form Θ of any $(1, 0)$-connection ω on $P(M, G)$ becomes a $(2, 0)$-form, and is given by $\Theta = T\theta \wedge \theta$, where T is a C^∞ tensor on $P(M, G)$ of type $(\rho_2, E^1_{(2)})$.

Proposition 6.3. *Assume that a principal bundle $P(M, G)$ gives a (G, ρ)-structure θ on M, and take a connection ω_0 on $P(M, G)$ with (G, ρ)-torsion T_0. Then, any connection ω on $P(M, G)$ with (G, ρ)-torsion T is given by $\omega = \omega_0 + f\theta$, where f is a tensor on $P(M, G)$ of type $(\rho_1, E^* \otimes \mathfrak{g})$ such that $\alpha f = T - T_0$.*

Proof. Take respectively the (G, ρ)-torsion forms $\Theta_0 = T_0\theta \wedge \theta$, $\Theta = T\theta \wedge \theta$ of two connections ω_0, ω on $P(M, G)$ and set, $\omega = \omega_0 + f\theta$. Then, we have the relation $\Theta - \Theta_0 = \bar{\rho}(f\theta)\theta$ which is equivalent to $T - T_0 = \alpha f$ by the Proposition 6.2.

Assume that $P(M, G)$ gives a (G, ρ)-structure on M with structure tensor S, and consider the exact sequence of vector bundles

$$0 \to K(M) \xrightarrow{\lambda} L_1(M) \xrightarrow{\alpha} V_2(M) \xrightarrow{\kappa} J(M) \to 0 .$$

Setting $Im\,\alpha = I(M)$, we have exact sequences of vector bundles over M

$$0 \to K(M) \xrightarrow{\lambda} L_1(M) \xrightarrow{\alpha} I(M) \to 0 ,$$

$$0 \to I(M) \xrightarrow{\iota} V_2(M) \xrightarrow{\kappa} J(M) \to 0 .$$

Taking sheaves of germs of their local sections, we obtain exact sequences of sheaves

$$0 \to \boldsymbol{K} \xrightarrow{\lambda} \boldsymbol{L}_1 \xrightarrow{\alpha} \boldsymbol{I} \to 0 ,$$

$$0 \to \boldsymbol{I} \xrightarrow{\iota} \boldsymbol{V}_2 \xrightarrow{\kappa} \boldsymbol{J} \to 0 ,$$

and their cohomology sequences

$$\begin{array}{l} 0 \to \Gamma(M, K) \xrightarrow{\lambda} \Gamma(M, L_1) \xrightarrow{\alpha} \Gamma(M, I) \xrightarrow{\delta} H^1(M, \boldsymbol{K}) \to \\ \qquad\qquad\qquad\quad f \longrightarrow t \longrightarrow \delta t, \end{array}$$

$$\begin{array}{l} 0 \to \Gamma(M, I) \xrightarrow{\iota} \Gamma(M, V_2) \xrightarrow{\kappa} \Gamma(M, J) \to H^1(M, \boldsymbol{I}) \to \\ \qquad\quad t \qquad\qquad T \longrightarrow S. \end{array}$$

Let T be a tensor on $P(M, G)$ of type $(\rho_2, E^1_{(2)})$, namely $T \in \Gamma(M, V_2)$. If T becomes the (G, ρ)-torsion of any connection on $P(M, G)$, it is necessary that $\kappa T = S$ by definition of the structure tensor. Take a connection ω_0 on $P(M, G)$ with (G, ρ)-torsion T_0, and set $t = T - T_0 \in \Gamma(M, I)$. Then, there exists a connection $\omega = \omega_0 + f\theta$ on $P(M, G)$ with given (G, ρ)-torsion T, if and only if there exists a tensor $f \in \Gamma(M, L_1)$ such that $\alpha f = t$. Therefore, we obtain the obstruction class $\delta t \in H^1(M, \boldsymbol{K})$ of the existence of connection with given (G, ρ)-torsion T.

Proposition 6.4. *The obstruction class $\delta t \in H^1(M, \boldsymbol{K})$ does not depend on the choice of connection ω_0 on $P(M, G)$.*

Proof. Let ω_0, ω_1 be two connections on $P(M, G)$ with (G, ρ)-torsions T_0, T_1 respectively. Then, we can set $\omega_1 = \omega_0 + g\theta$, $g \in \Gamma(M, L_1)$. For a given tensor $T \in \Gamma(M, V_2)$ such that $\kappa T = S$, setting $t = T - T_0$, $t' = T - T_1$, we see that

$$t-t' = T_1 - T_0 = \alpha g, \quad \text{and} \quad \delta t - \delta t' = \delta\circ\alpha g = 0.$$

This proves that $\delta t = \delta t' \in H^1(M, \boldsymbol{K})$.

In the case of a C^∞ (G, ρ)-structure, the class δt always vanishes, since $\boldsymbol{K}_\infty$ is fine and so $H^1(M, \boldsymbol{K}_\infty)=0$. Thereby, we obtain the following result.

Assume that a C^∞ principal budle $P(M, G)$ gives a C^∞ (G, ρ)-structure on M with structure tensor S. Let T be a C^∞ tensor on $P(M, G)$ of type $(\rho_2, E^1_{[2]})$ such that $\kappa T = S$. Then, there exists a C^∞ connection on $P(M, G)$ with given (G, ρ)-torsion T. In particular, there exists a C^∞ connection on $P(M, G)$ without (G, ρ)-torsion, if and only if $S=0$.

Now, we assume that a complex analytic principal bundle $P(M, G)$ gives an analytic (G, ρ)-structure on M. Then, its basic form θ is analytic. Moreover, the (G, ρ)-torsion T of any holomorphic connection ω is also holomorphic. Since $P(M, G)$ has a local holomorphic connection ω over a neighborhood of any point $x \in M$, and since the structure tensor $S = \kappa T$ does not depend on the choice of connection, S can be defined globally and becomes an analytic tensor.

Theorem 6. 2. *Assume that a complex analytic principal bundle $P(M, G)$ gives an analytic (G, ρ)-structure θ on M with structure tensor S.*

1° *Let T be a C^∞ tensor on $P(M, G)$ of type $(\rho_2, E^1_{[2]})$ such that $\kappa T = S$. Then, there exists a $(1, 0)$-connection on $P(M, G)$ with given (G, ρ)-torsion T.*

2° *Assume that there exists an analytic connection on $P(M, G)$. Let T be an analytic tensor on $P(M, G)$ of type $(\rho_2, E^1_{[2]})$ such that $\kappa T = S$. Take any $(1, 0)$-connection ω on $P(M, G)$ with given (G, ρ)-torsion T. Then, the $(1, 1)$-component Ω^{11} of its curvature form is given by $\Omega^{11} = \psi\theta$, where ψ is a tensorial $(0, 1)$-form on $P(M, G)$ of type $(\rho_1, \mathrm{Ker}\,\alpha)$, and the class $[-\psi] \in H^1(A^{0*}(M, K))$ corresponds to the obstruction class $\delta t \in H^1(M, \boldsymbol{K}_h)$ under the Dolbeault isomorphism.*

Proof. 1°. Since $\boldsymbol{K}_\infty$ is fine, we have $\delta t = 0$.

2°. Take an analytic connection ω_0 on $P(M, G)$ with (G, ρ)-torsion T_0, and set $\omega = \omega_0 + f\theta$, $f \in A^{00}(M, L_1)$. Since the forms θ, ω_0

are analytic, we have $\Omega^{11}=d''f\theta$ and hence $d''f=\psi$. Moreover, since T, T_0 are analytic, it holds that $\alpha\psi=d''\alpha f=d''(T-T_0)=0$. Therefore, we have a cohomology class

$$[-\psi]=[-d''f]\in H^1(A^{0*}(M, K)).$$

By the theorem 5.1, the class $[-d''f]$ corresponds to the obstruction class δt under the Dolbeault isomorphism. The theorem has been thus proved.

Moreover, we can see directly the class $[\psi]\in H^1(A^{0*}(M, K))$ express the obstruction of the existence of analytic connection with given (G, ρ)-torsion T. If there exists an analytic connection $\omega=\omega_0+f\theta$ with given (G, ρ)-torsion T, then we have $\psi=d''f=0$, since f is an analytic tensor. Obviousely, it holds that $[\psi]=0$. Conversely, if $[\psi]=0$, then there exists a C^∞ tensor $g\in A^{00}(M, K)$ such that $d''g=\psi$, and we obtain a (1, 0)-connection $\tilde{\omega}=\omega_0+(f-g)\theta$. Since $d''\tilde{\omega}=(d''f-d''g)\theta=0$, the connection $\tilde{\omega}$ becomes analytic, and its (G, ρ)-torsion $\tilde{T}$ is given by $\tilde{T}=T-\alpha g=T$.

§ 7. Soudures

Let $\tilde{P}$(M, $\tilde{G}$) be a pincipal bundle, and let G be a closed subgroup of $\tilde{G}$. Take an associated bundle $B=\tilde{P}/G$ of $\tilde{P}(M, \tilde{G})$ with fibre $F=\tilde{G}/G$, and assume that the following conditions are satisfied.

(a) There exists a section $s: M\rightarrow B$.

(b) $dim\, M = dim\, F$.

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "\tilde{s}"] \arrow[d, dash, "G"] & \tilde{P} \arrow[d, dash, "G"] \\
& M \arrow[r, "s"] \arrow[dr, equal] & B=\tilde{P}/G \arrow[d, dash, "F=\tilde{G}/G"] \\
& & M
\end{tikzcd}$$

The condition (a) implies that the structural group $\tilde{G}$ of $\tilde{P}$ can be reduced to its subgroup G, and we have a restriction P(M, G) of $\tilde{P}(M, \tilde{G})$ induced by the map s. Let us denote the injection by $\tilde{s}: P\rightarrow\tilde{P}$. For any point $x\in M$, we denote by V_x the vertical vector space of B at the point $s(x)\in B$. Then, we have an associated vector bundle

$$V(M)=\bigcup_{x\in M} V_x = P\times_{is(G)}\mathfrak{f}$$

of $P(M, G)$ determined by the isotropy representation $is: G\rightarrow GL(\mathfrak{f})$.

In fact, $V(M)$ can be regarded as an induced bundle of the vertical vector bundle $V(B)=\tilde{P}\times_{is(G)}\mathfrak{k}$ of B by the map s. Take the tangent vector bundle $T(M)$ of M. A bijection of vector bundles

$$\theta:\ T(M)\simeq V(M)=P\times_{is(G)}\mathfrak{k}$$

is called a *soudure* of B on M. If we identify a point $x\in M$ with $s(x)\in F_x$, and a tangent vector space $T_x(M)$ with $T_{s(x)}(F_x)=V_x$ by θ, we can suppose a fibre F_x of B as a tangent space of M at $x\in M$. In this sense, the fibre bundle B with a soudure on M is called a *tangent space bundle* over M.

By definition, a soudure of B on M can be regarded as a (G, is)-structure on M. Hence, a soudure is determined by the basic form $\theta:\ T(P)\rightarrow\mathfrak{k}$, characterized by the properties:

(i) $\theta\circ R_g=is(g^{-1})\theta\,,\quad g\in G\,,$

(ii) $\theta(X)=0$ for $X\in T(P)$, if and only if X is vertical.

By mean of the basic form θ, to any tensorial k-form $\varphi:\ T^k(P)\rightarrow W$ on $P(M, G)$ of type (σ, W) corresponds uniquely a tensor

$$f:\ P\rightarrow Hom(\mathfrak{k}^{[k]}, W)\simeq\mathfrak{k}_{[k]}\otimes W$$

on $P(M, G)$ of type $(is_{[k]}\otimes\sigma,\ \mathfrak{k}_{[k]}\otimes W)$ such that

$$\varphi=f\theta\wedge\cdots\wedge\theta\,.$$

Moreover, we have the linear map of vector spaces

$$\alpha:\ \mathfrak{k}^*\otimes\mathfrak{g}\rightarrow\mathfrak{k}^1_{[2]}\,,$$

and the associated vector bundles of $P(M, G)$

$$L_1(M)=P\times_{\rho_1(G)}(\mathfrak{k}^*\otimes\mathfrak{g}),\ V_2(M)=P\times_{\rho_2(G)}\mathfrak{k}^1_{[2]}\,.$$

Then, the map α induces the linear map of vector bundles $\alpha:\ L_1(M)\rightarrow V_2(M)$. Thus, we may consider the *(G, is)-torsion* T of a connection ω on $P(M, G)$, and we can define the *structure tensor* S of a soudure. Rerated to them, the results in the preceding section also hold.

§ 8. Cartan connections

Let $B=\tilde{P}/G$ be a fibre bundle satisfying the conditions (a), (b)

in the preceding section, and take the restriction $\tilde{s}: P \to \tilde{P}$ induced by the section $s: M \to B$. The natural projection $\tau: \tilde{G} \to F = \tilde{G}/G$ induces a projection $\tau: \tilde{\mathfrak{g}} \to \mathfrak{f}$, and we have an exact and commutative diagram of vector spaces

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{g} & \xrightarrow{\iota} & \tilde{\mathfrak{g}} & \xrightarrow{\tau} & \mathfrak{f} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle ad(g)} & & \downarrow{\scriptstyle ad(g)} & & \downarrow{\scriptstyle is(g)} & & \\ 0 & \longrightarrow & \mathfrak{g} & \xrightarrow{\iota} & \tilde{\mathfrak{g}} & \xrightarrow{\tau} & \mathfrak{f} & \longrightarrow & 0 \end{array}$$

for $g \in G$, where $\tilde{\mathfrak{g}} = T_e(\tilde{G})$, $\mathfrak{g} = T_e(G)$ and $\mathfrak{f} = T_{\tau e}(F)$. Taking associated vector bundles of $P(M, G)$

$$L(M) = P \times_{ad(G)} \mathfrak{g}\,, \quad \tilde{L}(M) = P \times_{ad(G)} \tilde{\mathfrak{g}}\,, \quad V(M) = P \times_{is(G)} \mathfrak{f}\,,$$

we get an exact sequence of vector bundles

$$0 \to L(M) \xrightarrow{\iota} \tilde{L}(M) \xrightarrow{\tau} V(M) \to 0\,.$$

On the other hand, we have the fundamental sequence of $P(M, G)$

$$0 \to L(M) \xrightarrow{\lambda} Q(M) \xrightarrow{\pi} T(M) \to 0\,.$$

The *Cartan connection* of $B = \tilde{P}/G$ is by definition an isomorphism of these sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L(M) & \xrightarrow{\lambda} & Q(M) & \xrightarrow{\pi} & T(M) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \theta} & & \\ 0 & \longrightarrow & L(M) & \xrightarrow{\iota} & \tilde{L}(M) & \xrightarrow{\tau} & V(M) & \longrightarrow & 0\,, \end{array}$$

namely, ω and θ are bijections of vector bundles such that $\tau\circ\omega = \theta\circ\pi$ and $\omega\circ\lambda = \iota$. We may suppose that a Cartan connection of B is defined by a bijection of vector bundles $\omega: Q(M) \to \tilde{L}(M)$ such that $\omega\circ\lambda = \iota$. In fact, if such a bijection ω is given, then it induces uniquely a bijection $\theta: T(M) \to V(M)$ such that $\tau\circ\omega = \theta\ \pi$. The bijection ω is regarded as a contravariant 1-form $\omega: T(P) \to \tilde{\mathfrak{g}}$ on $P(M, G)$ of type $(ad, \tilde{\mathfrak{g}})$ called the *Cartan connection form.*

Proposition 8.1. *The Cartan connection form $\omega: T(P) \to \tilde{\mathfrak{g}}$ is characterized by the following properties,*

(i) $\omega\circ R_g = ad(g^{-1})\omega$, $g\in G$.

(ii) $\omega(pA) = A$, $p\in P$, $A\in\mathfrak{g}$.

(iii) *If* $\omega(X) = 0$ *for* $X\in T(P)$, *then* $X = 0$.

Proof. The property (i) shows that ω is a contravariant 1-form on $P(M, G)$ of type $(ad, \tilde{\mathfrak{g}})$. Hence, we can regard ω as a linear map of vector bundles $\omega: Q(M)\to\tilde{L}(M)$. Then, the property (ii) means that $\omega\circ\lambda=\iota$, and the property (iii) proves that ω becomes a bijection.

Assume that a Cartan connection $\omega: T(P)\to\tilde{\mathfrak{g}}$ of B is given. Then, we have uniquely a connection $\tilde{\omega}: T(\tilde{P})\to\tilde{\mathfrak{g}}$ on $\tilde{P}(M, \tilde{G})$ such that $\omega=\tilde{\omega}\circ\tilde{s}$, and a soudure θ: $T(P)\to\mathfrak{f}$ of B on M such that $\theta=\tau\circ\omega$. Let $\tilde{\Omega}$ be the curvature form of the connection $\tilde{\omega}$ on $\tilde{P}(M, \tilde{G})$. Then, we have a tensorial 2-form

$$
\begin{tikzcd}
T(P) \arrow[r, "\omega"] \arrow[d, "\tilde{s}"'] & \tilde{\mathfrak{g}} \\
T(\tilde{P}) \arrow[ur, "\tilde{\omega}"'] &
\end{tikzcd}
$$

$$\Omega = \tilde{\Omega}\circ\tilde{s}: T^2(P)\to\tilde{\mathfrak{g}}$$

on $P(M, G)$ of type $(ad, \tilde{\mathfrak{g}})$ called the *Cartan curvature form* of the Cartan connection ω. Clearly, it is given by the structure equation

$$\Omega = d\omega+\frac{1}{2}[\omega, \omega].$$

Now we research into Cartan connections with a given soudure. Assume that a principal bundle $P(M, G)$ gives a soudure θ of B on M, and consider the exact sequences of vector bundles

$$
\begin{tikzcd}
\mathfrak{S}: 0 \arrow[r] & L(M) \arrow[r, "\lambda"] \arrow[d, equal] & Q(M) \arrow[r, "\pi"] & T(M) \arrow[r] \arrow[d, "\theta"] & 0 \\
\tilde{\mathfrak{S}}: 0 \arrow[r] & L(M) \arrow[r, "\iota"] & \tilde{L}(M) \arrow[r, "\tau"] & V(M) \arrow[r] & 0.
\end{tikzcd}
$$

Then, we have the obstruction classes of their splittings

$$a(\mathfrak{S})\in H^1(M, \boldsymbol{Hom}\,(T, L)),\quad a(\tilde{\mathfrak{S}})\in H^1(M, \boldsymbol{Hom}\,(V, L)).$$

The bijection θ: $T(M)\cong V(M)$ induces a bijection of cohomology groups

$$\theta^*: H^1(M, \boldsymbol{Hom}\,(V, L))\cong H^1(M, \boldsymbol{Hom}\,(T, L)).$$

Then, by the Propositions 2. 4 and 2. 5, the class

$$a(\mathfrak{S}) - \theta^* a(\tilde{\mathfrak{S}}) \in H^1(M, \boldsymbol{Hom}\,(T, L))$$

is regarded as the obstruction class of the existence of Cartan connection with given soudure θ.

In the case of a C^∞ soudure, since the sheaf $\boldsymbol{Hom}\,(T, L)_\infty$ is fine, the obstruction class $a(\mathfrak{S}) - \theta^* a(\tilde{\mathfrak{S}})$ always vanishes. Thereby, we obtain the following result.

Assume that a C^∞ principal bundle $P(M, G)$ gives a C^∞ soudure θ of B on M. Then, there exists a C^∞ Cartan connection of B with given soudure θ.

Now, let us consider the case of a complex analytic soudure.

Theorem 8.1. *Assume that a complex analytic principal bundle $P(M, G)$ gives an analytic soudure θ of B on* M.

1° *There exists a* (1, 0) *Cartan connection of B with given soudure θ.*

2° *Take any* (1, 0) *Cartan connection ω of B with given soudure θ. Then the* (1, 1)*-component Ω^{11} of its Cartan curvature form becomes a tensorial* (1, 1)*-form on $P(M, G)$ of type $(ad, \mathfrak{g})$, and the class $[\Omega^{11}] \in H^1(A^{1*}(M, L))$ corresponds to the obstruction class*

$$a(\mathfrak{S}) - \theta^* a(\tilde{\mathfrak{S}}) \in H^1(M, \boldsymbol{Hom}\,(T, L)_h)$$

of the existence of analytic Cartan connection of B with given soudure θ under the Dolbeault isomorphism.

Proof. 1°. Since $\boldsymbol{Hom}(T, L)_\infty$ is fine, we have $a(\mathfrak{S}) - \theta^* a(\tilde{\mathfrak{S}}) = 0$.

2°. For a (1, 0) Cartan connection ω of B with given soudure θ, since θ is analytic, it holds that

$$\Omega^{11} = d''\omega\,, \quad \tau\Omega^{11} = d''\tau\omega = d''\theta = 0\,,$$

and hence $\Omega^{11} \in A^{11}(M, L)$, $d''\Omega^{11} = 0$. Therefore, we have a cohomology class $[\Omega^{11}] \in H^1(A^{1*}(M, L))$. Let ω, ω_1 be two (1, 0) Cartan connections of B with given soudure θ. Then, we have a (1, 0)-form $\psi = \omega_1 - \omega \in A^{10}(M, L)$, since $\tau\psi = \theta - \theta = 0$, and it holds that $\Omega_1^{11} - \Omega^{11} = d''\psi$. This proves that the class $[\Omega^{11}]$ does not depend on the choice of (1, 0) Cartan connection ω of B with given soudure θ. Let us take a (1, 0)-connection ω_0 on $P(M, G)$ and a (1, 0)-extension φ of θ over $\tilde{L}(M)$. Then, we have a (1, 0) Cartan connection $\omega=\omega_0+\varphi$ of B with given soudure θ, and it holds that

$$\Omega^{11} = \Omega_0^{11}+d''\varphi ,$$

where $\Omega_0^{11}=d''\omega_0$ denotes the (1, 1)-component of the curvature form of ω_0. By the theorems 4.1 and 5.1, the classes

$$[\Omega_0^{11}], \quad [-d''\varphi]\in H^1(A^{1*}(M, L))$$

correspond respectively the obstruction classes

$$a(\mathfrak{S}), \ \delta\theta\in H^1(M, \boldsymbol{Hom}(T, L)_h)$$

under the Dolbeault isomorphism. Since $\delta\theta=\theta^*a(\tilde{\mathfrak{S}})$ by the Proposition 2.4, the class $[\Omega^{11}]$ corresponds to the class $a(\mathfrak{S})-\theta^*a(\tilde{\mathfrak{S}})$ under the Dolbeault isomorphism. The theorem has been thus proved.

Moreover, we can see directly the class $[\Omega^{11}]\in H^1(A^{1*}(M, L))$ express the obstruction of the existence of analytic Cartan connection with given soudure θ. If there exists an analytic Cartan connection ω with given soudure θ, then $\Omega^{11}=d''\omega=0$, and hence $[\Omega^{11}]=0$. Conversely, if $[\Omega^{11}]=0$, then there exists a tensorial (1, 0)-form $\psi\in A^{10}(M, L)$ such that $d''\psi=\Omega^{11}$, and we obtain a (1, 0) Cartan connection $\tilde{\omega}=\omega-\psi$. Then, we have

$$d''\tilde{\omega} = \Omega^{11}-\Omega^{11} = 0 , \quad \tau\tilde{\omega} = \tau\omega = \theta .$$

This proves that $\tilde{\omega}$ becomes an analytic Cartan connection with given soudure θ.

§ 9. The Cartan structure tensor of a soudure

Let $\omega: T(P)\to\tilde{\mathfrak{g}}$ be a Cartan connection of $B=\tilde{P}/G$, and let $\Omega: T^2(P)\to\tilde{\mathfrak{g}}$ denote its Cartan curvature form. Then, we have a tensorial 2-form

$$\tilde{\Theta} = \tau\Omega : T^2(P) \to \mathfrak{f}$$

on $P(M, G)$ of type $(is, \mathfrak{f})$ called the *Cartan torsion form* of Cartan connection ω. Take the basic form of soudure

$$\theta = \tau\omega : T(P) \to \mathfrak{f} .$$

Then, the Cartan torsion form of ω is given by the structure equation

$$\tilde{\Theta} = d\theta + \frac{1}{2}\tau[\omega, \omega].$$

Moreover, we have uniquely a tensor $\tilde{T}: P \to \mathfrak{k}^1_{(2)}$ on $P(M, G)$ of type $(\rho_2, \mathfrak{k}^1_{(2)})$ such that $\tilde{\Theta} = \tilde{T}\theta \wedge \theta$. The tensor $\tilde{T}$ is called the *Cartan torsion* of ω. Let us consider the exact sequence of vector bundles

$$0 \to K(M) \overset{\lambda}{\to} L_1(M) \overset{\alpha}{\to} V_2(M) \overset{\kappa}{\to} J(M) \to 0.$$

Assume that a principal bundle $P(M, G)$ gives a soudure θ of B on M, and take a Cartan connection ω with given soudure θ. Let $\tilde{T}$ be its Cartan torsion. Then, we obtain a tensor $\tilde{S} = \kappa\tilde{T}$ on $P(M, G)$ of type $(\rho_2, \mathit{Coker}\,\alpha)$ called the *Cartan structure tensor* of soudure θ.

Theorem 9.1. *The Cartan structure tensor $\tilde{S}$ of soudure θ does not depend on the choice of Cartan connection of B with given soudure θ.*

Proof. Let ω, ω_1 be two Cartan connections of B with given soudure θ, and let $\tilde{\Theta} = \tilde{T}\theta \wedge \theta$, $\tilde{\Theta}_1 = \tilde{T}_1\theta \wedge \theta$ denote respectively their Cartan torsion forms. Then, the form $\varphi = \omega_1 - \omega$ becomes a tensorial 1-form on $P(M, G)$ of type $(ad, \mathfrak{g})$, since $\tau\varphi = \theta - \theta = 0$. It holds that

$$\begin{aligned}\tilde{\Theta}_1 - \tilde{\Theta} &= \left(d\theta + \frac{1}{2}\tau[\omega_1, \omega_1]\right) - \left(d\theta + \frac{1}{2}\tau[\omega, \omega]\right) \\ &= \frac{1}{2}\tau[\varphi, \omega_1] + \frac{1}{2}\tau[\varphi, \omega] \\ &= \frac{1}{2}\,\overline{is}\,(\varphi)\tau\omega_1 + \frac{1}{2}\,\overline{is}\,(\varphi)\tau\omega = \overline{is}\,(\varphi)\theta.\end{aligned}$$

Therefore, setting $\varphi = f\theta$, we have $\tilde{T}_1 - \tilde{T} = \alpha f$ and so $\kappa\tilde{T}_1 - \kappa\tilde{T} = \kappa\circ\alpha f = 0$. This proves that $\kappa\tilde{T}_1 = \kappa\tilde{T} = \tilde{S}$.

Proposition 9.1. *Let θ be a soudure of B on M, and let S, $\tilde{S}$ denote respectively its structure tensor and its Cartan structure tensor. Take any tensorial 1-form ψ on $P(M, G)$ of type $(ad, \tilde{\mathfrak{g}})$ such that $\tau\psi = \theta$, and set $\tau[\psi, \psi] = U\theta \wedge \theta$. Then, it holds that*

$$\tilde{S} = S + \frac{1}{2}\kappa U .$$

Proof. Let ω_0 be a connection on $P(M, G)$. Then, any Cartan connection ω of B with soudure θ is given by $\omega = \omega_0 + \psi$, where ψ is any tensorial 1-form on $P(M, G)$ of type $(ad, \tilde{\mathfrak{g}})$ such that $\tau\psi = \theta$. Let Θ, $\tilde{\Theta}$ denote respectively the (G, is)-torsion form of ω_0 and the Cartan torsion form of ω. Then, it holds that

$$\begin{aligned}\tilde{\Theta} - \Theta &= \left(d\theta + \frac{1}{2}\tau[\omega, \omega]\right) - (d\theta + \overline{is}\,(\omega_0)\theta) \\ &= \frac{1}{2}\tau[\omega, \omega] - \tau[\omega_0, \omega] = \frac{1}{2}\tau[\psi, \psi] .\end{aligned}$$

This proves that $\tilde{S} - S = \frac{1}{2}\kappa U$.

Accordingly, if the homogeneous space $F = \tilde{G}/G$ is symmetric, then the Cartan structure tensor $\tilde{S}$ of soudure θ coincides with the structure tensor S of soudure θ.

Now, we research into Cartan connections with a given Cartan torsion. Assume that a principal bundle $P(M, G)$ defines a soudure θ of $B = \tilde{P}/G$ on M. Take a Cartan connection ω_0 of B with soudure θ, and then any Cartan connection ω of B with soudure θ is uniquely given by $\omega = \omega_0 + g\theta$, where g is any tensor on $P(M, G)$ of type $(\rho_1, E^* \otimes \mathfrak{g})$.

Proposition 9. 2. *Assume that a principal bundle $P(M, G)$ gives a soudure θ of B on M, and take a Cartan connection ω_0 of B with soudure θ and Cartan torsion $\tilde{T}_0$. Then, any Cartan connection ω of B with soudure θ and Cartan torsion $\tilde{T}$ is given by $\omega = \omega_0 + g\theta$, where g is any tensor on $P(M, G)$ of type $(\rho_1, E^* \otimes \mathfrak{g})$ such that $\alpha g = \tilde{T} - \tilde{T}_0$.*

Proof. Take respectively the Cartan torsion forms $\tilde{\Theta}_0 = \tilde{T}_0\theta \wedge \theta$, $\tilde{\Theta} = \tilde{T}\theta \wedge \theta$ of two Cartan connections ω_0, ω of B with soudure θ, and set $\omega = \omega_0 + g\theta$. Then, we have the relation

$$\tilde{\Theta} - \tilde{\Theta}_0 = \overline{is}\,(g\theta)\theta ,$$

which is equivalent to $\tilde{T} - \tilde{T}_0 = \alpha g$.

Consider the exact sequences of sheaves on M

$$0 \to \boldsymbol{K} \xrightarrow{\lambda} \boldsymbol{L}_1 \xrightarrow{\alpha} \boldsymbol{I} \to 0\,,$$

$$0 \to \boldsymbol{I} \xrightarrow{\iota} \boldsymbol{V}_2 \xrightarrow{\kappa} \boldsymbol{J} \to 0\,,$$

and their cohomology sequence

$$\begin{array}{l} 0 \to \Gamma(M, K) \xrightarrow{\lambda} \Gamma(M, L_1) \xrightarrow{\alpha} \Gamma(M, I) \xrightarrow{\delta} H^1(M, \boldsymbol{K}) \to \\ \qquad\qquad\qquad\quad g \longrightarrow \tilde{t} \longrightarrow \delta\tilde{t}, \end{array}$$

$$\begin{array}{l} 0 \to \Gamma(M, I) \xrightarrow{\iota} \Gamma(M, V_2) \xrightarrow{\kappa} \Gamma(M, J) \to H^1(M, \boldsymbol{I}) \to \\ \qquad\quad \tilde{t} \qquad\qquad \tilde{T} \longrightarrow \tilde{S}. \end{array}$$

If a tensor $\tilde{T} \in \Gamma(M, V_2)$ becomes the Cartan torsion of any Cartan connection of B with soudure θ, it is necessary that $\kappa\tilde{T}=\tilde{S}$. Take a Cartan connection ω_0 of B with soudure θ and Cartan torsion $\tilde{T}_0$, and set $\tilde{t}=\tilde{T}-\tilde{T}_0 \in \Gamma(M, I)$. Then, we obtain the obstruction class $\delta\tilde{t} \in H^1(M, \boldsymbol{K})$ of the existence of Cartan connection with soudure θ and given Cartan torsion $\tilde{T}$.

Proposition 9. 3. *The obstruction class $\delta\tilde{t} \in H^1(M, \boldsymbol{K})$ does not depend on the choice of Cartan connection ω_0 of B with given soudure θ.*

Proof. Let ω_0, ω_1 be two Cartan connections of B with given soudure θ. Take their Cartan torsions $\tilde{T}_0$, $\tilde{T}_1$ respectively. Then, we can set $\omega_1=\omega_0+h\theta$, $h \in \Gamma(M, L_1)$. For a given tensor $\tilde{T} \in \Gamma(M, V_2)$ such that $\kappa\tilde{T}=\tilde{S}$, setting $\tilde{t}=\tilde{T}-\tilde{T}_0$, $\tilde{t}'=\tilde{T}-\tilde{T}_1$, we see that

$$\tilde{t}-\tilde{t}' = \tilde{T}_1-\tilde{T}_0 = \alpha h\,, \quad \text{and} \quad \delta\tilde{t}-\delta\tilde{t}' = \delta\circ\alpha h = 0\,.$$

This proves that $\delta\tilde{t}=\delta\tilde{t}' \in H^1(M, \boldsymbol{K})$.

In the case of a C^∞ soudure, the class $\delta\tilde{t}$ always vanishes, since $\boldsymbol{K}_\infty$ is fine. Thereby we obtain the following result.

Assume that a C^∞ principal bundle $P(M, G)$ gives a C^∞ soudure θ of B on M with Cartan structure tensor $\tilde{S}$. Let $\tilde{T}$ be a C^∞ tensor on $P(M, G)$ of type $(\rho_2, \mathfrak{f}^1_{[2]})$ such that $\kappa\tilde{T}=\tilde{S}$. Then, there exists a C^∞ Cartan connection of B with soudure θ and given Cartan torsion $\tilde{T}$. In particular, there exists a C^∞ Cartan connection of B without Cartan torsion, if and only if $\tilde{S}=0$.

In the case of a complex analytic soudure, we obtain the result like the Theorem 6. 2.

Theorem 9. 2. *Assume that a complex analytic principal bundle $P(M, G)$ gives an analytic soudure θ of B on M with Cartan structure tensor $\tilde{S}$.*

1° *Let $\tilde{T}$ be a C^∞ tensor on $P(M, G)$ of type $(\rho_2, \mathfrak{f}^1_{(2)})$ such that $\kappa\tilde{T}=\tilde{S}$. Then, there exists a $(1, 0)$ Cartan connection of B with soudure θ and given Cartan torsion $\tilde{T}$.*

2° *Assume that there exists an analytic Cartan connection of B with soudure θ. Let $\tilde{T}$ be an analytic tensor on $P(M, G)$ of type $(\rho_2, \mathfrak{f}^1_{(2)})$ such that $\kappa\tilde{T}=\tilde{S}$. Take any $(1, 0)$ Cartan connection ω of B with soudure θ and given Cartan torsion $\tilde{T}$. Then, the $(1, 1)$-component Ω^{11} of its Cartan curvature form is given by $\Omega^{11}=\psi\theta$, where ψ is a tensorial $(0, 1)$-form on $P(M, G)$ of type $(\rho_1, Ker\,\alpha)$, and the class $[-\psi]\in H^1(A^{0*}(M, K))$ corresponds to the obstruction class $\delta\tilde{t}\in H^1(M, \boldsymbol{K}_h)$ under the Dolbeault isomorphism.*

Proof. 1°. Since $\boldsymbol{K}_\infty$ is fine, we have $\delta\tilde{t}=0$.

2°. Take an analytic Cartan connection ω_0 of B with soudure θ and Cartan torsion $\tilde{T}_0$, and set $\omega=\omega_0+g\theta$, $g\in A^{00}(M, L_1)$. Then, we have $\Omega^{11}=d''\omega=d''g\theta$ and hence $d''g=\psi$. Moreover, it holds that $\alpha\psi=d''\alpha g=d''(\tilde{T}-\tilde{T}_0)=0$. Therefore, we have a class

$$[-\psi] = [-d''g]\in H^1(A^{0*}(M, K)).$$

By the Theorem 5. 1, the class $[-d''g]$ corresponds to the class $\delta\tilde{t}$ under the Dolbeault isomorphism. The theorem has been proved.

Moreover, we can see directly the class $[\psi]$ express the obstruction of the existence of analytic Cartan connection with soudure θ and given Cartan torsion $\tilde{T}$. If there exists an analytic Cartan connection $\omega=\omega_0+g\theta$ with soudure θ and given Cartan torsion $\tilde{T}$, then we have $\psi=d''g=0$ and $[\psi]=0$. Conversely, if $[\psi]=0$, then there exists a C^∞ tensor $h\in A^{00}(M, K)$ such that $d''h=\psi$, and we obtain an analytic Cartan connection $\tilde{\omega}=\omega_0+(g-h)\theta$ with soudure θ and given Cartan torsion $\tilde{T}$.

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